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SPECTRAL CONVERGENCE ON DEGENERATING SURFACES

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1. Introduction. The study of the spectrum of the Laplace operator has produced an extensive literature. (See [Cha] and the references therein.) Of special interest to recent applications has been the behavior of spectra on two-dimensional surfaces with degenerating metrics; for example, the case of hyperbolic metrics on Riemann surfaces is already quite complicated. (See [Hj], [Ji], [W1], [W2].) In this paper we show that, for a wide variety of degenerating metrics which have, however, quite different behavior from that of the hyperbolic metric, the spectrum converges to the spectrum of the surface with the degenerate metric.

Specifically, we consider surfaces M_0 with a singular metric, where the singularity in local coordinates is quasi-isometrically a cone. (See Sect. 2 for our model.) Such singularities were studied first by Cheeger [Che1] and subsequently by various authors, particularly in the context of $\bar{\partial}$, Dirac, and other first-order operators. (See [Chou], [BS], [S1], [S2].) It is a fundamental fact about metrics with cone singularities that the Laplacian Δ_0 on M_0 still has a discrete spectrum $\mathrm{Spec}(\Delta_0) = \{\lambda_i(0)\}_{i=0}^{\infty}$, which we order $0 = \lambda_0(0) < \lambda_1(0) \leqslant \lambda_2(0) \leqslant \dots$. The natural question which then arises is the following: suppose we are given compact surfaces M_t with degenerating metrics g_t converging as $t \to 0$ to a metric on M_0 which has a cone singularity p. The singularity is assumed to be a double point; that is, locally we have two cones joined at their vertices. The noncompact surface $M_0 \backslash \{p\}$ may or may not be connected, and we shall refer to these two possibilities as the nonseparating and separating cases, respectively. We are interested in when Spec $(\Delta_t) = \{\lambda_i(t)\}_{i=0}^{\infty}$ converges to $\mathrm{Spec}(\Delta_0)$. To state the results precisely, we fix some notation: let $\{\varphi_i(t)\}_{i=0}^{\infty}$ denote a complete orthonormal basis of eigenfunctions with eigenvalues $\lambda_i(t)$, and for $\lambda > 0$ define the kernel function

$$K_t(x, y; \lambda) = \sum_{\lambda_i(t) < \lambda} \varphi_i(t)(x)\varphi_i(t)(y).$$

By spectral convergence we mean the following:

(∗) *Spectral convergence*

(i) For all $i \geqslant 1$, $\lim_{t \to 0} \lambda_i(t) = \lambda_i(0)$;

(ii) for any sequence $t_j \to 0$ there exists a subsequence $t_j' \to 0$ such that for all $i \geqslant 1$

$$\lim_{j \to \infty} \varphi_i(t_j') = \varphi_i(0)$$

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uniformly on compact subsets of $M_0\backslash\{p\}$ for some choice of complete orthonormal basis $\{\varphi_i(0)\}$ of eigenfunctions for M_0;

(iii) for any $\lambda > 0$, $\lambda \notin \mathrm{Spec}(\Delta_0)$,

$$\lim_{t\to 0} K_t(x, y; \lambda) = K_0(x, y; \lambda)$$

uniformly on compact subsets of $M_0\backslash\{p\} \times M_0\backslash\{p\}$.

In Section 2 we recall the definition of a cone metric and some basic results. We shall construct a model for M_t, $0 \leqslant t \leqslant 1$, degenerating as $t \to 0$ to a surface with a cone metric, and in the subsequent two sections we prove the following theorem.

THEOREM A. *For M_t a family of compact Riemannian surfaces degenerating as $t \to 0$ to a surface with cone metric, we have spectral convergence* (*).

Our main tool for the proof of Theorem A is the result of P. Li (Theorem 2.5 below) giving L^∞ estimates on eigenfunctions in terms of the inverse squared of the isoperimetric constant. For our degeneration model, the constant, localized to the degenerating neighborhood, is bounded away from zero (Prop. 2.6), and Li's estimate may then be used to extract a converging subsequence of eigenfunctions. The theorem then follows by a min-max argument. All this occupies Sections 3 and 4.

In Section 5 we construct analytic families of compact Riemann surfaces M_t of genus $g \geqslant 2$ (where t is now in the unit disk $D \subset \mathbb{C}$) degenerating as $t \to 0$ to a surface M_0 with a node p. If μ_t denotes the Bergman metric on M_t (see Def. 5.1), then M_0 has the metric $g_i/g\ \mu$ on M_i, $i = 1, 2$ if p is separating. If p is nonseparating, then M_0 has the metric $(g-1)/g\ \mu$. However, in this case the elliptic tail becomes a "long, thin cylinder" as $t \to 0$. We prove the following theorem.

THEOREM B. *Let M_t be a degenerating family of compact Riemann surfaces endowed with Bergman metrics μ_t.*

(i) *If $M_0\backslash\{p\}$ has two components, then as $t \to 0$, we have spectral convergence* (*);

(ii) *if $M_0\backslash\{p\}$ is connected, then the set of limit points of* $\mathrm{Spec}(\Delta_t)$ *as $t \to 0$ is dense in* $[0, +\infty)$.

Finally, in Section 6 we study the admissible metrics of Arakelov [A], normalized to have unit area. (See Def. 6.2.) In Proposition 6.6 we show that these metrics degenerate to "admissible cone metrics" which are supported on the component of $M_0\backslash\{p\}$ with the larger genus. In the equal-genus separating case, we again have a long, thin cylinder.

THEOREM C. *Let M_t be a degenerating family of compact Riemann surfaces with normalized admissible metrics.*

(i) *If $M_0\backslash\{p\}$ has one component or has two components of unequal genus, then we have spectral convergence* (*). ($\mathrm{Spec}(\Delta_0)$ *is the spectrum of the cone metric on the component of larger genus.*)

(ii) *If M_t degenerates to two surfaces of equal genus, joined at a separating node, then the set of limit points of* $\mathrm{Spec}(\Delta_t)$ *as $t \to 0$ is dense in* $[0, +\infty)$.

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2. Cone metrics and isoperimetric constants. In this section we recall the definition of a cone metric and its spectrum. We present a model for a surface with smooth metric degenerating to a cone metric. Finally, we introduce the isoperimetric constant $\mathscr{I}(C)$, and in Proposition 2.6 we show that, for the degeneration model, the constant is bounded away from zero.

Definition 2.1. Let $(N, \tilde{g})$ be a closed, smooth $(n-1)$-dimensional Riemannian manifold. The cone $C(N)$ on N is defined as the space $(0, 1) \times N$ with metric

$$ds_C^2(r, x) = dr \otimes dr + r^2\tilde{g}(x).$$

An n-dimensional manifold M with metric g defined on $M\backslash\{p\}$ is called a *cone manifold*, and g is called a *cone metric* with conical singularity at p if, for some choice of N and some neighborhood U of p, $U\backslash\{p\}$ is isometric to $C(N)$.

Of course, we may generalize this definition to include the case of several conical singularities. For simplicity, however, we shall always deal with one.

Let $g = g_{ij}dx_i \otimes dx_j$ have conical singularity at p. Then

$$\Delta = -\frac{1}{\sqrt{\det g_{ij}}}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\sqrt{\det g_{ij}}\,g^{ij}\frac{\partial}{\partial x_j}\right)$$

(where $g^{ij} = (g^{-1})^{ij}$) is a second-order differential operator acting on $C_0^\infty(M\backslash\{p\})$. We wish to extend Δ as an operator acting on the Hilbert space $L^2(M)$. For this we take the domain of Δ to consist of L^2 functions f such that $|\nabla f|$, $\Delta f \in L^2(M)$. Since Stokes's theorem holds for cone manifolds ([Che2], Theorem 2.2), then by a theorem of Gaffney [G], the L^2-closure of Δ is selfadjoint. We call this closure the Laplacian of M and continue to denote it by Δ. Furthermore, we have the following theorem.

THEOREM 2.2 ([Che1], Theorem 3.1).
(a) *Δ acting on $L^2(M)$ has discrete spectrum, and each eigenvalue has finite multiplicity.*
(b) *An eigenfunction φ of Δ with eigenvalue λ is characterized by $\Delta\varphi - \lambda\varphi = 0$, with φ, $|\nabla\varphi| \in L^2(M)$. The eigenvalues may be ordered with multiplicity* $0 = \lambda_0(M) \leqslant \lambda_1(M) \leqslant \cdots$.

In this paper we are interested in the case of two-dimensional manifolds and a slight generalization of the notion of conic singularity—namely, the case where p is a *double point*. This may be regarded locally as the union of two cone surfaces with the singularity identified. It is natural to view such a singularity as arising from a pinched cylinder or annulus. Consider the following family C_t of annuli with a metric: for $0 \leqslant t \leqslant 1$,

$$C_t = \{(x, y) \mid -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 2\pi\}/\{(x, 0) \sim (x, 2\pi)\}$$

with metric $ds_t^2 = dx^2 + (t + (1 - t)x^2)\, dy^2$. From Definition 2.1 the metric on C_0 has a cone singularity with a double point as described above. In the following we shall refer to a family of compact, connected surfaces M_t with Riemannian metrics g_t, $0 < t \leqslant 1$, as a *conic degenerating family* if M_t contains a cylinder (henceforth referred to as the *pinching annulus*) which is uniformly quasi-isometric to C_t and g_t converges on $M_t \backslash C_t$ to a smooth Riemannian metric. In the limit we have a singular metric g_0 on M_0, quasi-isometric to one with a cone singularity at a double point p—notice that $M_0 \backslash \{p\}$ may or may not be connected. We refer to these two possibilities as the nonseparating and separating cases, respectively. We now turn to the isoperimetric constant.

Definition 2.3. For a compact Riemannian manifold M^n of dimension n without boundary, the *Sobolev constant* $\mathscr{S}(M)$ is defined to be supremum over all constants c such that

$$\left(\int_M |\nabla f|\right)^n \geqslant c \inf_{\alpha \in \mathbb{R}} \left(\int_M |f - \alpha|^{n/(n-1)}\right)^{n-1}$$

for all functions f on M, and the *isoperimetric constant* is defined as

$$\mathscr{I}(M) = \inf_S \frac{\mathrm{Area}(S)^n}{(\min\{\mathrm{Vol}(N_1), \mathrm{Vol}(N_2)\})^{n-1}},$$

where S ranges over all hypersurfaces in M which divide M into two components N_1 and N_2 with $\partial(N_1) = \partial(N_2) = S$.

On the other hand, for a compact Riemannian manifold M^n with nonempty boundary ∂M, the Sobolev constant $\mathscr{S}(M)$ is defined as

$$\mathscr{S}(M) = \inf_f \frac{(\int_M |\nabla f|)^n}{\int_M |f|^{n/(n-1)}}$$

where $f \neq 0 \in C_0^\infty(M \backslash \partial M)$, and the isoperimetric constant is defined as

$$\mathscr{I}(M) = \inf_D \frac{\{A(\partial D)\}^n}{\{V(D)\}^{n-1}}$$

where $D \subset M$ ranges over all open submanifolds of M having smooth boundary satisfying $\overline{D} \cap \partial M = \varnothing$, and A and V denote the area and volume, respectively.

THEOREM 2.4. (See [Cha], Theorems 4 and 12 in Chap. IV.) *For any compact Riemannian manifold M with boundary ∂M (∂M may be empty),*

$$\mathscr{I}(M) \leqslant \mathscr{S}(M).$$

If we treat the cone singularity as an interior point, then the definitions of $\mathscr{S}(M)$, $\mathscr{I}(M)$, and Theorem 2.4 easily generalize for surfaces ($n = 2$) with cone singularities. For the proof of Theorems A, B, and C, we need the following L^∞ bounds on eigenfunctions. (For simplicity we only state the case $n = 2$.)

THEOREM 2.5 (P. Li).

(i) *If M is a two-dimensional compact Riemannian manifold without boundary, then there is a constant c independent of M such that, for any eigenfunction φ on M with eigenvalue $\lambda \neq 0$,*

$$\|\varphi\|_{L^\infty} \leqslant c\left(\frac{4\lambda}{\mathscr{S}(M)}\right)^2 V(M)\|\varphi\|_{L^2}.$$

(ii) *If M is compact with nonempty boundary, then for any eigenfunction φ on M of eigenvalue λ with respect to Dirichlet boundary conditions, the same inequality holds for φ.*

(iii) *In either of the above cases, if we assume M has a cone singularity, then the same inequality holds for eigenfunctions on M.*

Proof. For parts (i) and (ii), see [Li] and [Cha, Sect. 4 in Chap. IV]. For part (iii) we note that Stokes's theorem holds for manifolds with cone singularities. (See [Che2].) Then the same proof works in this case as well.

We have the following uniform lower bound for $\mathscr{I}(C_t)$.

PROPOSITION 2.6. *There exists a constant $c > 0$ independent of t such that for $0 < t \leqslant 1$*

$$\mathscr{I}(C_t) \geqslant c > 0.$$

To estimate isoperimetric constants, we need the following theorem.

THEOREM 2.7 (F. Fiala, see [Fa]). *Let M be a Riemannian surface, K its Gaussian curvature, and $K^+ = \max\{0, K\}$. Then for any simply connected domain D in M,*

$$L^2(\partial D) - 4\pi A(D) + 2\pi \int_D K^+ \geqslant 0$$

where $L(\partial D)$ is the length of the boundary ∂D and $A(D)$ is the area of the domain D. In particular, if $K \leqslant 0$, then

$$L^2(\partial D) \geqslant 4\pi A(D).$$

LEMMA 2.8. *For $0 < t \leqslant 1$ the Gaussian curvature of C_t is nonpositive.*

Proof. This follows by direct computation.

Proof of Proposition 2.6. For $1/2 \leqslant t \leqslant 1$ the cones C_t form a compact family of compact surfaces. Thus, to bound the isoperimetric constants of C_t away from zero, it suffices to consider the case $0 < t \leqslant 1/2$. By the definition of isoperimetric constants, we need to estimate $L^2(\partial D)/A(D)$ for all domains $D \subset C_t$. According to a theorem of S.-T. Yau, however, it suffices to consider the situation where the domain D is connected. (See [Yau].) There are two cases to consider: (a) no component of ∂D is homotopic to a boundary component of C_t; (b) at least one component of ∂D is homotopic to a boundary component of C_t.

Case (a). In this case every component of ∂D is contractible in C_t. Thus, we can assume that the domain D is simply connected. Otherwise, D may be embedded in the universal covering space $\tilde{C}_t$ of C_t, which is homeomorphic to $\mathbb{R}^2$. Fill in the interior holes of $D \subset \mathbb{R}^2$ and replace D by the newly filled one. In this way we increase the area of the domain, while decreasing the length of the boundary. Since the domain D is simply connected, by Theorem 2.7 and Lemma 2.8

$$\frac{L^2(\partial D)}{A(D)} \geqslant 4\pi.$$

Case (b). Since ∂D has at least one component homotopic to one component of the boundary of C_t and since D is connected, then ∂D has two components which are homotopic to the boundaries of C_t, and all other components are contractible in C_t. Filling in the holes bounded by the latter boundaries, we increase the area and decrease the length. Thus, we can assume that D is homeomorphic to a cylinder and that it has two boundaries, denoted by γ_1, γ_2 (γ_1 lies to the left of γ_2), which are homotopic to the boundaries of C_t.

Step (i). First, we assume that γ_1 and γ_2 are rotationally symmetric, that is, for some $-1 < \varepsilon_1 < \varepsilon_2 < 1$,

$$D = \{(x, y) \in C_t | \varepsilon_1 < x < \varepsilon_2\}.$$

It can be seen easily that it suffices to consider the case $\varepsilon_1 = 0$, and $0 < \varepsilon_2 = \varepsilon < 1$. Then

$$L(\partial D) = 2\pi\{(t + (1 - t)\varepsilon)^{1/2} + t^{1/2}\},$$

$$A(D) = \int_0^{2\pi} dy \int_0^{\varepsilon} (t + (1 - t)x^2)^{1/2}\, dx.$$

We are now going to estimate $A(D_{\varepsilon,t})$ from above and $L^2(\partial D_{\varepsilon,t})/A(D_{\varepsilon,t})$ from below. Depending on the relative size of t and ε, there are two cases to consider.

First, we assume $t \geqslant \varepsilon^2 > 0$. From the inequality $\sqrt{1 + x} \leqslant 1 + x/2$ for $x \geqslant 0$,

$$
\begin{aligned}
A(D) &= 2\pi \int_0^\varepsilon (t + (1-t)x^2)^{1/2}\, dx \\
&\leqslant 2\pi\sqrt{t} \int_0^\varepsilon \left(1 + \frac{1-t}{2t}x^2\right) dx \\
&= 2\pi\sqrt{t}\left(\varepsilon + \frac{1-t}{2t}\frac{\varepsilon^3}{3}\right) \\
&\leqslant 2\pi\left\{t + \frac{1}{6}(1-t)t\right\} \leqslant \frac{7}{3}\pi t, \\
L^2(\partial D) &\geqslant 4\pi^2\{(t + (1-t)\varepsilon^2) + t\} \geqslant 8\pi^2 t.
\end{aligned}
$$

Thus, for $t \geqslant \varepsilon^2 > 0$,

$$
\frac{L^2\partial(D)}{A(D)} \geqslant \frac{8\pi^2 t}{\frac{7}{3}\pi t} = \frac{24}{7}\pi.
$$

Next, assume $0 < t \leqslant \varepsilon^2$. Since $\sqrt{1+x} \leqslant 1 + \sqrt{x}$ for $x \geqslant 0$,

$$
\begin{aligned}
A(D) &= 2\pi \int_0^\varepsilon (t + (1-t)x^2)^{1/2}\, dx \\
&\leqslant 2\pi\sqrt{t} \int_0^\varepsilon \left(1 + \sqrt{\frac{1-t}{t}}x\right) dx \\
&= 2\pi\sqrt{t}\left(\varepsilon + \sqrt{\frac{1-t}{t}}\frac{\varepsilon^2}{2}\right) \\
&\leqslant 2\pi\left\{\varepsilon^2 + \frac{1}{2}\sqrt{1-t}\varepsilon^2\right\} \leqslant 3\pi\varepsilon^2, \\
L^2(\partial D) &\geqslant 4\pi^2\{t + (1-t)\varepsilon^2\} \\
&\geqslant 4\pi^2(1-t)\varepsilon^2 \geqslant 2\pi^2\varepsilon^2,
\end{aligned}
$$

since $0 < t \leqslant 1/2$. Thus, for $0 < t \leqslant \varepsilon^2$, $L^2(\partial D)/A(D) \geqslant 2\pi/3$, and so for rotationally symmetric domain D, we certainly have $L^2(\partial D)/A(D) \geqslant 1/3$.

Step (ii). Second, we consider the case where γ_1 and γ_2 may not be rotationally symmetric but neither of them intersects the pinching geodesic $\gamma(t) = \{(0, y) \in C_t\}$ in C_t. Let γ' be the rotationally symmetric closed curve lying between γ_1, $\gamma(t)$, and touching γ_1. Let D_1 be the domain bounded by γ_1, $\gamma(t)$ and let D' be the domain bounded by γ', $\gamma(t)$. Then $D_1 \backslash D'$ is a union of several simply connected domains in C_t. By Theorem 2.7 and Lemma 2.8

$$(L(\gamma_1) + L(\gamma'))^2 \geqslant 4\pi A(D_1 \backslash D').$$

On the other hand, since D' is rotationally symmetric, by Step (i)

$$(L(\gamma') + L(\gamma(t)))^2 \geqslant \frac{2}{3}\pi A(D').$$

Since $L(\gamma_1) \geqslant L(\gamma')$, $L(\gamma_1) \geqslant L(\gamma(t))$,

$$8L^2(\gamma_1) \geqslant 4\pi A(D_1 \backslash D') + \frac{2}{3}\pi A(D') \geqslant \frac{2}{3}\pi A(D_1).$$

Similarly, let D_2 be the domain bounded by γ_2, $\gamma(t)$. Then $L^2(\gamma_2) \geqslant \pi A(D_2)/12$. Since $A(D) \leqslant A(D_1) + A(D_2)$ (if γ_1 and γ_2 lie on different sides of $\gamma(t)$, then the equality holds), and $L(\partial D) = L(\gamma_1) + L(\gamma_2)$,

$$L^2(\partial D) \geqslant L^2(\gamma_1) + L^2(\gamma_2) \geqslant \frac{\pi}{12}(A(D_1) + A(D_2)) \geqslant \frac{\pi}{12}A(D).$$

Step (iii). Third, we assume that only one of γ_1, γ_2 intersects $\gamma(t)$. Suppose γ_1 intersects $\gamma(t)$. Then the subdomain $D_1 = D \cap \{(x, y) \in C_t | x \leqslant 0\}$ of D lying to the left of the pinching geodesic $\gamma(t)$ is a union of simply connected domains. Then by Theorem 2.7 and Lemma 2.8

$$(L(\gamma_1) + L(\gamma(t)))^2 \geqslant (L(\partial D_1))^2 \geqslant 4\pi A(D_1).$$

The right subdomain, $D_2 = D \cap \{(x, y) \in C_t | x \geqslant 0\}$, is contained in the domain $\tilde{D}_2$ bounded by γ_2, $\gamma(t)$. Since γ_2 does not intersect $\gamma(t)$, by Step (ii)

$$(L(\gamma_2) + L(\gamma(t)))^2 \geqslant \frac{\pi}{12}A(\tilde{D}_2) \geqslant \frac{\pi}{12}A(D_2).$$

Notice that for $i = 1, 2$, $L(\gamma_i) \geqslant L(\gamma(t))$; so

$$\begin{aligned}(L(\partial D))^2 &\geqslant (L(\gamma_1))^2 + (L(\gamma_2))^2 \\ &\geqslant \frac{1}{4}\{(L(\gamma_1) + L(\gamma(t)))^2 + (L(\gamma_1) + L(\gamma(t)))^2\}\end{aligned}$$

$$\geqslant \frac{1}{4}\{4\pi A(D_1) + 1/12\pi A(D_2)\}$$

$$\geqslant \frac{\pi}{48} A(D).$$

Step (iv). Finally, we assume that both γ_1, γ_2 intersect $\gamma(t)$. Then $D\backslash\gamma(t)$ is a union of several simply connected domains. By Theorem 2.7 and Lemma 2.8

$$(L(\gamma_1) + L(\gamma_2) + L(\gamma(t)))^2 \geqslant (L(\partial D\backslash\gamma(t)))^2 \geqslant 4\pi A(D\backslash\gamma(t)) = 4\pi A(D).$$

Since $L(\gamma_1) + L(\gamma_2) \geqslant 2L(\gamma(t))$,

$$(L(\partial D))^2 \geqslant \frac{1}{4}(L(\gamma_1) + L(\gamma_2) + L(\gamma(t)))^2 \geqslant \pi A(D).$$

Combining cases (a) and (b), we get that, for $0 \leqslant t \leqslant 1/2$, $\mathscr{I}(C_t) \geqslant \pi/48$. As stated at the beginning of the proof, for $1/2 \leqslant t \leqslant 1$ the cone C_t forms a compact family of compact surfaces. Therefore, the proof of Proposition 2.6 is complete.

COROLLARY 2.9. *For any conic degenerating family M_t of surfaces, if the pinching geodesic is nonseparating, then there exists a constant $c > 0$ depending only on the family such that, for $0 < t \leqslant 1$,*

$$\mathscr{I}(M_t) \geqslant c > 0.$$

Proof. For the family M_t the complement of the pinching annulus C_t forms a compact family of compact surfaces. By assumption, the pinching geodesic $\gamma(t)$ is nonseparating; thus, it suffices to consider the isoperimetric constants for the pinching annuli C_t. Since the metrics on the pinching cones C_t are uniformly quasi-isometric to the standard metrics ds_t^2 on the cones C_t above and the isoperimetric constants are determined up to some multiple by the quasi-isometric class of the metrics, the corollary follows immediately from Proposition 2.6.

Remark 2.10. For a degenerating family M_t of surfaces with hyperbolic metrics, whether the pinching geodesics in M_t are separating or not, the isoperimetric constant of M_t (or of the pinching annulus) converges to zero as $t \to 0$. Because of this fact, the spectral degeneration for hyperbolic surfaces is more complicated. (See [Hj], [Ji], [W1], [W2].)

3. Spectral degeneration for cones. The proof of Theorem A is divided into two steps.

1. For all $i \geqslant 1$, $\overline{\lim}_{t\to 0} \lambda_i(t) \leqslant \lambda_i(0)$.
2. For all $i \geqslant 1$, $\underline{\lim}_{t\to 0} \lambda_i(t) \geqslant \lambda_i(0)$.

In this section we are going to prove step (1), that is, the following proposition.

PROPOSITION 3.1. *Given any conic degenerating family M_t of surfaces,*

$$\overline{\lim_{t\to 0}}\, \lambda_i(t) \leqslant \lambda_i(0), \qquad \text{for all } i \geqslant 1.$$

Let M be any Riemannian surface with cone singularities. For simplicity we assume that M has only one cone singular point, and unlike the degeneration model described in Section 2 we take it to be a "single" point, as opposed to a double point. Hence, we may write $M = K \cup C$, where K is a compact, connected surface with boundary and $C = \{(x, y) | 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 2\pi\}/\{(x, 0) \sim (x, 2\pi)\}$ endowed with a metric quasi-isometric to the standard one $ds^2 = dx^2 + x^2 dy^2$. For any $0 < \varepsilon < 1$ let $M_\varepsilon = K \cup \{(x, y) \in C | x \geqslant \varepsilon\}$ be the submanifold of M obtained by cutting off a subcylinder.

Let $\{\lambda_{i,\varepsilon}\}_1^\infty$ be all the eigenvalues of M_ε (counted with multiplicity) with respect to the Dirichlet boundary condition and let $\{\lambda_i\}_1^\infty$ be all the eigenvalues of M. Then we have the following proposition.

PROPOSITION 3.2. *For all $i \geqslant 1$, $\lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = \lambda_i$.*

Before proving Proposition 3.2, we establish some lemmas whose statements and proofs are models for arguments later on. By a proof similar to that of Proposition 2.6, we immediately have the following lemma.

LEMMA 3.3. *For a surface M with only cone singularities and subdomains M_ε as above, there exists a constant $c > 0$ such that for all $0 \leqslant \varepsilon < 1$*

$$\mathscr{I}(M_\varepsilon) \geqslant c > 0.$$

LEMMA 3.4. *For any sequence $\varepsilon_j \to 0$ let $\varphi_{n_1,\varepsilon_j}, \ldots, \varphi_{n_m,\varepsilon_j}$ be orthonormal eigenfunctions on M_{ε_j} with eigenvalues $\lambda_{n_1,\varepsilon_j}, \ldots, \lambda_{n_m,\varepsilon_j}$. Assume that, for $1 \leqslant i \leqslant m$, $\lim_{\varepsilon_j \to 0} \lambda_{n_i,\varepsilon_j} = \lambda_{n_i}$ and $\varphi_{n_i,\varepsilon_j}$ converges smoothly over compact subsets of M to a function φ_{n_i} on M. Then the limit functions $\varphi_{n_1}, \ldots, \varphi_{n_m}$ are orthonormal eigenfunctions of M with eigenvalues $\lambda_{n_1}, \ldots, \lambda_{n_m}$.*

Proof. First of all, it is clear that for $1 \leqslant i \leqslant m$

$$(\Delta - \lambda_{n_i})\varphi_{n_i} = 0, \qquad \|\varphi_{n_i}\|_{L^2} \leqslant 1$$

$$\text{and} \qquad \|\nabla\varphi_{n_i}\|_{L^2} \leqslant \lambda_{n_i} < +\infty$$

since $\|\varphi_{n_i,\varepsilon_j}\|_{L^2} = 1$ and $\|\nabla\varphi_{n_i,\varepsilon_j}\|_{L^2} = \lambda_{n_i,\varepsilon_j}$. Therefore, it suffices to prove that for $1 \leqslant i, k \leqslant m$,

$$\langle \varphi_{n_i}, \varphi_{n_k} \rangle = \delta_{ik}.$$

For any $0 < \varepsilon < \delta < 1$ define a subdomain $C_{\varepsilon,\delta} = \{(x, y) \in C | \varepsilon \leqslant x \leqslant \delta\} \subset C \subset M$ and $C_\delta = C_{0,\delta}$ a subcylinder of C. It is clear that $C_{\varepsilon,\delta} \subset C_\delta$. Then for any $0 < \varepsilon_j <$

$\delta < 1$ and any $1 \leqslant i, k \leqslant m$,

$$\begin{aligned}\langle \varphi_{n_i}, \varphi_{n_k} \rangle_M &= \int_M \varphi_{n_i} \varphi_{n_k} \\ &= \int_{M \setminus C_\delta} \varphi_{n_i} \varphi_{n_k} + \int_{C_\delta} \varphi_{n_i} \varphi_{n_k}, \\ \delta_{ik} &= \langle \varphi_{n_i, \varepsilon_j}, \varphi_{n_k, \varepsilon_j} \rangle_{M_{\varepsilon_j}} \\ &= \int_{M \setminus C_\delta} \varphi_{n_k, \varepsilon_j} \varphi_{n_k, \varepsilon_j} + \int_{C_{\varepsilon_j, \delta}} \varphi_{n_k, \varepsilon_j} \varphi_{n_k, \varepsilon_j}.\end{aligned}$$

Therefore,

$$\begin{aligned}|\langle \varphi_{n_i}, \varphi_{n_k} \rangle_M - \delta_{ik}| \leqslant & \left| \int_{M \setminus C_\delta} \varphi_{n_i} \varphi_{n_k} - \int_{M \setminus C_\delta} \varphi_{n_k, \varepsilon_j} \varphi_{n_k, \varepsilon_j} \right| \\ & + \left| \int_{C_\delta} \varphi_{n_i} \varphi_{n_k} \right| + \left| \int_{C_{\varepsilon_j, \delta}} \varphi_{n_k, \varepsilon_j} \varphi_{n_k, \varepsilon_j} \right|.\end{aligned}$$

By Theorems 2.4, 2.5, and Lemma 3.3, $\varphi_{n_1, \varepsilon_j}, \ldots, \varphi_{n_m, \varepsilon_j}$ are bounded from above independent of $\{\varepsilon_j\}$; $\varphi_{n_1}, \ldots, \varphi_{n_m}$ are therefore bounded as well by the assumption of convergence of $\{\varphi_{n_i, \varepsilon_j}\}$ and the uniform bound on the latter. Furthermore, $\lim_{\delta \to 0} A(C_\delta) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} A(C_{\varepsilon, \delta}) = 0$. Then for any $\delta' > 0$ there exists $\delta_0 > 0$ such that for $0 < \delta \leqslant \delta_0$

$$\left| \int_{C_\delta} \varphi_{n_i} \varphi_{n_k} \right| + \lim_{\varepsilon_j \to 0} \left| \int_{C_{\varepsilon_j, \delta}} \varphi_{n_i, \varepsilon_j} \varphi_{n_k, \varepsilon_j} \right| \leqslant \delta'.$$

Notice the uniform convergence of $\varphi_{n_i, \varepsilon_j}, \varphi_{n_k, \varepsilon_j}$ to $\varphi_{n_i}, \varphi_{n_k}$, respectively, over compact subsets of M. It follows that

$$\begin{aligned}|\langle \varphi_{n_i}, \varphi_{n_k} \rangle_M - \delta_{ik}| &\leqslant \overline{\lim_{\varepsilon_j \to 0}} \left| \int_{M \setminus C_\delta} \varphi_{n_i} \varphi_{n_k} - \int_{M \setminus C_\delta} \varphi_{n_i, \varepsilon_j} \varphi_{n_k, \varepsilon_j} \right| + \delta' \\ &= \delta'.\end{aligned}$$

Since $\delta' > 0$ is arbitrary, for $1 \leqslant i, k \leqslant n$, we have

$$\langle \varphi_{n_i}, \varphi_{n_k} \rangle_M - \delta_{ik} = 0.$$

LEMMA 3.5. *Given any sequence $\varepsilon_j \to 0$ and a normalized eigenfunction φ_{ε_j} on M_{ε_j} with eigenvalue λ_{ε_j}, assume that $\overline{\lim}_{\varepsilon_j \to 0} \lambda_{\varepsilon_j} < +\infty$. Then there exists a subsequence $\varepsilon'_j \to 0$ such that $\lim_{\varepsilon'_j \to 0} \lambda_{\varepsilon'_j}$ exists and $\varphi_{\varepsilon'_j}$ converges smoothly over compact subsets of M to a function φ, which is a normalized eigenfunction on M with eigenvalue $\lim_{\varepsilon'_j \to 0} \lambda_{\varepsilon'_j}$.*

Proof. We have

$$(\Delta - \lambda_{\varepsilon_j})\varphi_{\varepsilon_j} = 0 \qquad \text{and}$$

$$\int_{M_{\varepsilon_j}} |\nabla \varphi_{\varepsilon_j}|^2 = \lambda_{\varepsilon_j} \int_{M_{\varepsilon_j}} |\varphi_{\varepsilon_j}|^2 = \lambda_{\varepsilon_j}.$$

By regularity theory (see Theorems 8.8, 8.10 in [GT]), for any compact subset $\bar{D} \subset M$ and any $k \in \mathbb{N}$, there exists a constant $c = c(\bar{D}, \overline{\lim}_{\varepsilon_j \to 0} \lambda_{\varepsilon_j})$ such that

$$\|\varphi_{\varepsilon_j}\|_{W^{k,2}(\bar{D})} \leqslant c.$$

Then by the Sobolev embedding theorem (see Theorem 5.4 in [Ad]) and a diagonal argument, there exists a subsequence $\varepsilon'_j \to 0$ such that $\lim_{\varepsilon'_j \to 0} \lambda_{\varepsilon'_j}$ exists and $\varphi_{\varepsilon'_j}$ converges over compact subsets of M to a function φ. By Lemma 3.4 the limit function φ is a normalized eigenfunction on M with eigenvalue $\lim_{\varepsilon'_j \to 0} \lambda_{\varepsilon'_j}$.

Proof of Proposition 3.2. By domain monotonicity for Dirichlet eigenvalues, $\lambda_{i,\varepsilon} \leqslant \lambda_{i,1/2}$ for $0 < \varepsilon \leqslant 1/2$ and $i \geqslant 1$. For any sequence $\varepsilon_j \to 0$ let $\{\varphi_{i,\varepsilon_j}\}_1^\infty$ be a complete system of orthonormal eigenfunctions with eigenvalues $\{\lambda_{i,\varepsilon_j}\}_1^\infty$. By Lemmas 3.4, 3.5, and a diagonal argument, there exists a subsequence $\varepsilon'_j \to 0$ such that, for all $i \geqslant 1$, $\lambda_i^* = \lim_{\varepsilon'_j \to 0} \lambda_{i,\varepsilon'_j}$ exists, $\varphi_{i,\varepsilon'_j}$ converges smoothly over compact subsets of M to an eigenfunction φ_i^* on M with eigenvalues λ_i^*, and the limit functions $\{\varphi_i^*\}_1^\infty$ are orthonormal.

Claim. The limit functions $\{\varphi_i^*\}_1^\infty$ form a complete system of orthonormal eigenfunctions with eigenvalues $\{\lambda_i^*\}_1^\infty$.

Assuming the claim, it is clear that, for all $i \geqslant 1$, $\lambda_i^* = \lambda_i$. By the arbitrary choice of $\varepsilon_j \to 0$, for all $i \geqslant 1$

$$\lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = \lambda_i.$$

This completes the proof of Proposition 3.2.

Proof of claim. Assume the contrary. Then there exists a normalized eigenfunction φ on M with eigenvalue λ such that, for all $i \geqslant 1$, $\langle \varphi, \varphi_i^* \rangle = 0$. Let $\eta_\varepsilon = \eta_\varepsilon(x)$ be a cutoff function on M such that $\eta_\varepsilon = 1$ on $M_{3\varepsilon} = M \backslash C_{3\varepsilon}$, $\eta_\varepsilon = 0$ on $C_{2\varepsilon}$, and

$|\nabla \eta_\varepsilon| \leqslant 2/\varepsilon$. Then

$$\lim_{\varepsilon \to 0} \int_M |\varphi \eta_\varepsilon|^2 = \int_M |\varphi|^2 = 1 \qquad \text{and}$$

$$\begin{aligned}\int_M |\nabla(\varphi \eta_\varepsilon)|^2 &\leqslant \int_M |\nabla \varphi|^2 + \int_M |\varphi|^2 |\nabla \eta_\varepsilon|^2 \\ &\leqslant \lambda \int_M |\varphi|^2 + \frac{4}{\varepsilon^2} \int_{C_{\varepsilon,2\varepsilon}} |\varphi|^2.\end{aligned}$$

By Theorems 2.4, 2.5, and Lemma 3.3, there exists some constant $e_0 > 0$ such that $\|\varphi\|_{L^\infty} \leqslant e_0 < +\infty$. Then

$$\begin{aligned}\int_{C_{\varepsilon,2\varepsilon}} |\varphi|^2 &\leqslant e_0 \int_{C_{\varepsilon,2\varepsilon}} 1 \\ &= e_0 A(C_{\varepsilon,2\varepsilon}) = \frac{3}{2} e_0 \varepsilon^2.\end{aligned}$$

Therefore,

$$\int_M |\nabla(\varphi \eta_\varepsilon)|^2 \leqslant \lambda + 6 e_0. \tag{3.6}$$

Expanding the function $\varphi \eta_{\varepsilon'_j}$ in terms of the complete system of orthonormal eigenfunctions $\{\varphi_{i,\varepsilon'_j}\}_1^\infty$ on $M_{\varepsilon'_j}$,

$$\varphi \eta_{\varepsilon'_j} = \sum_{i=1}^{\infty} a_i(\varepsilon'_j) \varphi_{i,\varepsilon'_j} \tag{3.7}$$

where for $i \geqslant 1$

$$a_i(\varepsilon'_j) = \langle \varphi \eta_{\varepsilon'_j}, \varphi_{i,\varepsilon'_j} \rangle_{M_{\varepsilon'_j}} \qquad \text{and}$$

$$\sum_{i=1}^{\infty} a_i^2(\varepsilon'_j) = \|\varphi \eta_{\varepsilon'_j}\|^2.$$

Similarly, from equation 3.7

$$\begin{aligned}\|\nabla(\varphi \eta_{\varepsilon'_j})\|^2 &= \sum_{i=1}^{\infty} a_i^2(\varepsilon'_j) \lambda_{i,\varepsilon'_j} \\ &\geqslant \sum_{i=1}^{\infty} a_i^2(\varepsilon'_j) \lambda_i\end{aligned}$$

where the inequality $\lambda_{i,\varepsilon_j'} \geqslant \lambda_i$ follows from the domain monotonicity for Dirichlet eigenvalues. For any $N \in \mathbb{N}$

$$\|\nabla(\varphi\eta_{\varepsilon_j'})\|^2 \geqslant \lambda_N \sum_{i>N}^{\infty} a_i^2(\varepsilon_j').$$

Since $\lambda_N \to +\infty$ as $N \to +\infty$, by equation 3.6, for any $0 < \delta < 1$ there exists N_0 independent of ε_j' such that

$$\sum_{i>N_0}^{\infty} a_i^2(\varepsilon_j') \leqslant \delta \qquad \text{and}$$

$$\sum_{i=1}^{N_0} a_i^2(\varepsilon_j') \geqslant \|\varphi\eta_{\varepsilon_j'}\|^2 - \delta. \tag{3.8}$$

On the other hand, for $1 \leqslant i \leqslant N_0$

$$\begin{aligned}\lim_{\varepsilon_j'\to 0} a_i(\varepsilon_j') &= \lim_{\varepsilon_j'\to 0} \langle \varphi\eta_{\varepsilon_j'}, \varphi_{i,\varepsilon_j'}\rangle_{M_{\varepsilon_j'}} \\ &= \langle \varphi, \varphi_i^* \rangle = 0\end{aligned}$$

where in the second equality, we use the fact that $\varphi_{i,\varepsilon_j'}$ is bounded independent of ε_j'; this follows from Theorems 2.4, 2.5, Lemma 3.3, and was used in the proof of Lemma 3.4. Then letting $\varepsilon_j' \to 0$ in equation 3.8, we get

$$\begin{aligned}0 = \lim_{\varepsilon_j'\to 0} \sum_{i=1}^{N_0} a_i^2(\varepsilon_j') &\geqslant \lim_{\varepsilon_j'\to 0} \|\varphi\eta_{\varepsilon_j'}\|^2 - \delta \\ &= \|\varphi\|^2 - \delta = 1 - \delta.\end{aligned}$$

Since $\delta < 1$, this is a contradiction! Thus, we have proven the claim and thence Proposition 3.2.

Remark 3.9. The basic philosophy here is that, since in the limiting process of $M_\varepsilon \to M$ as $\varepsilon \to 0$ no mass of the eigenfunctions of M_ε is lost (see Lemma 3.4), it is reasonable that all eigenfunctions on M should come from eigenfunctions on M_ε.

Remark 3.10. A special case of Proposition 3.2 and its proof is the following fact. (See [CF1].) Let M^n be a compact Riemannian manifold of dimension $n \geqslant 2$, p be a distinguished point in M, and M_ε (for $\varepsilon > 0$ small) be the complement of the geodesic ball around p with radius ε. Then the Dirichlet eigenvalues of M_ε converge to eigenvalues of M as $\varepsilon \to 0$.

Proof of Proposition 3.1. For $1 > \varepsilon > 0$ let $M_{t,\varepsilon} = M_t \backslash \{(x, y) \in C_t \,|\, |x| < \varepsilon\}$ and let $\{\lambda_i(t, \varepsilon)\}_{i=1}^{\infty}$ be all the eigenvalues of $M_{t,\varepsilon}$ with respect to the Dirichlet boundary condition. Then by the domain monotonicity for eigenvalues, for all $i \geqslant 1$

$$\lambda_i(t) \leqslant \lambda_i(t, \varepsilon).$$

For any fixed $\varepsilon > 0$, $M_{t,\varepsilon}$ $(0 \leqslant t \leqslant 1)$ forms a compact family of compact surfaces. Thus, for all $i \geqslant 1$

$$\lim_{t \to 0} \lambda_i(t, \varepsilon) = \lambda_i(0, \varepsilon).$$

By Proposition 3.2, for all $i \geqslant 1$, $\lim_{\varepsilon \to 0} \lambda_i(0, \varepsilon) = \lambda_i(0)$. Therefore, for all $i \geqslant 1$

$$\begin{aligned} \overline{\lim_{t \to 0}}\, \lambda_i(t) &\leqslant \lim_{\varepsilon \to 0} \lim_{t \to 0} \lambda_i(t, \varepsilon) \\ &= \lim_{\varepsilon \to 0} \lambda_i(0, \varepsilon) \\ &= \lambda_i(0). \end{aligned}$$

This completes the proof.

To prepare for the proof of $\underline{\lim}_{t \to 0} \lambda_i(t) \geqslant \lambda_i(0)$ $(i \geqslant 1)$ in the next section, we study the spectral degeneration for the cone family C_t first. With respect to the Dirichlet boundary condition on ∂C_t, let $\{\mu_i(t)\}_{i=1}^{\infty}$ be all the eigenvalues of C_t counted with multiplicity.

PROPOSITION 3.11

(i) *For all* $i \geqslant 1$, $\lim_{t \to 0} \mu_i(t) = \mu_i(0)$.

(ii) *For any sequence* $t_j \to 0$ *let* $\{\psi_i(t_j)\}_{i=1}^{\infty}$ *be a complete system of orthonormal (Dirichlet) eigenfunctions on* C_{t_j} *with eigenvalues* $\{\mu_i(t_j)\}_{i=1}^{\infty}$. *Then there exists a subsequence* $t_j' \to 0$ *such that, for all* $i \geqslant 1$, $\psi_i(t_j')$ *converges smoothly over compact subsets of* C_0 *to an eigenfunction* $\psi_i(0)$ *with eigenvalue* $\mu_i(0)$, *and* $\{\psi_i(0)\}_{i=1}^{\infty}$ *is a complete system of orthonormal Dirichlet eigenfunctions on* C_0.

Proof. By the same proof as that of Proposition 3.1, for all $i \geqslant 1$

$$\overline{\lim_{t \to 0}}\, \mu_i(t) \leqslant \mu_i(0).$$

On the other hand, from Proposition 2.6 there exists a constant $c > 0$ such that, for $0 \leqslant t \leqslant 1$, $\mathscr{I}(C_t) \geqslant c > 0$. Then by arguments similar to those in the proofs of Lemmas 3.3 and 3.4, there exists a subsequence $t_j' \to 0$ such that, for all $i \geqslant 1$, $\psi_i(t_j')$ converges smoothly over compact subsets of C_0 to a Dirichlet eigenfunction $\psi_i(0)$ with eigenvalue $\mu_i(0)$, and $\{\psi_i(0)\}_{i=1}^{\infty}$ are orthonormal Dirichlet eigenfunctions on C_0. It is clear then that for all $i \geqslant 1$

$$\underline{\lim}_{t_j' \to 0}\, \mu_i(t_j') \geqslant \mu_i(0).$$

By the arbitrary choice of $t_j \to 0$

$$\varliminf_{t\to 0} \mu_i(t) \geqslant \mu_i(0).$$

Therefore, for all $i \geqslant 1$

$$\lim_{t\to 0} \mu_i(t) = \mu_i(0),$$

and $\{\psi_i(0)\}_{i=1}^{\infty}$ is a complete system of orthonormal Dirichlet eigenfunctions on C_0 with eigenvalues $\{\mu_i(0)\}_{i=1}^{\infty}$.

Remark 3.12. By Corollary 2.9, if the pinching geodesic $\gamma(t)$ in the conic family M_t is nonseparating, then the isoperimetric constant $\mathscr{I}(M_t) \geqslant c > 0$ for some constant c independent of t, and the proof above works for Theorem A in this case also. But for the case of the separating pinching geodesic, we need another argument. Instead, in Section 4 we prove Theorem A simultaneously for the pinching geodesic separating or not, thus justifying the philosophy that, to understand general degenerating families, it suffices to understand the degeneration of the pinched part. (see [Ji].)

Remark 3.13. The above proof for Proposition 3.11 gives a new, elementary proof of Theorem B in [CF2].

4. Proof of Theorem A.

In this section we prove that for all $i \geqslant 1$

$$\varliminf_{t\to 0} \lambda_i(t) \geqslant \lambda_i(0),$$

and we finish the proof of Theorem A.

By Proposition 3.1, for all $i \geqslant 1$

$$\varlimsup_{t\to 0} \lambda_i(t) \leqslant \lambda_i(0).$$

Then by arguments similar to those in the proofs of Lemmas 3.3 and 3.4, for any sequence $t_j \to 0$ there exists a subsequence $t_j' \to 0$ such that, for all $i \geqslant 1$, $\varphi_i(t_j')$ converges smoothly over compact subsets of M_0 to a function φ_i^* on M_0, and $\lambda_i^* = \lim_{t_j'\to 0} \lambda_i(t_j')$ exists. The limit function φ_i^* satisfies

$$(\Delta - \lambda_i^*)\varphi_i^* = 0,\ \|\varphi_i^*\|_{L^2} \leqslant 1 \qquad \text{and}$$

$$\|\nabla\varphi_i^*\|_{L^2} \leqslant \lambda_i^* < +\infty.$$

LEMMA 4.1. *The limit functions $\{\varphi_i^*\}_{i=1}^{\infty}$ are orthonormal eigenfunctions on M_0 with eigenvalues $\{\lambda_i^*\}_{i=1}^{\infty}$; that is, for all $i, k \geqslant 1$*

$$\langle \varphi_i^*, \varphi_k^* \rangle = \delta_{ik}.$$

Assume Lemma 4.1 first. Then the limit functions $\{\varphi_i^*\}_{i=1}^{\infty}$ are, in particular, linearly independent, and thus by min-max we have for all $i \geqslant 1$

$$\lim_{t_j' \to 0} \lambda_i(t_j') = \lambda_i^* \geqslant \lambda_i(0).$$

By the arbitrary choice of the sequence $t_j \to 0$, for all $i \geqslant 1$

$$\varliminf_{t \to 0} \lambda_i(t) \geqslant \lambda_i(0).$$

Therefore, combined with Proposition 3.1,

$$\lim_{t \to 0} \lambda_i(t) = \lambda_i(0), \qquad \text{for all } i \geqslant 1.$$

The limit functions $\{\varphi_i^*\}_{i=1}^{\infty}$ are a complete system of orthonormal eigenfunctions on M_0 with eigenvalues $\{\lambda_i(0)\}_{i=1}^{\infty}$ and thus may be denoted by $\{\varphi_i(0)\}_{i=1}^{\infty}$. This proves parts (i) and (ii) of Theorem A. For part (iii), $\lambda_i(0) < \lambda \notin \operatorname{Spec}(\Delta_0)$ if and only if $\lambda_i(t) < \lambda$ for small t; so by parts (i) and (ii)

$$\lim_{t_j' \to 0} K_{t_j'}(x, y; \lambda) = K_0(x, y; \lambda).$$

By the arbitrary choice of the sequence $t_j \to 0$,

$$\lim_{t \to 0} K_t(x, y; \lambda) = K_0(x, y; \lambda).$$

This completes the proof of Theorem A under the assumption of Lemma 4.1.

Proof of Lemma 4.1. Let $\eta = \eta(x)$ be a cutoff function on M_0 such that $\eta = 0$ on $M_{0,1/2} = M_0 \backslash \{(x, y) \in C_0 \,|\, |x| \leqslant 1/2\}$, $\eta = 1$ on $M_0 \backslash M_{0,1/4} = \{(x, y) \in C_0 \,|\, |x| \geqslant 1/4\}$, and $|\nabla \eta| \leqslant 8$ on M_0. For any fixed $i_0 \geqslant 1$ consider the function $\varphi_{i_0}(t_j')\eta$ on $C_{t_j'}$. We want to show that $\varphi_{i_0}(t_j')\eta$(or $\varphi_{i_0}(t_j')$) does not lose any mass inside the pinching annulus during degeneration. More precisely, define

$$m_0 = \varlimsup_{t_j' \to 0} \int_{M_{t_j'}} |\varphi_{i_0}(t_j')\eta|^2;$$

then we have the following claim.

Claim. (i) The mass of the limit function $\varphi_{i_0}^*\eta$ is

$$\int_{M_0} |\varphi_{i_0}^*\eta|^2 = m_0;$$

(ii) The mass lost during degeneration is

$$\lim_{\varepsilon\to 0}\overline{\lim_{t_j'\to 0}}\int_{M_t\setminus M_{t,\varepsilon}}|\varphi_{i_0}(t_j')|^2=0.$$

Proof of claim. Actually, by taking a subsequence, if necessary, we may assume that $\lim_{t_j'\to 0}\int_{M_{t_j'}}|\varphi_{i_0}(t_j')\eta|^2=m_0$. Expanding the function $\varphi_{i_0}(t_j')\eta$ on $C_{t_j'}$ in terms of a complete system of orthonormal Dirichlet eigenfunctions $\{\psi_i(t_j')\}_{i=1}^{\infty}$ on $C_{t_j'}$ with eigenvalues $\{\mu_i(t_j')\}_{i=1}^{\infty}$,

$$\varphi_{i_0}(t_j')\eta=\sum_{i=1}^{\infty}b_i(t_j')\psi_i(t_j')$$

where

$$\sum_{i=1}^{\infty}b_i^2(t_j')=\int_{M_{t_j'}}|\varphi_{i_0}(t_j')\eta|^2,$$

$$\sum_{i=1}^{\infty}b_i^2(t_j')\mu_i(t_j')=\int_{M_{t_j'}}|\nabla\varphi_{i_0}(t_j')\eta|^2.$$

Now

$$\begin{aligned}\int_{M_{t_j'}}|\nabla\varphi_{i_0}(t_j')\eta|^2&\leqslant\int_{M_{t_j'}}|\nabla\varphi_{i_0}(t_j')|^2+\max|\nabla\eta|\int_{M_{t_j'}}|\varphi_{i_0}(t_j')|^2\\&\leqslant\lambda_{i_0}(t_j')+8\leqslant c_0\end{aligned}$$

for some constant $c_0<+\infty$ independent of t_j', using $\overline{\lim}_{t\to 0}\lambda_{i_0}(t)\leqslant\lambda_{i_0}(0)$. Thus, for any $N\in\mathbb{N}$

$$\begin{aligned}\mu_N(t_j')\sum_{i\geqslant N}^{\infty}b_i^2(t_j')&\leqslant\sum_{i\geqslant N}^{\infty}b_i^2(t_j')\mu_i(t_j')\\&\leqslant\sum_{i=1}^{\infty}b_i^2(t_j')\mu_i(t_j')\\&=\int_{M_{t_j'}}|\nabla\varphi_{i_0}(t_j')\eta|^2\leqslant c_0.\end{aligned}$$

Since $\lim_{t\to 0}\mu_N(t)=\mu_N(0)$ and $\lim_{N\to+\infty}\mu_N(0)=+\infty$, for any $\delta>0$ there exists an

$N_0 \in \mathbb{N}$ such that

$$\sum_{i \geqslant N_0}^{\infty} b_i^2(t_j') \leqslant \delta,$$

$$\sum_{i=1}^{N_0} b_i^2(t_j') \geqslant \int_{M_{t_j'}} |\varphi_{i_0}(t_j')\eta|^2 - \delta,$$

$$\int_{C_{t_j'}} \left| \varphi_{i_0}(t_j')\eta - \sum_{i=1}^{N_0} b_i^2(t_j')\psi_i(t_j') \right|^2 \leqslant \sum_{i > N_0}^{\infty} b_i^2(t_j') \leqslant \delta.$$

By further taking a subsequence we assume that, for $1 \leqslant i \leqslant N_0$, $\lim_{t_j' \to 0} b_i(t_j') = b_i(0)$. Then

$$\begin{aligned} \sum_{i=1}^{N_0} b_i^2(0) &\geqslant \overline{\lim_{t_j' \to 0}} \sum_{i=1}^{N_0} b_i^2(t_j') \\ &\geqslant \overline{\lim_{t_j' \to 0}} \int_{M_{t_j'}} |\varphi_{i_0}(t_j')\eta|^2 - \delta = m_0 - \delta. \end{aligned}$$

For any $1 > \varepsilon > 0$ and $0 \leqslant t \leqslant 1$, let $C_t(\varepsilon) = \{(x, y) \in C_t \,|\, |x| \geqslant \varepsilon\}$. We then have

$$\begin{aligned} \int_{C_0(\varepsilon)} |\varphi_{i_0}^* \eta|^2 &= \lim_{t_j' \to 0} \int_{C_{t_j'}(\varepsilon)} |\varphi_{i_0}(t_j')\eta|^2 \\ &\geqslant \underline{\lim}_{t_j' \to 0} \int_{C_{t_j'}(\varepsilon)} \left| \sum_{i=1}^{N_0} b_i(t_j')\psi_i(t_j') \right|^2 \\ &\quad - \overline{\lim_{t_j' \to 0}} \int_{C_{t_j'}} \left| \varphi_{i_0}(t_j')\eta - \sum_{i=1}^{N_0} b_i(t_j')\psi_i(t_j') \right|^2 \\ &\geqslant \underline{\lim}_{t_j' \to 0} \int_{C_{t_j'}(\varepsilon)} \left| \sum_{i=1}^{N_0} b_i(t_j')\psi_i(t_j') \right|^2 - \delta \\ &\geqslant \underline{\lim}_{t_j' \to 0} \int_{C_{t_j'}} \left| \sum_{i=1}^{N_0} b_i(t_j')\psi_i(t_j') \right|^2 - \delta \\ &\quad - \overline{\lim_{t_j' \to 0}} \int_{C_{t_j'} \setminus C_{t_j'}(\varepsilon)} \left| \sum_{i=1}^{N_0} b_i(t_j')\psi_i(t_j') \right|^2 - \delta \\ &\geqslant \underline{\lim}_{t_j' \to 0} \sum_{i=1}^{N_0} b_i^2(t_j') - N_0 \sum_{i=1}^{N_0} b_i^2(0)\, \overline{\lim_{t_j' \to 0}} \int_{C_{t_j'} \setminus C_{t_j'}(\varepsilon)} |\psi_i(t_j')|^2 - \delta. \end{aligned}$$

By Proposition 3.11, $\|\psi_i(t_j')\|_{L^2} = \|\psi_i(0)\|_{L^2} = 1$, and $\psi_i(t_j')$ converges to $\psi_i(0)$ uniformly over compact subsets of C_0; so

$$\lim_{\varepsilon\to 0} \overline{\lim_{t_j'\to 0}} \int_{C_{t_j'}\setminus C_{t_j'}(\varepsilon)} |\psi_i(t_j')|^2 = \lim_{\varepsilon\to 0} \int_{C_0\setminus C_0(\varepsilon)} |\psi_i(0)|^2 = 0.$$

Thus, for any $\delta > 0$ given above, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \leqslant \varepsilon_0$

$$N_0 \sum_{i=1}^{N_0} b_i^2(0) \overline{\lim_{t_j'\to 0}} \int_{C_{t_j'}\setminus C_{t_j'}(\varepsilon)} |\psi_i(t_j')|^2 \leqslant \delta.$$

And for any $\delta > 0$ and $\varepsilon \leqslant \varepsilon_0 < 1/4$

$$\int_{C_0} |\varphi_{i_0}^* \eta|^2 \geqslant \int_{C_0(\varepsilon)} |\varphi_{i_0}^* \eta|^2 \geqslant m_0 - 2\delta.$$

Since $\delta > 0$ is arbitrary,

$$\int_{C_0} |\varphi_{i_0}^* \eta|^2 = m_0.$$

This completes part (i) of the claim. For part (ii) we have for $\varepsilon \leqslant \varepsilon_0 < 1/4$

$$\begin{aligned}
\overline{\lim_{t_j'\to 0}} \int_{M_{t_j'}\setminus M_{t_j',\varepsilon}} |\varphi_{i_0}(t_j')|^2 &= \overline{\lim_{t_j'\to 0}} \int_{C_{t_j'}\setminus C_{t_j'}(\varepsilon)} |\varphi_{i_0}(t_j')\eta|^2 \\
&\leqslant \overline{\lim_{t_j'\to 0}} \int_{C_{t_j'}} |\varphi_{i_0}(t_j')\eta|^2 - \underline{\lim_{t_j'\to 0}} \int_{C_{t_j'}(\varepsilon)} |\varphi_{i_0}(t_j')\eta|^2 \\
&= \overline{\lim_{t_j'\to 0}} \int_{C_{t_j'}} |\varphi_{i_0}(t_j')\eta|^2 - \int_{C_0(\varepsilon)} |\varphi_{i_0}^* \eta|^2 \\
&\leqslant m_0 - (m_0 - 2\delta) = 2\delta.
\end{aligned}$$

Since $\delta > 0$ is arbitrary, part (ii) of the claim follows immediately, and the proof of the claim is complete.

We now use the claim to prove the orthonormality of the limit functions $\{\varphi_i^*\}_{i=1}^{\infty}$. Combining this with the equations preceding Lemma 4.1, we will have shown that the limit functions $\{\varphi_i^*\}_{i=1}^{\infty}$ are orthonormal eigenfunctions on M_0 with eigenvalues $\{\lambda_i^*\}_{i=1}^{\infty}$.

For any $i, k \geqslant 1$,

$$\begin{aligned}
|\langle \varphi_i^*, \varphi_k^* \rangle - \delta_{ik}| &= \lim_{\varepsilon \to 0} \left| \int_{M_{0,\varepsilon}} \varphi_i^* \varphi_k^* - \delta_{ik} \right| \\
&= \lim_{\varepsilon \to 0} \lim_{t_j' \to 0} \left| \int_{M_{t_j',\varepsilon}} \varphi_i(t_j')\varphi_k(t_j') - \delta_{ik} \right| \\
&\leqslant \lim_{\varepsilon \to 0} \lim_{t_j' \to 0} \left| \int_{M_{t_j'}} \varphi_i(t_j')\varphi_k(t_j') - \delta_{ik} \right| + \lim_{\varepsilon \to 0} \lim_{t_j' \to 0} \left| \int_{M_{t_j'} \setminus M_{t_j',\varepsilon}} \varphi_i(t_j')\varphi_k(t_j') \right| \\
&\leqslant \lim_{\varepsilon \to 0} \lim_{t_j' \to 0} \left(\int_{M_{t_j'} \setminus M_{t_j',\varepsilon}} |\varphi_i(t_j')|^2 \right)^{1/2} \left(\int_{M_{t_j'} \setminus M_{t_j',\varepsilon}} |\varphi_k(t_j')|^2 \right)^{1/2} \\
&= 0
\end{aligned}$$

where in the last equality we use part (ii) of the claim. This completes the proof of Lemma 4.1.

5. Proof of Theorem B. In this section and the next, we present examples of metrics defined on Riemann surfaces whose spectra converge by the results of the previous sections. As we have seen, the key estimate needed is a lower bound on the isoperimetric constant.

We shall consider a specific construction of an analytic family of Riemann surfaces $\pi: \mathcal{M} \to D$, where D is the unit disk in $\mathbb{C}$. This construction is standard, and we refer to [F1] for more details. Briefly, there are two cases to consider: (i) we start with two compact Riemann surfaces M_1, M_2 of genus $g_1 = g - j, g_2 = j$ respectively (we always take $j \leqslant g/2$, $g \geqslant 2$), and local coordinates z_1, z_2 centered at points $p_1 \in M_1$, $p_2 \in M_2$. For $t \in D \backslash \{0\}$ remove the disks $|z_i| < |t|$ and glue together the remaining surfaces by means of the identification $z_1 z_2 = t$. The resulting surfaces may be completed to form an analytic family $\pi: \mathcal{M} \to D$, where $M_t = \pi^{-1}(t)$ has genus g for $t \neq 0$ and $\pi^{-1}(0)$ is stable in the sense of Deligne-Mumford. Alternately, (ii) we could start with a single surface M of genus $g - 1 > 0$ and coordinates about two points $a, b \in M$ and a similar construction adds a handle to M. Then the fiber $\pi^{-1}(t)$ would have genus g for $t \neq 0$. In both cases (i) and (ii), we shall use the notation $M_0 = \pi^{-1}(0)$ and denote the identified double point (or "node") by p. The two types of degeneration are distinguished, as discussed in Section 2, by whether p separates the degenerate surface M_0.

We also introduce some notation: let $U_t = \{q \in M_1 \,|\, |z_1(q)| < |t|^{1/2}\}$ and suppose R is any region in M_1. Then there is a natural embedding $R \backslash U_t \to M_t$ under the identification described above. We shall denote the image $R \cap M_t$. This works as well for $R \subset M_2$ or $R \subset M$ in the nonseparating case. If R is, for example, an open submanifold of M_1 and $\bar{R} \subset M_1 \backslash \{p\}$, then for small $|t|$, R is embedded in M_t, and a metric ds_t^2 on M_t may be pulled back via this embedding and compared to a fixed metric on $R \subset M_1$. The estimates in the following two sections should be taken in this sense.

We now proceed to define the Bergman metric: let M be a compact Riemann surface of genus $g > 0$. Let $\omega_1, \ldots, \omega_g$ be a basis of abelian differentials, normalized with respect to the A-cycles of some symplectic homology basis for M, and denote by Ω_{ij} the associated period matrix.

Definition 5.1. The *Bergman metric* for M is defined by $ds^2 = \mu(z)|dz|^2$, where

$$\mu(z) = \frac{1}{g} \sum_{i,j=1}^{g} (\operatorname{Im} \Omega)_{ij}^{-1} \omega_i(z) \overline{\omega}_j(\bar{z}).$$

Remark 5.2. The Riemann surface M may be embedded into a g-dimensional complex torus $J(M)$, called the Jacobian variety on M. The metric μ is the one induced by this embedding from the natural Euclidean metric on $J(M)$. Since the scalar curvature of subvarieties decreases, we know that the scalar curvature of μ is nonpositive. (See [GH], p. 79.)

Now suppose we consider the Bergman metrics μ_t on the degenerating family $\mathcal{M}$ described above.

PROPOSITION 5.3 ([W], Lemmas 6.9 and 7.4).
(i) *For the degeneration* (*i*) *described above*

$$\mu_t \to \frac{g_i}{g} \mu_i$$

uniformly on compact subsets of $M_i \backslash \{p_i\}$, $i = 1, 2$, *where* μ_i *is the Bergman metric of* M_i. *Moreover, there is a constant* C *depending only on the family such that, in local coordinates about the node,*

$$\left| \mu_t(z) - \frac{g_i}{g} \mu_i(z) \right| \leqslant C|t|/|z|^2.$$

(ii) *For the degeneration* (*ii*) *described above*

$$\mu_t \to \frac{g-1}{g} \mu$$

uniformly on compact subsets of $M_0 \backslash \{p\}$. *Moreover, in local coordinates about the node,*

$$\left| \mu_t(z) - \frac{g-1}{g} \mu(z) - \frac{1}{-\log|t|} \frac{1}{2\pi g |z|^2} \right| \leqslant O\left(\frac{1}{-\log|t|}\right)$$

where the estimate is

$$\lim_{t\to 0}\sup_{|t|^{1/2}<|z|}(-|z|^2\log|t|)O\left(\frac{1}{-\log|t|}\right)=0.$$

Fix a geodesic disk C about the node p in M_0. Then for $t \neq 0$, $C_t = C \cap M_t$ is topologically a cylinder which contains the pinching region.

COROLLARY 5.4. *Let $\mathscr{M}$ be degenerating to a separating node, where M_t is equipped with the Bergman metric and C_t is as above. Then there exists a constant c depending only on $\mathscr{M}$ such that for all $t \in D\backslash\{0\}$*

$$\mathscr{I}(C_t) \geqslant c > 0.$$

Proof. By Remark 5.2 and Theorem 2.7, we may restrict our attention to homotopically nontrivial curves, rotationally symmetric as in Section 2. By Proposition 5.3, part (i), the error in estimating the lengths of such curves by the limiting metrics vanishes as $t \to 0$, and the corresponding subdomains clearly have finite area. Thus, $\mathscr{I}(C_t)$ may be bounded below by $\mathscr{I}(C_0)$ for the limiting metrics, which is clearly bounded away from zero.

Proof of Theorem B, part (i). The proof proceeds exactly as in Section 4, the crucial point being the bound of Corollary 5.4 and the discreteness of the spectrum for the limiting metric, which in this case is obvious. Note that by Remark 3.10 the limiting spectrum is indeed the spectrum for the closed problem on the disjoint union of M_1 and M_2 with a multiple of the Bergman metric.

As noted in the introduction, the pinching region for the nonseparating case becomes long and thin. This is easily seen from the result in part (ii) of Proposition 5.3. In order to prove part (ii) of Theorem B, we wish to compare the Bergman metric to one where the long, thin cylinder is actually flat. Set $l = \sqrt{-\log|t|}$ and construct a family of interpolating metrics $\tilde{\mu}_t$ satisfying

1. $\tilde{\mu}_t = \mu_t$ on the complement of the pinching annulus $\{z\,|\,|z| < l^{-1}\}$;
2. $\tilde{\mu}_t(z) = (-\log|t|)^{-1}\,|z|^{-2}$, for $|z| < \frac{1}{2}l^{-1}$;
3. $\sup_{(1/2)l^{-1}<|z|<l^{-1}}(\tilde{\mu}_t(z))$ is bounded independently of t.

Now choose $L > 0$, also independent of t, such that for $t \neq 0$

$$L^{-1}\tilde{\mu}_t \leqslant \mu_t \leqslant L\tilde{\mu}_t \tag{5.5}$$

on all of M_t. This is possible since by Proposition 5.3

$$0 < \inf_{|t|^{1/2}<|z|<l^{-1}}(-|z|^2\log|t|\mu_t(z)) \leqslant \sup_{|t|^{1/2}<|z|<l^{-1}}(-|z|^2\log|t|\mu_t(z)) < +\infty.$$

Let $\{\lambda_n(t)\}_{n=0}^{\infty}$ be the eigenvalues for μ_t and $\{\tilde{\lambda}_n(t)\}_{n=0}^{\infty}$ those for $\tilde{\mu}_t$. Then we have the following theorem.

THEOREM 5.6 (E. B. Davies, [D] Theorem 3). *Under the assumption equation* 5.5

$$L^{-4}\tilde{\lambda}_n(t) \leqslant \lambda_n(t) \leqslant L^4\tilde{\lambda}_n(t)$$

holds for all $n \geqslant 0$ *and* $t \neq 0$.

Proof of Theorem B, part (ii). By monotonicity it suffices to show that the Dirichlet and Neumann spectra of a subdomain become continuous as $t \to 0$ while the spectrum on the complement is controlled. By Theorem 5.6 we may equivalently consider the eigenvalue problem for $\tilde{\mu}_t$. But for $\tilde{\mu}_t$ the domain $|z| < \frac{1}{2}l^{-1}$ in local coordinates about the node is isometric to a flat cylinder of length $\sim l$ and circumference $2\pi l^{-1}$. The Dirichlet eigenvalues for the cylinder are

$$\lambda_{m,n} = \left(\frac{m\pi}{l}\right)^2 + (nl)^2, \qquad m = 1, 2, \ldots n = 0, 1, 2, \ldots$$

and the Neumann eigenvalues $\{\mu_{m,n}\}$ are the same, where we allow $m = 0$. As $l \to \infty$, $\{\lambda_{m,n}\}$ and $\{\mu_{m,n}\}$ clearly become dense on the entire interval $[0, +\infty)$. On the complement of the region $\{z \,|\, |z| < l^{-1}\}$, one can bound the isoperimetric constant away from zero, as in the proof of part (i), and by the results of Section 3, the Dirichlet and Neumann spectra converge. Finally, in the region $\{z \,|\, \frac{1}{2}l^{-1} < |z| < l^{-1}\}$ the annulus is collapsing to a circle. It is easy to see that Cheeger's constant diverges, and since it is a lower bound for the entire Dirichlet spectrum, the latter also diverges. The circumference of the annulus remains bounded away from zero. By decomposing into phases, we see that for small l there are only finitely many Neumann eigenvalues in any interval. This follows from the divergence of the Dirichlet spectrum, and the fact that, for each phase, Neumann eigenvalues can be bounded below by Dirichlet eigenvalues after shifting the index by two. (See [We].) Now the proof of part (ii) follows by monotonicity.

Remark 5.7. Heuristically, the fact that for small $|t|$ we have an embedded cylinder which is close to being flat means that in the limit we get the continuous spectrum of the real line, i.e., $[0, +\infty)$. This is in contrast to the case of the hyperbolic metric where this type of argument can be made rigorous; an embedded hyperbolic cylinder produces continuous spectrum only in the interval $[1/4, +\infty)$. (See [Ji].)

Remark 5.8. The long, thin cylinder may be understood geometrically—for the nonseparating case, the Jacobian variety $J(M_t)$ becomes a noncompact torus as $t \to 0$. Furthermore, from the embedding $M_t \to J(M_t)$, it can be seen (see [W]) that the pinching annulus wraps around that part of the torus which becomes unbounded. As $t \to 0$, we therefore produce a long, thin cylinder.

COROLLARY 5.9. *Let* $\lambda_1(t)$ *denote the first nonzero eigenvalue for the degenerating family* $\mathscr{M}$ *with Bergman metrics. Then* $\lambda_1 \to 0$ *for both the separating and nonseparating cases.*

Proof. For the nonseparating case this follows from monotonicity and the fact that infinitely many eigenvalues converge to zero for the long, thin cylinder. The limiting spectrum in the separating case is the union of the two spectra, and hence contains two zero eigenvalues. Since $\lambda_1(t)$ is the second eigenvalue in $\mathrm{Spec}(\Delta_t)$, $\lambda_1(t)$ must go to zero.

Remark 5.10. We may guess at how fast $\lambda_1 \to 0$ in the separating case. The method used suggests that $\lambda_1(t)$ should behave like the Dirichlet eigenvalue $\lambda_1(\varepsilon)$ for the complement of the set $\{q \| z(q)| < \varepsilon\}$ on one of the two surfaces M_i. By a result of Ozawa [O]

$$\lambda_1(\varepsilon) = -\frac{2\pi}{\mathrm{Area}(M_i)}(\log \varepsilon)^{-1} + O((\log \varepsilon)^{-2}).$$

Since we may take $\varepsilon \sim |t|^{1/2}$, we expect $\lambda_1(t)$ to be bounded above by a multiple of $(\log |t|)^{-1}$. Note that this is the behavior of λ_1 for the hyperbolic metric; however, in the nonseparating case λ_1 is bounded away from zero for the hyperbolic metric, in contrast to Corollary 5.9. (See [SWY].)

6. Proof of Theorem C. The Bergman metric of Section 5 degenerates to a smooth metric. In this section we study the admissible metrics introduced by Arakelov [A]; these degenerate to cone metrics. It will be convenient however to have a different description of cone metrics on surfaces. We have the following simple lemma.

LEMMA 6.1. *Let g be a metric on a two-dimensional manifold $M\backslash\{p\}$ such that, in local coordinates $\mathbf{x}$ centered at p, $g = \|\mathbf{x}\|^{-2\alpha}\mathbf{x}^* \, ds^2$, where ds^2 is the standard Euclidean metric and α is some number $0 \leqslant \alpha < 1$. Then g is a cone metric on M.*

Proof. Let (r, θ) be polar coordinates associated to $\mathbf{x}$, (τ, ϕ) coordinates on the cone $C(S^1_\alpha)$, where S^1_α is the circle of radius $1 - \alpha$ with the standard metric $\tilde{g}(\phi) = (1 - \alpha)^2 \, d\phi^2$. Consider the map

$$(r, \theta) \mapsto \left(\frac{r^{1-\alpha}}{1 - \alpha}, \theta\right).$$

This defines a smooth diffeomorphism from a deleted neighborhood of p to $C(S^1)$ and the standard metric $ds^2_C = d\tau^2 + \tau^2\tilde{g}(\phi)$ pulls back to g. Hence, by Definition 2.1, g is a cone metric.

As in Section 5, we fix a compact Riemann surface M of genus $g > 0$ and let μ denote the Bergman metric.

Definition 6.2. The *Arakelov-Green's function* on M, denoted $G(z, w)$, is characterized by the following

(i) $G(z, w)$ has a zero of order one on the diagonal in $M \times M$;
(ii) $G(z, w) = G(w, z)$;

(iii) for $z \neq w$, $\partial_z \partial_{\bar{z}} \log G(z, w) = -\frac{\pi}{2}\mu(z)$;
(iv) $\int_M \log G(z, w)\mu(z)|dz|^2 = 0$.

Definition 6.3. A metric $ds^2 = \rho(z)|dz|^2$ on M is called *admissible* if its Ricci form is proportional to the Kähler form of the Bergman metric; i.e.,

$$\partial_z \partial_{\bar{z}} \log \rho(z) = 2\pi(g-1)\mu(z).$$

Remark 6.4. The exact multiple follows from Gauss-Bonnet and the fact that $\int_M \mu(z)|dz|^2 = 1$. Note that an admissible metric always has negative curvature for $g \geqslant 2$ and that any two admissible metrics are proportional. For tori the metric is flat.

Definition 6.5. (i) Set

$$\rho(z) = \lim_{w \to z} \frac{G(z, w)^2}{|z - w|^2}.$$

Then ρ is an admissible metric and defines the *Arakelov metric* (see [A]).

(ii) The *normalized admissible metric* $\hat{\rho}$ is the admissible metric with unit area. By Remark 6.4, $\hat{\rho}(z) = \rho(z)/\text{Area}(M, \rho)$, where ρ is the Arakelov metric.

Now let $\mathscr{M}$ be an analytic family as described in Section 5. The Arakelov metrics ρ_t form a smooth family for $t \neq 0$, and their behavior as $t \to 0$ has been studied in [W]. We are interested in $\text{Spec}(\Delta_{\hat{\rho}_t})$.

PROPOSITION 6.6.

(i) *Let $\mathscr{M}$ be degenerating to a separating node with $j < g/2$. (see the beginning of Section 5 for notation.) Then*

$$\hat{\rho}_t(z) \to \rho_1(z) G_1(z, p_1)^{-4j/g}$$

uniformly on compact subsets of $M_1 \backslash \{p\}$. Here, ρ_1 is an admissible metric for M_1, and G_1 is the Arakelov-Green's function for M_1. Moreover, $\hat{\rho}_t$ vanishes to order $|t|^{2(1-2j/g)}$ uniformly on compact subsets of $M_2 \backslash \{p_2\}$.

(ii) *Let $\mathscr{M}$ be degenerating to a nonseparating node. Then*

$$\hat{\rho}_t(z) \to \rho(z)(G(z, a)G(z, b))^{-2/g}$$

uniformly on compact subsets of $M_0 \backslash \{p\}$. As above, ρ is an admissible metric for M, and a and b are as in Section 5.

Remark 6.7. The form of the limiting metrics (note that they are quasi-isometric to cone metrics by property (i) in Def. 6.2, Lemma 6.1, and the assumption on j) follows from the results in [W]. However, the asymptotic behavior for the Arakelov metric ρ_t studied there only gave pointwise results away from the node. To determine the limiting behavior of $\hat{\rho}_t = \rho_t/\text{Area}(\rho_t)$, we need estimates on the area as well.

Remark 6.8. The limiting metrics are "admissible cone metrics"; that is, their curvature is a multiple of the Bergman metric. In the cases considered above, the limiting metrics have nonpositive curvature bounded from below. (The curvature is negative if the limiting surfaces are not tori.)

To obtain the proposition we must control the behavior of the metric in the pinching region better than in [W]. To do this, we use Fay's expression for admissible metrics; let $\vartheta[f]$ denote the theta function with characteristic f associated to $J(M)$. Choose f to be an odd, nonsingular element of the theta divisor in $J(M)$.

PROPOSITION 6.9. *Let* $\rho(z) = |H_f(z)|^2\Psi(z)$, *where*

$$H_f(z) = \sum_{j=1}^{g} \frac{\partial}{\partial Z_j} \vartheta[f](0)\omega_j(z),$$

$$\Psi(z) = \exp\left\{\frac{4}{g}(g-1)\pi \sum_{i,j=1}^{g} (\operatorname{Im}\Omega)_{ij}^{-1} \operatorname{Im}\left(\int_{z_0}^{z} \omega_i - k_i\right) \operatorname{Im}\left(\int_{z_0}^{z} \omega_j - k_j\right) - \frac{2}{g}\sum_{j=1}^{g} \operatorname{Re}\int_{A_j} \omega_j(\xi) \log \frac{\vartheta[f](\int_\xi^z \vec{\omega})^2}{H_f(\xi)}\, d\xi\right\},$$

z_0 *is an arbitrary point of* M, *and* k *is a point in* $J(M)$ *depending upon* z_0. *Then* $ds^2 = \rho(z)\,|dz|^2$ *is an admissible metric.*

Sketch of proof. (See [F2] for details.) The zeros of $H_f(z)$, all of multiplicity two, coincide with the zeros of $\vartheta[f](\int_\xi^z \vec{\omega})$; so $\rho(z)|dz|^2$ is nonsingular. The factors of automorphy of ϑ cancel those for the first term in the exponential; so $\Psi(z)$ is indeed a single-valued function on M depending, however, on the choice of homology basis. Now by a simple computation

$$\partial_z\partial_{\bar{z}} \log \rho(z) = \partial_z\partial_{\bar{z}}\left\{\frac{4}{g}(g-1)\pi \sum_{i,j=1}^{g} (\operatorname{Im}\Omega)_{ij}^{-1} \operatorname{Im}\left(\int_{z_0}^{z} \omega_i - k_i\right) \operatorname{Im}\left(\int_{z_0}^{z} \omega_j - k_j\right)\right\}$$

$$= 2\pi(g-1)\mu(z).$$

Using this expression, we shall prove Proposition 6.6. For brevity we shall only prove part (i); part (ii) follows similarly.

LEMMA 6.10. *Let* $\mathcal{M}$ *be a degenerating family as in part* (i) *of Proposition* 6.6. *Then we may choose* f_t *analytic in* t *such that*

$$H_{f_t}(z) \to \alpha H_{f_0}(z)$$

uniformly on compact subsets of $M_1\backslash\{p_1\}$, *where* α *is some constant and* f_0 *is a nonsingular odd element of the theta divisor in* $J(M_1)$. *Moreover, if* G_2 *denotes the*

Arakelov-Green's function on M_2, *then* $|t|^{-1}G_2(z, p_2)^2 H_{f_t}$ *is uniformly bounded for* $t \in D\backslash\{0\}$ *and* $z \in M_2 \cap M_t$.

Proof. This is a simple consequence of the degeneration formulas in [F1]. The Jacobian variety degenerates to a product torus, and the theta divisor over the zero fiber

$$\Theta_0 = \Theta_1 \times J(M_1) \cup J(M_2) \times \Theta_2 .$$

(See [W].) Choose f_t such that $\lim_{t\to 0} f_t$ is in $\Theta_1 \times J(M_2)$ with f_0 in the first factor. Then for $i \leqslant g - j$

$$\frac{\partial}{\partial Z_i}\vartheta(f_t) \to \vartheta_2 \cdot \frac{\partial}{\partial Z_i}\vartheta_1(f_0)$$

and vanishes otherwise. Since the normalized abelian differentials are chosen such that $\omega_i(z, t)$ converges uniformly away from the node to the abelian differentials of the compact surface, and $\omega_i(z, t) \to 0$ for $i \leqslant g - j$ and $z \in M_2\backslash\{p_2\}$, we have the first part of the lemma with $\alpha = \vartheta_2$ evaluated on the second factor of $\lim_{t\to 0} f_t$. The second part follows from the fact that $H_{f_t} \to 0$ to order t on compact subsets of $M_2\backslash\{p_2\}$ and that near the node $H_{f_t}(z) \sim t\,dz/z^2$. (See Appendix A of [W].) Since $G_2(z, p_2)^2 \sim |z|^2$ near the node, the result follows.

LEMMA 6.11. *Let R be any region in* $M_1\backslash\{p\}$ *and* γ_t *any smooth family of curves in* $M_1 \cap M_t$. *Then there exists a positive constant C independent of t such that*

$$\text{(i)} \qquad \int_{R\cap M_t} |H_{f_t}(z) - \alpha H_{f_0}(z)|^2 \, |dz|^2 \leqslant -C|t| \log |t|,$$

$$\text{(ii)} \qquad \int_{\gamma_t} |H_{f_t}(z) - \alpha H_{f_0}(z)| \, |dz| \leqslant -C|t|^{1/2} \log|t| .$$

Proof. See [W], Propositions A.1 and A.4.

LEMMA 6.12. *Given* f_t *as in Lemma* 6.10, *there exists a bounded function* Ψ_0 *on* $M_0\backslash\{p\}$ *such that*

$$\Psi_t(z)|t|^{-2j/g} G_1(z, p_1)^{4j/g} \to \Psi_0(z) \qquad \textit{uniformly for } z \in M_1 \cap M_t;$$

$$\Psi_t(z)|t|^{2j/g} G_2(z, p_2)^{-4j/g} \to \Psi_0(z) \qquad \textit{uniformly for } z \in M_2 \cap M_t .$$

Moreover, $\Psi_0(z)|H_{f_0}(z)|^2$ *is an admissible metric on* M_1.

Proof. This may be proven by applying the degeneration formulas in [F1] to the explicit expression. We shall not go through the details since the answer was essentially obtained in [W]. Let us note only that uniformity on *all* of M_0 follows from the fact that the z dependence of Ψ_t is in terms of abelian integrals, and we may again apply Proposition A.1 of [W] to see that the limits are uniform.

Proof of Proposition 6.6, *part* (i). Let $C(t) = |t|^{-2j/g}$ and set

$$\rho_t(z) = C(t)\Psi_t(z)|H_{f_t}(z)|^2.$$

By Proposition 6.9, ρ_t is a smooth family of admissible metrics for $t \neq 0$. Furthermore,

$$\begin{aligned}\operatorname{Area}(M_t, \rho_t) = A_t &= \int_{M_t} C(t)\Psi_t(z)|H_{f_t}(z)|^2\,|dz|^2 \\ &= \int_{M_1\cap M_t}\cdots + \int_{M_2\cap M_t}\cdots.\end{aligned}$$

Treating first the second term,

$$\begin{aligned}\int_{M_2\cap M_t} C(t)\Psi_t|H_{f_t}|^2\,|dz|^2 &= \int_{M_2\cap M_t} [|t|^{2j/g}G_2(z, p_2)^{-4j/g}\Psi_t][|t|^{-2}G_2(z, p_2)^4|H_{f_t}|^2] \\ &\quad \times |t|^{2(1-2j/g)}G_2(z, p_2)^{-4(1-j/g)}|dz|^2.\end{aligned}$$

By Lemmas 6.10 and 6.12 the integrand is dominated (in local coordinates) by a multiple of $|t|^{2(1-2j/g)}G_2(z, p_2)^{-4(1-j/g)}$. For z near p_2, we estimate

$$\begin{aligned}\int_{|t|^{1/2}<|z|<1} |t|^{2(1-2j/g)}G_2(z, p_2)^{-4(1-j/g)} &\leqslant \text{const.}\,|t|^{2(1-2j/g)}\int_{|t|^{1/2}}^{1} dr r^{-3+4j/g} \\ &\leqslant \text{const.}\,|t|^{1-2j/g}.\end{aligned}$$

Since we assume $j/g < 1/2$, we conclude from the above that

$$\overline{\lim_{t\to 0}}\int_{M_2\cap M_t} C(t)\Psi_t|H_{f_t}|^2\,|dz|^2 = 0.$$

For the first term let $\hat{\Psi}_0(z) = \Psi_0(z)G_1(z, p_1)^{-4j/g}$. Then

$$\begin{aligned}\int_{M_1\cap M_t} C(t)\Psi_t|H_{f_t}|^2\,|dz|^2 &= \alpha^2\int_{M_1\cap M_t} \hat{\Psi}_0|H_{f_0}|^2\,|dz|^2 \qquad (6.13)\\ &\quad + \int_{M_1\cap M_t} \{C(t)\Psi_t|H_{f_t}|^2 - \alpha^2\hat{\Psi}_0|H_{f_0}(z)|^2\}|dz|^2.\end{aligned}$$

By uniform convergence, for any $\delta > 0$ the second term in equation 6.13 may be bounded by

$$\int_{M_1 \cap M_t} |(C(t)\Psi_t - \hat{\Psi}_0)|H_{f_t}|^2 + \hat{\Psi}_0(|H_{f_t}|^2 - \alpha^2|H_{f_0}|^2)||dz|^2$$

$$\leqslant \delta \int_{M_1 \cap M_t} \alpha^2 |H_{f_0}|^2 |dz|^2 + \delta \int_{M_1 \cap M_t} |H_{f_t} - \alpha H_{f_0}|^2 |dz|^2$$

$$+ \sup_{z \in M_1 \cap M_t} (\hat{\Psi}_0(z)) \int_{M_1 \cap M_t} |H_{f_t} - \alpha H_{f_0}|^2 |dz|^2$$

for sufficiently small $|t|$. The last two terms $\to 0$ as $t \to 0$ by Lemma 6.11 and the assumption $j < g/2$, and since δ was arbitrary, we conclude that the second term on the right-hand side of equation 6.13 vanishes as $t \to 0$. We have shown

$$\overline{\lim_{t \to 0}} \left\{ A_t - \alpha^2 \int_{M_1 \cap M_t} \hat{\Psi}_0 |H_{f_0}|^2 |dz|^2 \right\} = 0 .$$

Hence, the normalized admissible metric $\hat{\rho}_t = \rho_t / A_t$ converges as in Proposition 6.6, part (i), completing the proof.

PROPOSITION 6.14. *Let $\mathcal{M}$ be as in Proposition 6.6. Then there exists a constant c depending only on $\mathcal{M}$ such that for all $t \in D \backslash \{0\}$*

$$\mathcal{I}(M_t) \geqslant c > 0 .$$

Remark 6.15. Note that we make no assumption on $\mathcal{M}$. In particular, $\mathcal{I}(M_t)$ is bounded away from zero even in the separating case. (Compare with Remark 3.12.) The reason for this is that by Proposition 6.6 one entire side of the degenerating surface is collapsing, and a separating curve in the pinching annulus has squared length comparable to the area of this collapsing piece. As in Section 5, we also note that Proposition 6.6 and 6.14, combined with the arguments in Section 4 immediately prove Theorem C, part (i).

Proof of Proposition 6.14. Again, we only consider the separating case. Let γ_t be a smooth family of closed curves in M_t. Suppose that $\gamma_t \subset M_1 \cap M_t$. Then we estimate

$$|L(\gamma_t, \hat{\rho}_t) - L(\gamma_t, \rho_0)| = \int_0^1 ds |\dot{\gamma}_t(s)| |\sqrt{\hat{\rho}_t} - \sqrt{\rho_0}|$$

$$= \int_0^1 ds |\dot{\gamma}_t| | |H_{f_t}| (C(t)\Psi_t)^{1/2} - \alpha |H_{f_0}| (\hat{\Psi}_0)^{1/2} |$$

$$\leqslant \int_0^1 ds|\dot{\gamma}_t|\{\alpha|H_{f_0}||(C(t)\Psi_t)^{1/2} - (\hat{\Psi}_0)^{1/2}|$$

$$+ |H_{f_t} - \alpha H_{f_0}|(C(t)\Psi_t)^{1/2}\}.$$

The first term $\to 0$ as $t \to 0$ by the uniform convergence of $C(t)\Psi_t$. The second term is bounded by a mutiple of $-|t|^{1/2-j/g}\log|t|$ by Lemmas 6.11 and 6.12. Area estimates follow as in the proof of Proposition 6.6 above. Thus, restriciting γ_t in M_1, the isoperimetric quotient may be bounded by that of the metric ρ_0, which by the general arguments of Section 2 is bounded away from zero. Notice that, by Remark 6.4 and Theorem 2.7, we restrict to homotopically nontrivial loops. For γ_t restricted to a compact subset of $M_2\backslash\{p_2\}$, the metric $\hat{\rho}_t$ may be scaled by $|t|^{-2(1-2j/g)}$ to converge to a smooth metric whose isoperimetric constant is bounded away from zero, and hence the same for $\hat{\rho}_t$ by the scale invariance of $\mathscr{I}(M)$. The last case to consider is when γ_t is in the pinching annulus in M_2. Again, by Remark 6.4 and the arguments of Section 2, we may restrict ourselves to rotationally symmetric curves. One has from Lemmas 6.10 and 6.12 that, in $M_2 \cap M_t$, $\hat{\rho}_t$ is uniformly quasi-isometric to $|t|^{2(1-2j/g)}|z|^{-4(1-j/g)}$, where z is the local coordinate about the node. It is easy to see that the isoperimetric quotient is bounded away from zero for this metric as well. This completes the proof of Proposition 6.14.

It remains to prove Theorem C in the case of degeneration to a separating node where both surfaces have the same genus $j = g/2$.

LEMMA 6.16. *Let $\mathscr{M}$ be degenerating to a separating node where the surfaces M_1, M_2 both have genus $g/2$. Let ρ_t denote the Arakelov metric on M_t. Then*

$$|t|^{-1/2}\rho_t(z) \to \rho_i(z)(G_i(z, p_i))^{-2}$$

uniformly on compact subsets of $M_i\backslash\{p_i\}$. Here, ρ_i is the Arakelov metric of M_i.

Proof. This is just equation 8.1 of [W].

LEMMA 6.17. *Let $A_t = \mathrm{Area}(M_t, \rho_t)$. Then*

$$A_t = O(|t|^{1/2}\log|t|).$$

Proof. That this naive guess is correct follows from the explicit expression, Proposition 6.9, and Lemmas 6.10, 6.11 and 6.12. (The proofs of these did not depend on $j < g/2$.) Note especially the "extra" factor of $|t|^{1/2}$ in Lemma 6.11, part (i).

Proof of Theorem C, part (ii). Fix a $\delta > 0$. Then for small $|t|$ we construct a family of interpolating metrics $\tilde{\mu}_t$ as in Section 5, satisfying

1. $\tilde{\mu}_t = \hat{\rho}_t$ on the complement of the pinching annulus $\{z||z| < \delta\}$ in local coordinates about the node;
2. $\tilde{\mu}_t(z) = (-\log|t|)^{-1}|z|^{-2}$ for $|z| < \delta/2$;
3. $\sup_{\delta/2<|z|<\delta}(-\log|t|\tilde{\mu}_t(z))$ is bounded independently of t.

Then by Lemmas 6.16 and 6.17 (and their proofs) we can find an $L > 0$, independent of t, such that for $t \neq 0$

$$L^{-1}\tilde{\mu}_t \leqslant \hat{\rho}_t \leqslant L\tilde{\mu}_t$$

on all of M_t. By Theorem 5.6, $\mathrm{Spec}(\Delta_{\hat{\rho}_t})$ is bounded above and below by $\mathrm{Spec}(\Delta_{\tilde{\mu}_t})$. The region $\{z \mid |z| < \delta/2\}$ is a flat cylinder with respect to $\tilde{\mu}_t$, and its spectrum becomes dense in $[0, +\infty)$ as $t \to 0$. (See Section 5.) On the complement of the region, we can, by assumption 1 above and Lemma 6.16, rescale by a factor of $-\log|t|$ to obtain a smoothly converging metric with converging spectrum. Hence, the Dirichlet and Neumann spectra for $\tilde{\mu}_t$ diverge on this piece. The proof now follows from monotonicity.

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