SOLUTIONS: PROBLEM SET 22 FROM SECTION 9.4

2. 5 is a primitive root $\pmod{23}$ and $\text{ind}_5 3 = 16$ Thus if we set $y = \text{ind}_5 x$, we obtain the equivalent congruences

   (a) $5y + 16 \equiv 0 \pmod{22}$ which has the unique solution $y \equiv 10 \pmod{22}$ so that $x \equiv 5^{10} \equiv 9 \pmod{23}$.

   (b) $14y + 16 \equiv \text{ind}_5 2 = 2 \mod 22$, which has the solutions $y \equiv 10, 21 \pmod{22}$, giving $x \equiv 9, 14 \pmod{23}$.

4. 2 is a primitive root $\pmod{13}$. $\text{ind}_2 2 = 1$, so setting $b = \text{ind}_2 a$, $y = \text{ind}_2 x$, and taking indices on both sides, we get the equivalent congruence $4y + b \equiv 1 \pmod{12}$. This will have solutions if and only if $b \equiv 1 \pmod{4}$, so that $b \equiv 1, 5, 9 \pmod{12}$ and $a \equiv 2, 6, 5 \pmod{13}$.

10. Following the hint, if Q is as given, then $p_1 p_2 \cdots p_n$ is a solution of $x^4 \equiv -1 \pmod{Q}$. Consquently, if p is any prime divisor of Q, then $p_1 p_2 \cdots p_n$ is also a solution of $x^4 \equiv -1 \pmod{p}$. Hence p has the form $8k+1$ by problem 9. Since all the p_i are relatively prime to Q, it follows that p is distinct from all the p_i. Hence there are more than n primes of the form $8k + 1$, and hence infinitely many since n was arbitrary.