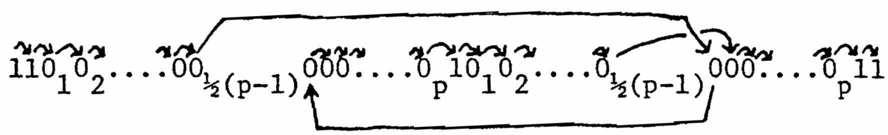
not allowed. Thus the 0^k block in x must be part of an E or an O block of zeros in some EEE or EOE block which cannot increase in length by one; that is, of a left or right E block of zeros. For concreteness, suppose right. Then the 0^j block in x must also be a right E block in some block of the form (*). But now, increase J by one in y, changing y with its OOE block to y' with an (allowed) EOE block. Then $g^{-1}y'$ must be the point obtained from x by increasing j by one. But this means that some EEO or EOO block occurs in $g^{-1}y'$, a contradiction.

So, S and $\bar{T}$ are not conjugate, and therefore they are not flip conjugate.

(4.6) Example

We define a mixing sofic shift S on symbols 0 and 1 by disallowing any block $C(n,k) = 110^n10^k11$ for which n and k are even and positive. Now define a homeomorphism $T = S^{n(x)}$ with the same orbits as S. Let $n(x) = 1$, except in the middle of the zero blocks of those $C(p,p)$ for which p is an odd prime; there, determine the jump pattern from



Define $f_o: T \to \bar{T}$, as in example 4.1, by $(f_o x)_j = (T^j x)_0$. Here, if $x_i \ldots x_j = C(p,p)$, where p is an odd prime, then $(f_o x)_i \ldots (f_o x)_j = C(p+1,p-1)$. That is, $(f_o x)$ is obtained by replacing any occurrence of such $C(p,p)$ in x with an occurrence of $C(p+1,p-1)$. Since $C(p+1,p-1)$ is not allowed in S, f_o is injective, and the subshift $\bar{T}$ is orbit equivalent to S. But $\bar{T}$ is obviously not sofic.

(4.7) Example

A homeomorphism T on a compact metric space X has the specification property if the following holds: X contains more than one point; and for any positive ε, there exists an integer $M(\varepsilon)$ such that for any $k \geqq 2$, for any k points $x_1,\ldots,x_k$ in X, for any integers $a_1 \leqq b_1 < a_2 \leqq b_2 < \ldots < a_k \leqq b_k$ with $a_i - b_{i-1} \geqq M(\varepsilon)$ for $2 \leqq i \leqq k$ and for any integer p with $p \geqq M(\varepsilon) + b_k - a_1$, there exists a point x in X with $T^p x = x$ such that $\mathrm{dist}(T^n x, T^n x_i) \leqq \varepsilon$ for $a_i \leqq n \leqq b_i$, $1 \leqq i \leqq k$.

This complicated but useful property is satisfied by interesting examples (see[2]): in particular, by mixing sofic shifts (which do not consist of just a single point). Specification implies positive topological entropy; specification and expansiveness together imply intrinsic ergodicity. Specification is preserved under the operations of taking powers, factors (modulo the trival factor) and finite products. Since orbit equivalence respects periodic orbits in particular, one might hope orbit equivalence would respect specification; but, rather decisively, it does not.

Define a mixing sofic shift S on symbols 0 and 1 as in (4.7) by disallowing any block $C(n,k) = 110^n10^k11$ for which n and k are even and positive. Let $D(i,j,k) = 110^i10^j110^k111$. As in (4.6), define a homeomorphism T with the same orbits as S, such that $f_o\colon T \downarrow \bar{T}$ is an isomorphism; but for this example, define the exceptional jumps between the 0^p blocks of D(p,p,2p) if p is an odd prime, so that if $x_i \ldots x_j = D(p,p,2p)$, with p an odd prime, then $(f_o x)_i \ldots (f_o x)_j = D(p+1,p-1,2p)$. That is, each $f_o x$ is obtained by replacing any occurrence of such $D(p,p,2p)$ in x with an occurrence of $D(p+1,p-1,2p)$.

Suppose $\bar{T}$ has the specification property. Then $\bar{T}$ certainly has a transition length. That is, there exists an integer M such that, given any two $\bar{T}$-words A and C, there is a $\bar{T}$ word B of length M such that ABC is a $\bar{T}$-word. Pick a prime p greater than M. Let $A = 110^{p+1}10^{p-1}11$; let $C = 1$. If $110^{p+1}10^{p-1}11$ occurs in $\bar{T}$, then it must occur in the block $110^{p+1}10^{p-1}110^{2p}111$; so, no B exists of length M such that ABC is a $\bar{T}$-word.

(4.8) Example

We will construct from the 2-shift S an isomorphic homeomorphism $Tx = S^{n(x)}x$ with the same orbits, such that uncountably many onesidedly transitive points of S are twosidedly transitive points of T, and vice versa (even though the jump function n(x) is continuous except at a single point). In particular, given an orbit conjugacy between isomorphic subshifts of finite type, it may not be possible for any conjugacy between them to respect the given correspondence of orbits.

Given k positive, let $C_k = 0110^{2k}10$. We will let E_k, F_k and G_k represent arbitrary blocks of length k. Let D(k) represent the set of all blocks of the form

$$(*)\quad (C_kE_kC_kF_k)(C_{k-1}E_{k-1}C_{k-1}F_{k-1})\cdots(C_1E_1C_1F_1)(C_1G_1)(C_2G_2)\cdots(C_kG_k).$$

For each block in D(k), k > 1, we will define n(x) not equal to one at three places not already defined via D(k-1):

$$0110_1\cdots0_k0_{k+1}\cdots0_{2k}10E_k0110_1\cdots0_k0_{k+1}\cdots0_{2k}10F_k\cdots0110_1\cdots0_k0\cdots G_k\ .$$

Consequently, if under the action of S a point travels through the zero-one partition in the order given by (*), then its itinerary under the action of T will be given by

$$(**)\quad (C_kF_k)(C_{k-1}F_{k-1})\cdots(C_1F_1)(C_1E_1C_1G_1)(C_2E_2C_2G_2)\cdots(C_kE_kC_kG_k)\ .$$

We have "robbed" the past of the blocks E_i, and taken them to the future. Similarly define jumps on the blocks of the form (**) so that the corresponding T-names have the form (*). The assignments of $n(x)$ not equal to one are consistent and locally constant. The jump function $n(x)$ is continuous except at 0^∞, where $T = S^{n(x)}$ is continuous by inspection. T is a bijection, hence a homeomorphism. Because any sequence of zeros and ones is the T name (with respect to the timezero partition) for at most one point, T is expansive; then because every sequence of 0's and 1's is the T-name of some point, T is isomorphic to the 2-shift. If the pattern of (*) is continued infinitely to both sides, it is easy to see that robbing the past of the E_i's may destroy left transitivity on uncountably many orbits.

Remark: by proposition 3.1, the set of orbits above which are bilaterally transitive for only one of S and T must have measure zero with respect to any ergodic measure on S.

(4.9) Example

Given S, let

$SF(x) = \{S^n x\colon n \text{ is positive}\}$,

$SP(x) = \{S^n x\colon n \text{ is negative}\}$.

Define TF(x), TP(x) similarly for T. Let D be the set of points x for which all of the following sets are infinite: $SF(x) \cap TF(x)$, $SF(x) \cap TP(x)$, $SP(x) \cap TF(x)$, and $SP(x) \cap TP(x)$.

It seems fair to say that for points in D, we do not have a natural way to specify the orientation of T with respect to S.

We will define T, sharing orbits with the 2-shift S, $T = S^{n(x)}x$, such that

(1) T is isomorphic to S,

(2) n(x) is discontinuous at just one point,

(3) D is a dense G_δ, and

(4) $\max_S D = 1$.

Let $A_n = 110^{2n}10$, $B_n = 010^{2n}11$. For some sequence of positive integers g_n (g_n for "gaps"), we will define jumps on each word $W = A_n E B_n$ which satisfies the following conditions:

(1) $1 \leq \text{length } E \leq g_n$;

(2) A_n occurs only once in W;

(3) B_n occurs only once in W.

For the jumps, A_n and B_n just "trade zeros" (so, the S-names and T-names of a point with respect to the time zero partition will be the same, and S and T will be isomorphic); we indicate the places where we define n(x) to be other than 1:

$$110_1\cdots 0_{n-1}0_n0_{n+1}\cdots 0_{2n}10E010_10_2\cdots 0_{n-1}0_n0_{n+1}\cdots 0_{2n}11 \quad .$$

A point x will be in D if x_0 occurs in x in blocks A_nEB_n, $1 \leq \text{length } E \leq g_n$, for infinitely many n (notice that x_0 can occur in A_n or B_n for at most one n). This makes D a dense G_δ.
Let D(n) be the intersection of the following two sets:

$\{x\colon \text{ if } \; 0 < i < \tfrac{1}{2}g_n, \; \text{ then } \; x_i \ldots x_{i+2n+3} \neq B_n\}$, and

$\{x\colon \text{ if } \; -\tfrac{1}{2}g_n < i < 0, \; \text{ then } \; x_i \ldots x_{i+2n+3} \neq A_n\}$.

It is clear that if g_n grows rapidly enough, then $\lim_n \max_S D(n) = 0$. But if x is not in D, then x is in all but finitely many of the D(n); the set of such x has measure zero.

Despite the size of D, the orbit conjugacy at hand has an innocuous look. One can formulate weaker notions of orientation in the search for some structure. However, one can also produce more vicious counterexamples.

(4.10) Example

Given how bad an orbit conjugacy can be if the jump function is allowed to be discontinuous at a single point, it is not clear that continuity of n(x) on a large set of orbits is of any use. (We know from Section 1 that the continuity set contains a dense open set and a dense invariant G_δ.) Still, we will give an example showing that n(x) may be unbounded on an uncountable set (it is easy to generalize the example and obtain n(x) unbounded on any proper subshift of a subshift of finite type). Proceed just as in example 4.2; but now, instead of defining jumps on the D_k specified by a single point of the two shift, define jumps on those D_k specified by any point of the 2-shift. The resulting T is isomorphic to the 3-shift, shares orbits with the 3-shift, and has n(x) unbounded at every point of the 2-shift.

With more work, one can probably get the continuity set to have less than full measure, so that the set of orbits on which n(x) is continuous has measure zero.

5. Questions

We conclude this chapter with a selection of open problems.

1. Does orbit equivalence respect topological entropy?
   Between expansive homeomorphisms?

2. If two mixing subshifts of finite type are orbit equivalent, must they be flip conjugate?

3. For mixing subshifts of finite type, does shift equivalence imply orbit equivalence?

4. Give a useful sufficient condition for orbit equivalence.

5. For mixing sofic shifts, does orbit equivalence by bounded jumps imply flip conjugacy?
   (In particular, are the sofic shifts of (3.2) conjugate?
   If they are not, it may require a subtle or penetrating invariant to separate them.)

(cont. next pg.)