SOLUTIONS: PROBLEM SET 33 FROM SECTION 13.2

4. This is a more difficult problem than I realized when I assigned it. One begins by noticing that $\{z, y^2, x^2\}$ is a Pythagorean triple and that if any two of the terms have a prime factor, p, then p divides all three terms, hence, in particular, both x and y. It then follows that p^4 divides x^4, y^4 and hence z^2, so that one can effect the descent by dividing all terms by p^4. to obtain

$$(\frac{x}{p})^4 - (\frac{y}{p})^4 = (\frac{z}{p^2})^2.$$

From this point, therefore, we may assume that $\{z, y^2, x^2\}$ is a primary Pythagorean triple. It follows that x is odd and that y and z have opposite parity. We need to treat two cases separately.

If y is even and z is odd, we have u and v relatively prime and of opposite parity with $y^2 = 2uv$ and $u^2 + v^2 = x^2$. We may assume, without loss of generality, that u is odd and v is even. Then u and $2v$ are perfect squares. Then we have s and t relatively prime and of opposite parity with $u = s^2 - t^2$ and $v = 2st$ so that $2v = 4st$ is a perfect square. It follows that both s and t are also perfect squares. Thus the equation $u = s^2 - t^2$ can be rewritten $r^2 = m^4 - n^4$, providing the descent, since $m \leq s < v < x = \sqrt{u^2 + v^2}$. Moreover, r is odd, so that the descent is to the same case that we started with, and we have shown that it is impossible that $x^4 - y^4 = z^2$ with z odd.

If y is odd and z is even, we must proceed differently. We observe that $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$ is a perfect square. Since x and y are relatively prime, the only common factor of $x^2 + y^2$ and $x^2 - y^2$ is 2. It follows that $x^2 + y^2 = 2s^2$ and $x^2 - y^2 = 2t^2$ with s odd, t even, and s and t relatively prime. By the same reasoning, it follows that one of $x + y$ and $x - y$ is twice an odd square, say $2u^2$ with u odd, and the other is an even square, say $4v^2$ with the parity of v undetermined. Moreover, u and v are relatively prime both to each other and to s, since $8u^2v^2 = 2t^2$. It now follows that $x = u^2 + 2v^2$ and $y = \pm(u^2 - 2v^2)$ where the sign will not matter because, in either case, $x^2 + y^2 = 2u^4 + 8v^4 = 2s^2$, so that we have $u^4 + 4v^4 = s^2$, with the three terms pairwise relatively prime. The generating formula for Pythagorean triples now gives $u^2 = m^2 - n^2$ and $2v^2 = 2mn$. It follows that m and n are perfect squares and that u^2 is the difference of two fourth powers. Since u is odd, we know this is impossible by the previous paragraph.

10. This is just a direct algebraic verification.

12. There is a misprint here: the Diophantine equation in question should be

$$x^3 + y^3 + w^3 = z^3.$$

Once this is corrected the proof is again a direct algebraic verification.