SOLUTIONS: PROBLEM SET 14 FROM SECTION 6.3

4. Let $b = 1+a+\cdots+a^{\phi(m)-1}$. Then $ab = a+a^2+\cdots+a^{\phi(m)-1}+a^{\phi(m)}$. By Euler's theorem, $ab \equiv b \pmod{m}$. Hence $(a-1)b \equiv 0 \pmod{m}$ and, since $(a-1, m) = 1$, it follows that $b \equiv 0 \pmod{m}$.

6. $\phi(10) = 4$, and $(7, 10) = 1$. Hence $7^4 \equiv 1 \pmod{10}$. Since $999,999 \equiv 3 \pmod{4}$, $7^{999,999} \equiv 7^3 \equiv 3 \pmod{10}$. Hence, the desired least digit is 3.

10. If a and b are relatively prime, then $a^{\phi(b)} \equiv 1 \pmod{b}$ and $b^{\phi(a)} \equiv 0 \pmod{b}$. Hence $a^{\phi(b)} + b^{\phi(a)} \equiv 1 \pmod{b}$. Similarly $a^{\phi(b)} + b^{\phi(a)} \equiv 1 \pmod{a}$, so by the Chinese remainder theorem, $a^{\phi(b)} + b^{\phi(a)} \equiv 1 \pmod{ab}$. Notice that this proof is essentially identical to a proof in the previous homework set.