Some terminology:
$\mathbb{C}^*$ is the group of nonzero complex numbers under multiplication.
$\mathbb{R}^*$ is the group of nonzero real numbers under multiplication.
$\mathbb{Z}_n$ is the group of integers (mod $n$) under addition (mod $n$).
For $n$ a positive integer, $S_n$ denotes the group of all permutations of $\{1, 2, \ldots, n\}$.
$\{k \in \mathbb{Z}_n : \gcd(k, n) = 1\}$ under multiplication (mod $n$).

1. (5 points) Let $H$ be the subgroup of $S_4$ generated by the cyclic permutation $(123)$. How many distinct cosets of $H$ are contained in $S_4$?

2. (5 points) Let $H$ and $K$ be normal subgroups of $G$. By definition, $G$ is the internal direct product of $H$ and $K$ if two statements are true. What are they?

3. (10 points) Suppose $\phi : G \to \overline{G}$ is a homomorphism of groups.
   (a) Give the definition of $\ker \phi$, the kernel of $\phi$.
   (b) Prove that $\ker \phi$ is a normal subgroup of $G$.

4. (5 points) State the First Isomorphism Theorem.

5. (10 points) Prove there is no homomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ onto $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

6. (20 points)
   (a) Let $H = \{(1), (123), (132)\}$. Is $H$ normal in $S_4$? Justify your answer.
   (b) Find a subgroup of $\mathbb{Z}_{12} \oplus \mathbb{Z}_{18}$ which is isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_3$.
   (c) Prove $S_5$ has no subgroup of order 7.
   (d) Suppose $n > 4$ and $H$ is a normal subgroup of $S_n$. What numbers can be the order of $S_n/H$? (No proof required.)

7. (10 points) Give a list of groups such that every abelian group of order 200 is isomorphic to exactly one group on your list.

8. (10 points) Suppose $p$ and $q$ are prime numbers and $G$ is a group.
   (a) Suppose $|G| = p$. Prove $G$ is cyclic.
(b) Suppose $|G| = pq$. Prove that every proper subgroup of $G$ is cyclic.

9. (10 points)  (a) What is the kernel of the homomorphism $\phi : \mathbb{C}^* \to \mathbb{C}^*$ defined by the rule $\phi(z) = z^4$?
(b) What is the kernel of the homomorphism $\psi : \mathbb{R}^* \to \mathbb{R}^*$ defined by the rule $\psi(x) = x^4$?

10. (15 points) For each statement below, write TRUE or FALSE. No proof necessary.
(a) Suppose $G = H \oplus K$ is a finite group. Then $|(h, k)| = |h| \cdot |k|$.
(b) If $G$ and $H$ are cyclic groups, then $G \oplus H$ is a cyclic group.
(c) If $G$ is an abelian group, then every subgroup of $G$ is normal.
(d) Let $H$ be the subgroup of $D_6$ consisting of its rotations. Then $H$ is a normal subgroup of $D_6$.
(e) If the center of $G$ is trivial, then $G$ is isomorphic to $\text{Inn}(G)$, the group of inner automorphisms of $G$. 