MATH 240 – Spring 2013 – Exam 3 Solutions

There are 5 questions. Answer each on a separate sheet of paper. Use the back side if needed.
On each sheet, put your name, your section leader’s name and your section meeting time.
When a question has a short final answer, put a BOX around that answer.

NOTATION: If $A$ is a matrix, then $A^T$ denotes the transpose of $A$.

1. (20 points) Let $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Find a diagonal matrix $D$ and an invertible matrix $P$ such that $P^{-1}AP = D$. (Do not compute $P^{-1}$.)

SOLUTION.

Because $A$ is triangular, its eigenvalues are the diagonal entries $1, 2, 0$. For each eigenvalue $\lambda$, solve $(A - \lambda I)x = 0$ to get an eigenvector for $\lambda$. We let the columns of $P$ be the eigenvectors and $D$ the diagonal matrix with their eigenvalues:

$$P = \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. (21 points) Determine which of the following matrices are diagonalizable:

$A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 3 \\ 4 & 2 & 0 & 1 \\ 5 & 3 & 1 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

Justify your answers.

SOLUTION.

- $A$ is diagonalizable because it is symmetric.
- The only eigenvalue of the matrix $B$ is 1; its algebraic multiplicity (multiplicity as a root of the characteristic polynomial) is four, but its geometric multiplicity (dimension of the eigenspace, i.e. the dimension of the null space of $B - I$) is less than four (it is one). Therefore $B$ is not diagonalizable.
- The characteristic polynomial of $C$ is $t^2 - 2t + 5$, and by the quadratic formula its roots are $(1/2)(2 \pm \sqrt{-16}) = 1 \pm 2i$, which are not real numbers. Therefore $C$ is not diagonalizable.
3. (a) (10 points) Compute the area of the parallelogram $P$ in $\mathbb{R}^3$ whose four corners are the points

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
-1 \\
3
\end{pmatrix}, \begin{pmatrix}
2 \\
1 \\
1
\end{pmatrix}, \begin{pmatrix}
2 \\
0 \\
4
\end{pmatrix}.
\]

**SOLUTION.**

The vectors in order have the form $0, u, v, u + v$. Let $A$ be the $3 \times 2$ matrix with columns $u, v$. Then we can compute

\[
A^T A = \begin{pmatrix}
0 & -1 & 3 \\
2 & 1 & 1
\end{pmatrix} \begin{pmatrix}
0 & 2 \\
-1 & 1 \\
3 & 1
\end{pmatrix} = \begin{pmatrix}
10 & 2 \\
2 & 6
\end{pmatrix}
\]

\[
\text{area}(P) = \sqrt{\det(A^T A)} = \sqrt{56} = 2\sqrt{14}.
\]

(b) (5 points) Give an example of two matrices with the same characteristic polynomial which are not similar. No justification required. **SOLUTION.**

For example: \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

(The identity matrix is diagonalizable and the other matrix is not.)
4. (a) (5 points) Let $W$ be the span of the vectors \[
\begin{pmatrix}
2 \\
3 \\
4 \\
5 \\
6
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
0 \\
1 \\
0 \\
1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
7 \\
7 \\
7 \\
7 \\
8
\end{pmatrix}.
\]

Write down a matrix $A$ such that $Ax = 0$ if and only if $x$ is in $W^\perp$. No justification necessary.

**SOLUTION.**

\[
A = \begin{pmatrix}
2 & 3 & 4 & 5 & 6 \\
1 & 0 & 1 & 0 & 1 \\
7 & 7 & 7 & 7 & 8
\end{pmatrix}
\]

(b) (5 points) Let $P = \begin{pmatrix} .2 & .8 \\ .4 & .6 \end{pmatrix}$. As $n$ goes to infinity, $P^n$ approaches a matrix $Q$. What is $Q$?

**SOLUTION.**

As in the MATLAB homework, $Q$ is the matrix who rows equal the stochastic left (row vector) eigenvector of $P$. For the eigenvalue $q$, first solve

\[
0 = v(P - I) = (v_1, v_2) \begin{pmatrix} -.8 & .8 \\ .4 & -.4 \end{pmatrix}.
\]

One solution is $v = (1, 2)$; divide by $1 + 2$ to get the stochastic eigenvector. Then

\[
Q = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix}.
\]
(c) (15 points) For each statement below, write TRUE or FALSE. No justification necessary.

i. If $W$ is a subspace of $\mathbb{R}^n$ and $W$ is not the trivial space $\{0\}$, then $W$ has an orthonormal basis.
   TRUE. The Gram-Schmidt algorithm will produce one.

ii. If $\mathbb{R}^n$ has an orthonormal basis of eigenvectors of a matrix $A$, then $A$ is a symmetric matrix.
   TRUE.

iii. The geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity.
   TRUE.

iv. If $A$ is a symmetric matrix, then $A$ is an orthogonal matrix.
   FALSE.

v. If $x$ is the closest vector in the column space of $A$ to $b$, then $x$ is a least squares solution for the equation $Ax = b$.
   FALSE.
   Let $\hat{b}$ denote the closest vector in the column space of $A$ to $b$. Then $x$ is a least squares solution for the equation $Ax = b$ iff $Ax = \hat{b}$ (not $x = \hat{b}$).
5. In this problem $A$ is an $m \times n$ matrix and $b$ is a vector in $\mathbb{R}^m$.

(a) (5 points) There is an equation involving $A^T$ whose solutions are precisely the least squares solutions to $Ax = b$. What is that equation?

**SOLUTION.**

$A^T A x = A^T b$.

(b) (12 points) Now consider the specific matrices

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad R = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let $A = QR$. Find a least squares solution to $Ax = b$.

**SOLUTION.**

Here $A = QR$ is indeed the $QR$ factorization of $A$. Recall that the least squares solutions are the solutions of $Rx = Q^T b$ – this follows easily by plugging into the equation of part (a):

$$A^T A x = A^T b$$

$$(QR)^T (QR) x = (QR)^T b$$

$$R^T Q^T QR x = R^T Q^T b$$

$$Q^T QR x = Q^T b$$

$$Rx = Q^T b.$$

Now substitute and solve:

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

(c) (4 points) Is the least squares solution in part (b) unique? Briefly justify your answer.

**SOLUTION.** The solution is unique; in general the least squares solution of $Ax = b$ is unique if and only if the columns of $A$ are linearly independent, as they are here.