Then D is a dense G_δ and $\max_S D = 1$.

10. S is the 3-shift, T is isomorphic to S.

Say n(x) is unbounded at y if n(x) is unbounded on any neighborhood of y. Then n(x) is unbounded at uncountably many points.

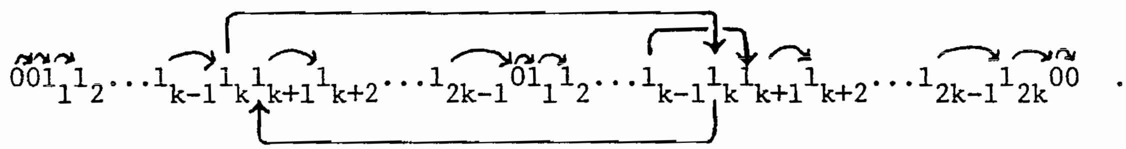
(This example is easily generalized to any mixing subshift of finite type S, with n(x) unbounded at every point of any given proper subshift of S.)

(4.1) Example

Let S be the 2-shift on symbols 0 and 1. We will define a homeomorphism T with the same orbits as S. We will have $Tx = Sx$ except on blocks $B_k = 001^{2k}01^{2k}00$, that is

$$B_k = 00\underbrace{111\ldots\ldots\ldots\ldots111}_{2k\ 1's}0\underbrace{111\ldots\ldots\ldots\ldots111}_{2k\ 1's}0, \quad k > 1;$$

there the jump pattern is like the shift S, except in the middle of the blocks of 1's, where the pattern is given by the following picture (in which 1_i represents the i^{th} time the symbol 1 appears in an uninterrupted block of 1's):



Follow the jumps through the block, and see that each symbol is hit exactly once.

Formally, define $Tx = S^{n(x)}x$ by defining $n(x)$:

$$\begin{aligned} n(x) &= 2k+1 && \text{if } x_{-(k+1)} \ldots x_{3k+3} = B_k, \quad k > 1, \\ &= 2 && \text{if } x_{-(3k+1)} \ldots x_{k+3} = B_k, \quad k > 1, \\ &= -2k && \text{if } x_{-(3k+2)} \ldots x_{k+2} = B_k, \quad k > 1, \\ &= 1 && \text{otherwise.} \end{aligned}$$

The function $n(x)$ is well-defined because the blocks B_k can overlap only on boundary blocks 00. Since $n(x)$ is continuous except at the fixed point $y = 1^{\infty}$, T is continuous except possibly at y; but examination of the definition of $n(x)$ reveals that if a sequence $\{y^k\}$ converges to y, then $\{Ty^k\}$ converges to $y = Ty$. So, T is everywhere continuous; therefore, since T is clearly a bijection, it must (by compactness) be a homeomorphism. Clearly S and T have the same orbits.

Given a nonnegative integer m, define a factor map $f_m\colon\ T \to T_m$, where T_m is the subshift with alphabet $B_{2m+1}(S)$ defined by $(f_m x)_j = (T^j x)_{-m} \ldots (T^j x)_m$. T is expansive if and only if some T_m is injective; but no T_m is. For example, let

$x = 0^\infty 1^{2m+4} \overset{\downarrow}{0} 1^{2m+4} 0^\infty$,

$y = 0^\infty 1^{2m+5} \overset{\downarrow}{0} 1^{2m+3} 0^\infty$,

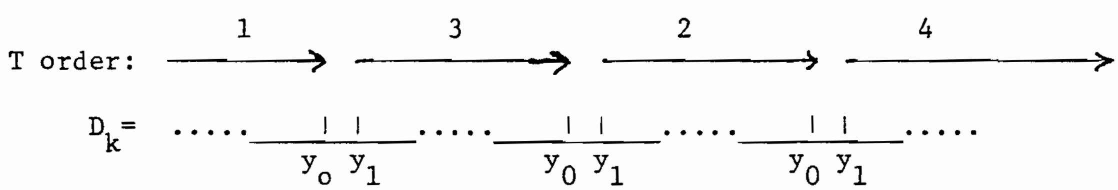
where the arrow indicates the zero coordinate; then $f_m x = f_m y$. In particular, $f_0 x = f_0 y = 0^\infty 1^{2m+5} \overset{\downarrow}{0} 1^{2m+3} 0^\infty$.

So, T is not expansive. Equivalently, since its domain is the Cantor discontinuum, T is not a subshift.

(4.2) Example

Let S be the full shift on symbols 0,1,2. Fix any y in S such that $y_i = 0$ or $y_i = 1$, for all i. We will construct $T = S^{n(x)}$ with the same orbits as S such that n(x) is unbounded at y but is continuous at every other point. In particular, such maps S constructed for different y are different, so there are uncountably many homeomorphisms with the same orbits on S.

Let $C_k = y_{-k} \cdots y_k$. Let $D_k = 2^{8k}C_k 2^{4k} C_k 2^{4k} C_k 2^{8k}$. We will define n(x) not equal to 1 only in the middle of the C_k's inside D_k, where k is positive. The blocks of 2's guarantee that a block D_k can be overlapped with a block D_j (where j is not equal to k) or with a translate of itself only to the extent of 2's on the right side of one overlapping 2's on the left side of the other. This provides sufficient disjointness for the jumps defined on the D_k to be consistent with one another. It is easiest to see the definition of n(x) on neighborhoods defined by the D_k by picturing the order in which T uses the symbols of D_k for zero coordinates:

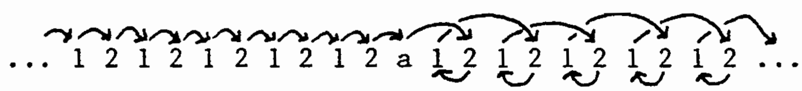


The highlighted symbols y_0 and y_1 in D_k are occuring in the middle of blocks C_k. Beneath each numbered arrow, the symbols of D_k are used for the zero coordinates by T in the same order as by S. After using the sumbols under arrow 1, T uses the symbols under arrow 2, and so on. Then n(x) is continuous at z if z is not equal to y, and n(x) is unbounded in any neighborhood of y. But T must be continuous at y: if $\{y^k\}$ is a sequence converging to y, then inspection of the rules defining the jump function shows that $\{Ty^k\}$ converges to Ty = Sy. So, T is everywhere continuous. Clearly T is a bijection, so T is a homeomorphism.

Remark: the T we constructed is isomorphic to S. By destroying expansiveness as in (4.1), it is easy to obtain T not isomorphic to S.

(4.3) <u>Example</u>

Let S be the subshift whose domain consists of the two orbits $(12)^{\infty}a(12)^{\infty}$ and $(12)^{\infty}$. Let T be the subshift whose domain consists of orbits $(12)^{\infty}a(21)^{\infty}$ and $(12)^{\infty}$. Then S and T are transitive sofic shifts, with a dense aperiodic orbit. T is naturally isomorphic to the homeomorphism sharing orbits with S determined by the following jump pattern:

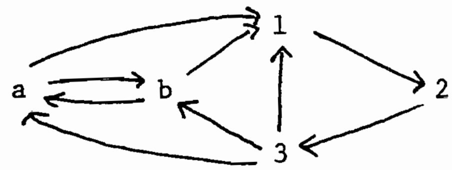


.

So, S and T are orbit equivalent. They cannot be flip conjugate; an aperiodic orbit of S^2 has two accumulation points, while an aperiodic orbit of T^2 has but one.

(4.4) Example

Let U be the subshift of finite type on symbols 1,2,3,a,b given by the following graph.



Define a subshift S of U by disallowing all blocks of the following forms:

I $a(123)^n b$, $n \geq 1$;

II $b(123)^n a$, $n \geq 1$;

III $(123)^n B(123)^n$, where

(i) $n \geqq 8L$, where L is the length of B,

(ii) the initial and terminal symbols of B come from {a,b}, and

(iii) L is not congruent to 0 modulo 3.

Define a homeomorphism T with the same orbits as S by setting S equal to T except on blocks $a(123)^\infty$, $(123)^\infty a$, and $a(123)^n a$, where the jump pattern is determined by the following picture.

Then T is naturally isomorphic to the subshift $\bar{T}$ whose domain is the set of all $\bar{x}$ which can be obtained from some x in S as follows:

(1) if $x_i \ldots x_j = a(123)^n a$,

then $\bar{x}_i \ldots \bar{x}_j = a(231)^n a$;

(2) if $x_i \ldots\ldots = a(123)^\infty$,

then $\bar{x}_i \ldots\ldots = a(231)^\infty$;

(3) if $\ldots\ldots x_i = (123)^\infty a$,

then $\ldots\ldots \bar{x}_i = (231)^\infty a$;

(4) otherwise, $\bar{x}_i = x_i$.

The orbit equivalent shifts $\bar{T}$ and S are mixing with periodic points dense, but they are not sofic. They cannot be flip conjugate. Any flip conjugacy must take the orbit $(123)^\infty$ of $\bar{T}$ to the orbit $(123)^\infty$ of S, hence must take the orbit $(123)^\infty a(231)^\infty$ of $\bar{T}$ to some orbit of the form $(123)^\infty B(123)^\infty$ of S, where the length of B is congruent to zero modulo 3. But for any $\bar{x}$ in the former orbit, $\{\bar{T}^{3n}\bar{x}$: n is an integer$\}$ has two accumulation points; for any y in the latter orbit, $\{S^{3n}y$: n is an integer$\}$ has but one accumulation point.

(4.5) Example

We will define a mixing sofic shift S with alphabet {0,1} by specifying its disallowed blocks. We disallow 111. Also, we disallow all blocks of the form

(*) $110^i10^j10^k10^\ell10^m11$: i,j,k,ℓ and m are all positive,

except those for which one of the following two conditions holds:

(1) j, k and ℓ are even

(and we say the block is EEE), or

(2) j and ℓ are even, k is odd

(and we say the block is EOE).

We will define a homeomorphism $Tx = S^{n(x)}x$ with the same orbits as S. The map f_o of example 4.1, $(f_ox)_j = (T^jx)_o$, will factor T isomorphically onto a subshift $\bar{T}$. Let $n(x) = 1$, except on the EEE blocks; there, define jumps between the middle of the blocks 0^j and 0^k as in example 4.1, so that T is a homeomorphism and also f_o takes blocks EEE to blocks OOE as follows:

if $x_0 \ldots x_n = 110^i10^j10^k10^\ell10^m11$, with i,j,k,ℓ,m even and positive

then $(f_ox)_0 \ldots (f_ox)_n = 110^i10^{j-1}10^{k+1}10^\ell10^m11$.

Then $\bar{T}$ is the shift on alphabet {0,1} defined by excluding 111 and also all blocks of the form (*) except those for which one of the following conditions hold:

(1) j is odd, k is odd, $k > 2$, ℓ is even

(the block is OOE with $k > 2$), or

(2) j and ℓ are even and k is odd

(the block is EOE).

Clearly S and T are mixing; from the viewpoint of sofic systems as finite automata, it is easy to see that they are sofic.

We claim that S and T are not flip conjugate. Since S and S^{-1} are obviously isomorphic, it is enough to suppose we have a conjugacy g from S to $\bar{T}$ and derive a contradiction. The map g must take the fixed point 0^∞ in S to 0^∞ in $\bar{T}$. Also, there exists N such that $(gx)_0$ depends only on $x_{-N}\ldots x_N$ and $(g^{-1}y)_0$ depends only on $y_{-N}\ldots y_N$. Construct a point y in $\bar{T}$,

$y = 0^\infty \overset{\downarrow}{1}10^I10^J10^K10^L10^M110^\infty$,

where the arrow indicates the zero coordinate of y,

such that the block of the form (*) occurring in y is OOE and also $I,J,K,L,M \geq 4N + 1$. Then $x = g^{-1}y$ must have the form

$x = 0^\infty \overset{\downarrow}{C}0^iB0^jB0^kB0^\ell B0^mC0^\infty$,

where the arrow indicates that x_0 lies in C in x,

the indicated zero blocks of x occupy coordinates of x contained in the coordinates of y occupied by the corresponding zero blocks of y, B and C are S-words and $k,j,k,\ell,m \geq 2N + 1$.

Now consider the sequence x' obtained by increasing k by one in x; that is,

$x' = 0^\infty \overset{\downarrow}{C}0^iB0^jB0^{k+1}B0^\ell B0^mC0^\infty$.

If x' were a point in S, then gx' would be obtained by increasing K by one in y; but then the (*) block of gx' would be OEE, which is