Topological Orbit Equivalence and Factor Maps in Symbolic Dynamics

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Abstract

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Chapter I, "Lower Entropy Factors of Sofic Systems", will appear as an article. The main result: if S and T are irreducible subshifts of finite type, the period of any periodic point of S is divisible by the period of some periodic point of T and the entropy of S is strictly greater than the entropy of T, then T is a factor of S. In Chapter II, homeomorphisms S and T on compact metric spaces are called (topologically) orbit equivalent if some homeomorphism takes orbits of S onto orbits of T. A topological analogue of Belinskaya's theorem is obtained: if S and T are transitive and orbit equivalent by continuous jumps, then S is isomorphic to T or T^{-1} by continuous jumps. If S and T are transitive subshifts of finite type orbit equivalent by bounded jumps, then S is isomorphic to T or T^{-1}; this result fails for sofic shifts. If S and T are orbit equivalent mixing sofic shifts, then S need not be isomorphic to T or T^{-1}.

However, the maximal measures of orbit equivalent mixing sofic shifts must have the same range of values on cylinder sets. Orbit equivalence does not respect expansiveness or specification, and displays other pathology. In Chapter III, an example is given of a sofic shift which is not "almost finite type", in the sense of Marcus.

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CHAPTER II

Topological Orbit Equivalence

generalization to the case where S and T are transitive homeomorphisms with the same orbits and $Tx = S^{n(x)}x$ for some bounded function $n(x)$. Here S and T need not be flip conjugate. However, they are flip conjugate after restriction to the complement X~P of a set P of periodic points whose periods are uniformly bounded, and the conjugating map g can be given the form $g(x) = T^{a(x)}x$, where $a(x)$ is continuous on X~P. We also show that T must be a subshift if S is a subshift.

In Section 3, we consider subshifts of finite type and other sofic shifts. Using pressure, we find that the range of the maximal measure on cylinder sets must be the same for orbit equivalent sofic shifts. This invariant of orbit equivalence distinguishes some mixing subshifts of finite type which are not distinguished by the zeta function. By example, we find orbit equivalence does not imply flip conjugacy for mixing sofic shifts (for subshifts of finite type we must leave the issue unresolved). By example, two isomorphic subshifts of finite type may be orbit equivalent in such a way that no conjugacy between them can respect the given correspondence of orbits. If two subshifts of finite type are orbit equivalent with the jump function $n(x)$ bounded, then they are flip conjugate by a map which preserves the given correspondence of orbits. By example, this result fails in the sofic case; here we do not know, if the sofic shifts are intrinsically ergodic with support, whether some other flip conjugacy must exist.

We quarantine most of the examples in Section 5. In addition to those mentioned above, we include the following: an orbit conjugate of the 2-shift which is not expansive; uncountably many homeomorphisms with the same orbits as the 3-shift; a sofic shift orbit conjugate to a subshift which is not sofic; a shift with specification orbit conjugate to a shift without specification; an orbit conjugacy which does not respect bilaterally transitive points, and other technical pathologies.

1. Background, Discussion and General Observations

First we give examples of orbit equivalence in various categories.

I. The measure-preserving case.

S and T are invertible ergodic transformations which preserve a Lebesgue probability. There is a measure-preserving bijection which takes S-orbits onto T-orbits.

II. The flow case.

S and T are continuous flows (actions of the reals) on a manifold which have the same orbits.

III. The connected case.

S and T are homeomorphisms of a topological space such that the complement of the periodic points of S and T is dense and path-connected. S and T have the same orbits.

IV. The homeomorphism case.

S and T are homeomorphisms of compact metric spaces. There is a homeomorphism which takes S-orbits onto T-orbits.

V. The measurable case.

S and T are nonsingular measurable invertible transformations of Lebesgue spaces. There is a nonsingular measurable bijection which takes S-orbits onto T-orbits.

VI. Actions of other groups.

G and H are groups acting measurably on a Lebesgue space which have the same orbits.

The measure-theoretic conditions above are only required to hold off some set of measure zero. We make no further mention of V and VI, but a lot has been done on each; see, for example, [17] for V and [18] for VI. A lot also have been done on II (see, for example, [11]); orbit equivalence with respected orientation is perhaps the most natural equivalence relation for real flows. The object of our study, IV, has received little attention.

We state three landmark theorems for subsequent comparisons.

<u>Dye's Theorem</u> [10]

Suppose S and T are invertible ergodic transformations which preserve a Lebesgue probability. Then S and T are orbit equivalent.

<u>Belinskaya's Theorem</u> [8]

Suppose S and T are ergodic automorphisms of a Lebesgue probability space (X,m) which have the same orbits, where $Tx = S^{n(x)}x$ and $n(x)$ is in $L^1(m)$. Then S is isomorphic to T or T^{-1}.