No proof is needed for TRUE-FALSE questions; just write clearly. You may assume given matrix expressions are well defined (i.e. the matrix sizes are compatible).

1. (a) Below are a matrix $A$ and the matrix $\text{rref}(A)$ produced by MATLAB.

$$A = \begin{pmatrix} 22 & -22 & 44 & 14 & 7 & 182 \\ 4 & -4 & 8 & 5 & 2 & 37 \\ 14 & -14 & 28 & 4 & 3 & 108 \\ 0 & 0 & 0 & 0 & 10 & 20 \\ 15 & -15 & 30 & 10 & 25 & 165 \end{pmatrix}, \quad \text{rref}(A) = \begin{pmatrix} 1 & -1 & 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Write down a basis for each of the following (no justification required).

i. (4 points) $\text{Row}(A)$, the row space of $A$.

**SOLUTION:** The first three rows of $\text{rref}(A)$.

(The first three rows of $A$ are NOT a basis.)

ii. (4 points) $\text{Col}(A)$, the column space of $A$.

**SOLUTION:** Columns 1, 4 and 5 of $A$.

iii. (6 points) $\text{Nul}(A)$, the null space of $A$.

**SOLUTION:** The following vectors form a basis:

$$\begin{pmatrix} -7 \\ 0 \\ -1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$ 

(b) (6 points) For each of the following, answer TRUE or FALSE.

i. **FALSE** $\text{rank}(A + B) \geq \text{rank}(A)$ whenever $A$ and $B$ are $10 \times 21$ matrices.

ii. **TRUE** $\text{rank}(AB) \leq \text{rank}(A)$ whenever $A$ and $B$ are $10 \times 10$ matrices.

2. (a) (15 points) Define

$$A = \begin{pmatrix} -3 & -2 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad D = \{x \in \mathbb{R}^2 : \sqrt{(x_1 - \pi)^2 + (x_2 - 17)^2} \leq 3\}.$$ 

Compute the area of the set $E = \{Ax : x \text{ is in } D\}$.

**SOLUTION:**

area($E$) = $|\text{det}(A)||\text{area}(D)| = |-7|(|\pi|2)^2 = 63\pi$.

(b) (5 points) Suppose $A$ and $B$ are $5 \times 5$ matrices with $\text{det}(A) = 10$ and $\text{det}(B) = 4$.

Compute the determinant of the matrix $M = -2A^3B^{-1}$.

**SOLUTION**

$\text{det}(M) = (-2)^5(\text{det}(A))^3(1/\text{det}(B)) = (-32)(10^3)(1/4) = -8000$. 

3. (a) (14 points) Let \( \mathbb{P}^1 \) denote the vector space of polynomials of degree at most 1; then \( B = \{3 + t, 5 + 5t\} \) is a basis of \( B \).

Find the coordinates vector \( x = [-7 + t]_B \) of the polynomial \(-7 + t\).

**SOLUTION.** This vector is the vector \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) such that \( x_1 (3 + t) + x_2 (5 + 5t) = -7 + t \). This \( x \) is the solution of

\[
\begin{pmatrix} 3 & 5 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}
\]

and this solution is \( x = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \).

(b) (6 points) For each of the following, answer TRUE or FALSE.

i. **FALSE.** \( \mathbb{R}^2 \) is a subspace of \( \mathbb{R}^3 \).

ii. **FALSE.** If \( C \) is a set of 14 vectors which span \( \mathbb{R}^5 \), then \( C \) contains every basis of \( \mathbb{R}^5 \).

4. (a) (8 points) Suppose that \( A \) and \( B \) are similar matrices. Prove that their characteristic polynomials are equal.

**SOLUTION:**

We have \( U^{-1}AU = B \) for some invertible \( U \). Therefore

\[
\det(tI - B) = \det(tI - U^{-1}BU) = \det(U^{-1}(tI - A)U) = \det(U^{-1})\det(tI - A)\det(U) = \frac{1}{\det(U)}\det(tI - A)\det(U) = \det(tI - A).
\]

(b) (12 points) Let \( V \) be the vector space of differentiable functions from \( \mathbb{R} \) to \( \mathbb{R} \).

For each of the following, answer TRUE or FALSE.

i. **TRUE** The set of functions \( \{\cos^2 t, \sin^2 t\} \) is a linearly independent subset of \( V \).

ii. **TRUE** The map \( T : V \to \mathbb{R} \) defined by the rule \( T(f) = f'(1) \) is a linear transformation.

iii. **FALSE** The map \( T : V \to \mathbb{R} \) defined by the rule \( T(f) = f(1) - 1 \) is a linear transformation.

iv. **TRUE** If \( S \) is a nonempty subset of a vector space \( V \), then the set of all linear combinations of \( S \) is a subspace of \( V \).

5. (a) (8 points) The determinant of an \( n \times n \) matrix \( A \) is a polynomial function of the entries of \( A \).

i. What is the degree of this polynomial, if \( n = 5 \)?

**Solution.** The degree is 5.

ii. This polynomial is a sum of monomials; how many monomials are there in this sum, if \( n = 5 \)?

**Solution.** The number of monomials here is \( 5! = 120 \).
(b) (12 points) For each of the following, answer TRUE or FALSE.

i. **TRUE** If $A$ is a $2 \times 2$ matrix with no eigenvalue, then $\det(A) > 0$.

ii. **TRUE** If 3 is an eigenvalue of a $4 \times 4$ matrix $A$, then 15 is an eigenvalue of $5A$.

iii. **FALSE** If $A$ and $B$ are $2 \times 2$ matrices with equal characteristic polynomials, then $A$ and $B$ are similar matrices.

iv. **TRUE** Two finite dimensional vector spaces are isomorphic vector spaces if they have the same dimension.