SOLUTIONS: PROBLEM SET 6 FROM SECTION 3.4

6. Let us represent the prime factorization of n by writing

$$n = \prod_{p|n} p^{\alpha(n,p)},$$

where $\alpha(n,p)$ denotes the highest power of p that divides n. We can extend this notation to the case where p does not divide n by setting $\alpha(n,p) = 0$ in this case. Then if $m^2 = n$, it follows that $\alpha(n,p) = 2\alpha(m,p)$ for all primes p, so that $\alpha(n,p)$ is even for all p. Conversely, if $\alpha(n,p)$ is even for all p, we can set

$$m = \prod_{p|n} p^{\frac{\alpha(n,p)}{2}},$$

and $n = m^2$.

8. Let n be an integer, and let m be the product of all those primes p for which $\alpha(n,p)$ is odd. Then m is square free by definition, $m|n$, and, by problem 6, $\frac{n}{m}$ is a perfect square.

10. Continuing the notation of problem 6, it is clear that $a|b$ if and only if $\alpha(a,p) \leq \alpha(b,p)$ for every prime p. By the hypothesis, $3\alpha(a,p) \leq 2\alpha(b,p$ for every prime p, from which it follows that, for every prime p, $\alpha(a,p) \leq \frac{2}{3}\alpha(b,p) \leq alpha(b,p)$.

14. It is clear that

$$\alpha(n!,p) = \sum_{k=1}^{n} \alpha(k,p).$$

Every integer less than or equal to n and divisible by p contributes at least 1 to this sum. There are $[n/p]$ such integers. Moreover, every integer k less than or equal to n makes a second contribution provided $p^2|k$. There are $[n/p^2]$ such k. Similarly, there are $[n/p^3]$ values of k that make a third contribution, and so on. The sum terminates because, for ℓ sufficiently large, $p^\ell > n$ so that $[n/p^\ell = 0$.

36. $18 = 2 \times 3^2$, and $540 = 2^2 \times 3^3 \times 5$. It follows that 18 divides both a and b, but that exactly one of a and b is divisible by 4, exactly one is divisible by 27, and exactly one is divisible by 5. The possibilities are: 18 and 540, 36 and 270, 54 and 180, or 80 and 108.