Following Justin’s Guide to MATLAB in MATH240 - Part 3

1. Method

You may want to review the first two guides whilst reading this one; the assumption is that you are comfortable with all those commands though not all are necessary.

2. New Commands

(a) Rank(A) will compute the rank of a matrix A.

(b) Eigenvalues can be found easily. If A is a matrix then:

\[ \text{eig}(A) \]

will return the eigenvalues. Note that it will return complex eigenvalues too, which we’re not so concerned about. So keep an i open for those.

(c) However the characteristic polynomial is interesting in its own right. To begin with note the useful command \text{eye}(n) which returns the \( n \times n \) identity (eye-dentity?) matrix:

\[ \text{eye}(5) \]

(d) So now let use L for \( \lambda \) and if we have a matrix like:

\[
\begin{bmatrix}
8 & -10 & -5 \\
2 & 17 & 2 \\
-9 & -18 & 4
\end{bmatrix}
\]

we can symbolically define L:

\[ \text{syms L} \]

and then:

\[ \text{det}(A-L\cdot \text{eye}(3)) \]

to get the characteristic polynomial for A.

(e) We can solve it using \text{solve}. One useful fact is that \text{solve} will assume the expression equals 0 unless specified and will solve for the single variable. Therefore we can do:

\[ \text{solve( det( L\cdot \text{eye}(3) - A ) )} \]

to get the solutions to the characteristic equation.

(f) Of course if we have an eigenvalue \( \lambda \) we can use \text{rref} on an augmented matrix \( [A - \lambda I|0] \) to lead us to the eigenvectors.

(g) Even better: MATLAB can do everything in one go. If you recall from class, \textit{diagonalizing} a matrix A means finding a diagonal matrix D and an invertible matrix P with A = PDP\(^{-1}\). The diagonal matrix D contains the eigenvalues along the diagonal and the matrix P contains eigenvectors as columns, with column \( i \) of P corresponding to the eigenvalue in column \( i \) of D.

To do this we use the \text{eig} command again but demand different output. The format is:

\[ [P, D] = \text{eig}(A) \]

which assigns P and D for A, if possible. If it’s not possible MATLAB returns very strange-looking output.
1. Let $A$ be the matrix \[
\begin{pmatrix}
1 & -3 & 7 \\
2 & 5 & 6 \\
7 & 1 & 33
\end{pmatrix}.
\]
(a) Compute $\text{rref}(A)$ and $\text{rank}(A)$.
(b) $\star$ What are the pivot positions of $A$?
(c) $\star$ For a general $m \times n$ matrix $B$ with $k$ pivot positions, what are $\text{dim}(\text{nul}B)$, $\text{rank}(B)$ and $\text{dim}(\text{range}B)$ in terms of $k, m, n$?
(We use Lay’s terminology for range: it is the space of outputs, not necessarily the codomain.)

2. Let $[x]_B$ denote the coordinate vector of $x$ with respect to a basis $B$.
For bases $B$ and $C$, $P_{c\leftarrow b}$ denotes the matrix $P$ such that $P[x]_B = [x]_C$.
($P$ is the change of coordinates matrix.) The following are bases for the vector space $\mathbb{P}_3$:
\[E = \{1, t, t^2, t^3\},\]
\[B = \{1, 2 - 2t, 2 - t - t^2, 1 + 2t + t^3\},\quad \text{and}\]
\[C = \{1 + 2t + t^3, 2 - t, 3t - 4t^2 + t^3, t\}.\]
(a) Let $\{b_1, b_2, b_3, b_4\}$ denote the vectors of $B$. Exhibit the $4 \times 4$ matrix $B$ for which column $i$ is $[b_i]_E$.
(b) Let $\{c_1, c_2, c_3, c_4\}$ denote the vectors of $C$. Exhibit the $4 \times 4$ matrix $C$ for which column $i$ is $[c_i]_E$.
(c) Compute the matrices $P = P_{E\leftarrow B}$ and $Q = P_{E\leftarrow C}$.
[Better alternative: compute a different $Q$, which is $Q = P_{C\leftarrow E}$. Otherwise the transition to the next step is a little mysterious. But you can compute the original $Q$ if you wish.]
(d) Compute the matrix $R$ such that $R[p]_B = [p]_C$ for every $p$ in $\mathbb{P}_3$.
(e) What is the $B$ coordinate vector of the polynomial $t$?

3. For this problem, we define \[
A = \begin{pmatrix}
-3 & -4 & 20 & -8 & -1 \\
14 & 11 & 46 & 18 & 2 \\
6 & 4 & -17 & 8 & 1 \\
11 & 7 & -37 & 18 & 2 \\
18 & 12 & -60 & 24 & 6
\end{pmatrix}.
\]
(a) Use the $\text{eig}$ command to find the eigenvalues of $A$.
(b) Write $p = \text{det}(\text{I} \cdot \text{eye}(n) - A)$ to find the characteristic polynomial of $A$.
(c) Use $\text{factor}(p)$ to factor this characteristic polynomial.
(Problems continue on the next page.)
4. Consider the matrix

\[
C = \begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 2 & 3 \\
3 & 2 & 5 & 0 \\
4 & 3 & 0 & 3
\end{bmatrix}.
\]

(This is a symmetric matrix – equal to its transpose – and we will have a theorem that such a matrix has a basis of eigenvectors.)

(a) Find the eigenvalues of \( C \) using \texttt{eig}.

(b) Find matrices \( P, D \) such that \( D \) is diagonal and \( P^{-1}CP = D \).

(c) The equation means \( CP = PD \), which is a way of writing that the columns of \( P \) are eigenvectors.

(Moreover, because the columns of \( P \) are linearly independent, they form a basis of eigenvectors.)

Exhibit \( CP \) and \( PD \).

(d) \star \text{MATLAB chose the eigenvectors (columns of } P) \text{ so that every column would have a certain length. What is it?}

(e) Exhibit \( P \ast P' \). What is the relation between the inverse and the transpose of \( P \)? Check by exhibiting \texttt{inv(P)}. (If we begin with any symmetric matrix \( C \), then Theorem 2 in Section 7.1 of Lay shows we can always find a \( P \) of this type.)

5. (a) Exhibit \( A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \).

(b) Exhibit \([P, D]=\text{eig}(A)\).

(c) \* Do we have \( P^{-1}AP = D \)?

(d) \* Does \( A \) have a basis of eigenvectors? Justify your answer.

6. (a) Exhibit \( A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \).

(b) Exhibit \texttt{eig(A)}.

(c) Exhibit \([P, D]=\text{eig}(A)\).

(d) Exhibit \texttt{inv(P)*A*P}.

(If the matrix is diagonalizable but with some nonreal eigenvalues, then MATLAB just goes to work with complex coefficients.)

The last problem is on the next page.
7. A **stochastic** matrix is a square matrix with every entry nonnegative and every row sum equal to one. (In Section 4.9, Lay uses a transposed convention that every column sum is 1, but the row sum definition is the more standard choice.

A stochastic matrix $P$ can be used to define a Markov process: the entry $P(i,j)$ is interpreted as the probability of going from state $i$ to state $j$, and $P^n(i,j)$ is interpreted as the probability of going from state $i$ to state $j$ in $n$ steps. For example, for $P \times 2$, state 1 might mean “sunny weather” and state 2 might mean “rainy weather”, with $P^n(1,2)$ interpreted as the probability of rainy weather after $n$ days given that today’s weather is sunny. Markov models are used a lot. There is more on this in Lay’s Section 4.9.

In this problem we examine the likelihood of moving from one state to another after a delay.

(a) Exhibit the stochastic matrix $P = \begin{pmatrix} .8 & .2 \\ .3 & .7 \end{pmatrix}$.

(b) Exhibit a column vector $v$ with positive entries such that $Av = v$ for every $2 \times 2$ stochastic matrix $A$. (This vector $v$ is a right eigenvector of $A$ for the eigenvalue 1.)

(c) Use MATLAB commands to compute a row vector $u$ with positive entries such that $uP = u$ and the entries of $u$ sum to 1. (This vector $u$ is a left eigenvector of $P$ for the eigenvalue 1.)

(You might use $[Q,D]=\text{eig}(P')$, define $w$ to be the transpose of an appropriate column of $Q$, and then multiply $w$ by a suitably defined scalar.)

(d) Exhibit the matrices $P, P^2, P^5, P^{10}, P^{20}$.

(e) Interpreting $P^n(i,j)$ as for a Markov model: you should be seeing in the example that the probability of being in state $j$ after $n$ steps approaches a constant independent of the initial state. What is that constant, in terms of an eigenvector?

(f) Repeat parts (a), (c) and (d) of the problem for the matrix $P = \begin{pmatrix} .6 & 0 & .2 & .2 \\ .1 & .7 & .1 & .1 \\ 0 & .2 & .5 & .3 \\ 0 & .3 & .1 & .6 \end{pmatrix}$.

Here is a remark for your information, in case you are interested.

The behavior above in the powers of a stochastic matrix $P$ is guaranteed, as long as that matrix $P$ has a power for which all entries are positive. Without that condition, the powers of a stochastic matrix $P$ won’t necessarily approach a matrix with equal rows (although it might). For examples, you could consider which of the stochastic matrices below have powers converging to a matrix with all rows equal.

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]