SOLUTIONS: PROBLEM SET 11 FROM SECTION 4.4

2. We set $f(x) = x^3 + 8x^2 - x - 1)$
   (a) The two solutions of $f(x) \equiv 0 \pmod{11}$ are $x \equiv 4$ and $x \equiv 5$.
   (b) We have $f'(5) \equiv 0 \pmod{11}$, but $f(5) \not\equiv 0 \pmod{121}$ it follows that there is no solution of $f \equiv 0 \pmod{121}$ that is congruent to 5 (mod 11). $f'(4) \not\equiv 0 \pmod{11}$. It follows that there is a unique solution of $f \equiv 0 \pmod{11^k}$ for each k and that solution is congruent to 4 (mod 11). We have $x \equiv 59 \pmod{121}$
   (c) $x \equiv 1148 \pmod{1331}$

4. Set $g(x) = x^2 + x + 34$. The only solution of $g(x) \equiv 0 \pmod 3$ is $x \equiv 1 \pmod 3$. $f'(1) = 3 \equiv 0 \pmod 3$. Since, in fact $f(1) = 36 \equiv 0 \pmod 9$, 1,4 and 7 are all solutions $\pmod 9$. Only $x \equiv 4$ is a solution $\pmod{27}$ consequently $x \equiv 13$ and $x \equiv 22$ are also solutions $\pmod{27}$, and there are no others. However, none of these is a solution $\pmod{81}$, and it follows that there is no solution $\pmod{81}$.

10. Set $h(x) = x^5 + x - 6$. By the Chinese remainder theorem, it suffices to find the numbers of solutions of $h(x) \equiv 0 \pmod{16}$ and $h(x) \equiv 0 \pmod 9$, and the number of solutions of $h(x) \equiv 0 \pmod{144}$ will be the product of these. The only solution $\pmod 3$ is $x \equiv 0$, and $h'(0) = 1 \not\equiv 0 \pmod 3$. It follows that there is exactly one solution $\pmod 9$. Both 0 and 1 are solutions $\pmod 2$, $h'(0) = 1 \not\equiv 0 \pmod 2$, and $h'(1) = 6 \equiv 0 \pmod 2$. It follows that there is a unique even solution $\pmod{2^k}$ for every k and, in particular $\pmod{16}$. We must investigate the odd solutions further. $x \equiv 1$ is a solution $\pmod 4$ but not $\pmod 8$. It follows that $x \equiv 3$ is also a solution $\pmod 4$ and, in fact $x \equiv 3$ is a solution $\pmod{16}$. It follows that $x \equiv 7$ is also a solution $\pmod 8$, but $x \equiv 7$ turns out not to be a solution $\pmod{16}$. It now follows that $x \equiv 3$ and $x \equiv 11$ are the only odd solutions $\pmod{16}$. With the unique even solution there three solutions in all to $h \equiv 0 \pmod{16}$. Since there is a unique solution $\pmod 9$, it follows that there are exactly 3 solutions $\pmod{144}$.