Error control of nonlinear evolution equations

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Abstract

We derive *a posteriori* error estimates for backward Euler approximations of evolution inequalities in Hilbert spaces. The underlying nonlinear (multivalued) monotone operator is subdifferential, or more generally angle-boundedy. The estimates depend solely on the discrete solution and data, impose no constraints between consecutive time-steps, exhibit explicit stability factors, and are optimal with respect to both order and regularity.

suboper **1. Subdifferential Operators.** Let $\mathcal{H}$ be a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Let $\phi : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous convex function with (not empty, convex) proper domain $D(\phi)$. Let $\mathfrak{F} = \partial\phi : \mathcal{H} \to 2^{\mathcal{H}}$ be the subdifferential of ϕ with proper domain $D(\mathfrak{F}) \subset D(\phi)$: $\mathfrak{F}$ is characterized in terms of ϕ by the inequalities

$$\sigma(w; v) := \inf_{w_* \in \mathfrak{F}(w)} \big(\phi(v) - \phi(w) - \langle w_*, v - w \rangle\big) \geq 0 \quad \forall\, w \in D(\mathfrak{F}),\ v \in D(\phi). \tag{1}$$ define_coerc

Given $u_0 \in \overline{D(\phi)}$ and $f \in L^1(0, T; \mathcal{H})$, we set $u(0) = u_0$ and consider the differential inclusion

$$u'(t) + \mathfrak{F}(u(t)) \ni f \quad \text{a.e. } t \in (0, T). \tag{2}$$ subCP

This is equivalent to the evolution variational inequality with coercivity term σ

$$\langle u'(t) - f(t), u(t) - v \rangle + \phi(u(t)) - \phi(v) + \sigma(u(t); v) \leq 0 \quad \forall\, v \in D(\phi). \tag{3}$$ cont_stronger

\df@label Let $\mathcal{P} = \{t_n\}_{n=0}^N$ be a partition of $[0, T]$ with variable step-size $\tau_n := t_n - t_{n-1}$, and set $\delta Z_n := \tau_n^{-1}(Z_n - Z_{n-1})$ for any $\{Z_n\}_{n=0}^N \subset \mathcal{H}$. Let $Z(t)$ denote the piecewise linear interpolant of $\{Z_n\}_{n=0}^N$ and $\bar{Z}(t)$ its piecewise constant interpolant. Given $U_0 \in D(\phi)$ and $\{F_n\}_{n=1}^N$ approximations of u_0 and f, we define $\{U_n\}_{n=1}^N$ to be the solution of the inclusion (backward Euler method)

$$\delta U_n + \mathfrak{F}(U_n) \ni F_n \quad \forall\, 1 \leq n \leq N, \tag{4}$$ VId

or equivalently of the strong form of a variational inequality including σ

$$\langle \delta U_n - F_n, U_n - v \rangle + \phi(U_n) - \phi(v) + \sigma(U_n; v) \leq 0 \quad \forall\, v \in D(\phi). \tag{5}$$ dis_stronger

Our aim is to derive *a posteriori* estimates for the error $|u(T) - U_N|$ which depend solely on U_n for $n \leq N$ and data, are optimal with respect to both order and regularity of u, require minimal regularity of $\mathfrak{F}$ (or ϕ), exhibit explicit stability constants, and impose no constraints between consecutive time-steps. To this end, we introduce a new approach based on evaluating the amount of energy dissipation $\mathcal{E}_n \geq 0$:

$$\mathcal{E}_n := \langle F_n - \delta U_n, \delta U_n \rangle - \delta\phi(U_n) \quad \forall\, 1 \leq n \leq N. \tag{6}$$ ene_diss

This quantity is a measure of the deviation of $\{U_n\}_{n=1}^N$ from satisfaction of the conservation form $|u'(t)|^2 + \frac{d}{dt}\phi(u(t)) = \langle f(t), u'(t)\rangle$. Duality techniques as in [2],[3] either fail in this context or yield suboptimal results: the adjoint linearized equation may exhibit no regularizing effect or, even worse, may make no sense if a differential of $\mathfrak{F}$ does not exist or is singular.

We first introduce the error notation

$$E(u,U;T) = \max\left(\max_{t\in[0,T]} |u(t)-U(t)|, 2\int_0^T \left(\sigma(u(t);U(t)) + \sigma(\bar{U}(t);u(t))\right) dt\right), \tag{7}$$

and state the following fundamental *a posteriori* error estimate.

main_uno **Theorem 1** $E(u,U;T) \le \left(|u_0-U_0|^2 + \sum_{n=1}^N \tau_n^2\mathcal{E}_n\right)^{1/2} + \|f-\bar{F}\|_{L^1(0,T;\mathcal{H})}$.

We provide a simplified proof with the essential ideas in §2. We now discuss several consequences of Theorem 1. If $\mathfrak{F}$ is *single-valued*, then we set

$$\varrho(v,w) = \langle \mathfrak{F}(v) - \mathfrak{F}(w), v-w\rangle \quad \forall\, v,w \in D(\mathfrak{F}), \tag{8}$$ Coerc

and note that $\tau_n\mathcal{E}_n = \sigma(U_n;U_{n-1}) \le \varrho(U_n,U_{n-1})$ for $n\ge 1$ provided $U_0 \in D(\mathfrak{F})$. We then obtain

$$E(u,U;T) \le \left(|u_0-U_0|^2 + \sum_{n=1}^N \tau_n\varrho(U_n,U_{n-1})\right)^{1/2} + \|f-\bar{F}\|_{L^1(0,T;\mathcal{H})}. \tag{9}$$ v_o

The inclusion (4) suggests a canonical way to replace $\mathfrak{F}(U_n)$ in (8) in the multivalued case, namely $F_n - \delta U_n \in \mathfrak{F}(U_n)$. If $U_0 \in D(\mathfrak{F})$ we define $U_{-1} := U_0 + \tau_0(\mathfrak{F}(U_0))^\circ$ and consider the *residual monotone* term

$$\mathcal{D}_n := \tau_n\langle \delta F_n - \delta^2 U_n, \delta U_n\rangle \quad \forall\, 1\le n\le N. \tag{10}$$ resdn

It is easy to check that $\mathcal{E}_n \le \mathcal{D}_n$. Therefore, this yields the following variant of Theorem 1

$$E(u,U;T) \le \left(|u_0-U_0|^2 + \sum_{n=1}^N \tau_n^2\mathcal{D}_n\right)^{1/2} + \|f-\bar{F}\|_{L^1(0,T;\mathcal{H})}. \tag{11}$$ est_uno_due

The error exhibit also an exponential decay provided $\mathfrak{F}$ is coercive with respect to $|\cdot|$.

main_due_coercH **Theorem 2** *Let $U_0 \in D(\phi)$ and $\mathfrak{F}$ satisfy $\varrho(w,v) \ge 2\alpha|w-v|^2$ for $w,v\in D(\mathfrak{F})$ with $\alpha>0$. Then*

$$|u(T)-U(T)| \le \left(\mathrm{e}^{-2\alpha T}|u_0-U_0|^2 + \sum_{n=1}^N \mathrm{e}^{-2\alpha(T-t_n)}\tau_n^2\mathcal{E}_n\right)^{1/2} + \int_0^T \mathrm{e}^{-\alpha(T-t)}|f(t)-\bar{F}(t)|\,dt.$$

The above estimates possess an optimal *a priori* rate of convergence depending on the regularity of $U_0 = u_0$ and require *minimal* regularity of $\mathfrak{F}$. Let τ be the largest time-step.

sqrt_tau_c **Theorem 3** *Let $U_0\in D(\phi)$ and $\phi\ge 0$. Then, $\mathcal{E}_n$ satisfies $\sum_{n=1}^N \tau_n\mathcal{E}_n \le \phi(U_0) + \frac14\|\bar{F}\|^2_{L^2(0,T;\mathcal{H})}$. If also $U_0 = u_0$, $f\in L^2(0,T;\mathcal{H})$, and $F_n := \tau_n^{-1}\int_{t_{n-1}}^{t_n} f(t)\,dt$ for $1\le n\le N$ and $F_0 := 0$, then*

$$E(u,U;T) \le \left(3\tau\left(\phi(u_0) + \|f\|^2_{L^2(0,T;\mathcal{H})}\right)\right)^{1/2}.$$

main_due **Theorem 4** *If $U_0\in D(\mathfrak{F})$, then $\mathcal{D}_n$ satisfies $\sum_{n=1}^N \mathcal{D}_n \le \left(\frac{1}{\sqrt{2}}|\delta U_0| + \mathrm{Var}\,\bar{F}\right)^2$. Moreover $U_0 := u_0$, $f\in BV(0,T;\mathcal{H})$ and $F_n := f(t_n^+)$ imply*

$$E(u,U;T) \le \tau\left(\tfrac{1}{\sqrt{2}}\left|(f(0^+) - \mathfrak{F}(u_0))^\circ\right| + 2\,\mathrm{Var}\, f\right). \tag{12}$$ c_tre

2. Basic Estimates. We derive Theorem 1 under the simplifying assumptions $f = 0$ and $\sigma(w;v) = 0$. We first rewrite (5) to obtain the *continuous* evolution variational inequality satisfied by U for all $t \in (t_{n-1}, t_n)$

$$\langle U'(t), U(t) - v\rangle + \phi(U(t)) - \phi(v) \le \mathcal{R}(t) \quad \forall\, v \in \mathcal{H}, \tag{13}$$

where $\mathcal{R}(t)$ is a residual term, independent of the test function v, given by

$$\mathcal{R}(t) := \langle U'(t), U(t) - U_n\rangle + \phi(U(t)) - \phi(U_n).$$

The second step consists in estimating $\mathcal{R}(t)$. Here we use the elementary identity

$$U(t) - U_n = \frac{t - t_n}{\tau_n}\,(U_n - U_{n-1}) = (t - t_n)U'(t),$$

and the convexity of ϕ to get the bound

$$\phi(U(t)) \le \frac{t_n - t}{\tau_n}\,\phi(U_{n-1}) + \frac{t - t_{n-1}}{\tau_n}\,\phi(U_n).$$

Hence, in view of (6), $\mathcal{R}(t)$ can be readily bounded by $\mathcal{R}(t) \le (t_n - t)\mathcal{E}_n$. Finally, choosing $v = U(t)$ in (3), $v = u(t)$ in (13), and adding the two inequalities, we see that the terms involving ϕ cancel out and we end up with the error inequality

$$\tfrac{1}{2}\tfrac{d}{dt}|u(t) - U(t)|^2 \le (t_n - t)\mathcal{E}_n \quad \text{a.e. } t \in (t_{n-1}, t_n).$$

Upon integration in time, this upper bound immediately yields Theorem 1.

We prove now Theorems 3 and 4, namely *a priori* upper bounds for the above estimators. A simple first result is directly suggested by (6), which implies

$$0 \le \tau_n \mathcal{E}_n \le \phi(U_{n-1}) - \phi(U_n).$$

Therefore, we end up with a telescopic sum

$$\sum_{n=1}^{N} \tau_n^2 \mathcal{E}_n \le \tau \sum_{n=1}^{N} \big(\phi(U_{n-1}) - \phi(U_n)\big) = \tau\big(\phi(U_0) - \phi(U_N)\big),$$

whence

$$\max_{t\in[0,T]} |u(t) - U(t)|^2 \le \phi(U_0)\,\tau + |u_0 - U_0|^2.$$

We then obtain the rate of convergence $O(\sqrt{\tau})$ asserted in Theorem 3 provided $U_0 = u_0 \in D(\phi)$. On the other hand, since $\mathcal{D}_n = \langle \delta U_{n-1} - \delta U_n, \delta U_n\rangle$, making use of the identity $2\langle v - w, v\rangle = |v|^2 - |w|^2 + |v - w|^2$ for $v, w \in \mathcal{H}$, we see that the sum on the right-hand side of (11) telescopes

$$2\sum_{n=1}^{N} \tau_n^2 \mathcal{D}_n \le \tau^2 \sum_{n=1}^{N} \big(|\delta U_{n-1}|^2 - |\delta U_n|^2\big) \le \tau^2 |\delta U_0|^2 = \tau^2 |\mathfrak{F}(U_0)^\circ|^2.$$

The following estimate thus holds

$$\max_{t\in[0,T]} |u(t) - U(t)|^2 \le \tfrac{1}{2}|\mathfrak{F}(U_0)|^2 \tau^2 + |u_0 - U_0|^2.$$

This implies the a priori linear rate of convergence asserted in Theorem 4 provided $U_0 = u_0 \in D(\mathfrak{F})$, and so that proved by Rulla [4] for constant time steps. We stress that a linear rate cannot be expected for general monotone operators [4].

angle

3. Angle-Bounded Operators. The intriguing question whether there is a larger class of monotone operators than merely subdifferentials possessing both optimal a posteriori and a priori error bounds is elucidated here. To motivate the discussion, we take $f=0$ and proceed formally to write (4) as $U' + \mathfrak{F}(U_n) = 0$ for $t_{n-1} < t < t_n$ and a suitable selection in $\mathfrak{F}(U_n)$. We next subtract this equation from (2), multiply by $u-U$, and use the definition of U to obtain

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}|u-U|^2 &\le \langle \mathfrak{F}(U_n)-\mathfrak{F}(u), u-U\rangle \\ &\le \tau_n^{-1}(t-t_{n-1})\langle \mathfrak{F}(U_n)-\mathfrak{F}(u), u-U_n\rangle + \tau_n^{-1}(t_n-t)\langle \mathfrak{F}(U_n)-\mathfrak{F}(u), u-U_{n-1}\rangle.\end{aligned}$$

We are thus led to investigate the residual term $\langle \mathfrak{F}(U_n)-\mathfrak{F}(u), u-U_{n-1}\rangle$.

The (multi-valued) operator $\mathfrak{F}$ is γ*-angle-bounded* if there exists $\gamma>0$ such that

$$\langle \mathfrak{F}(v)-\mathfrak{F}(w), w-z\rangle \le \gamma^2 \langle \mathfrak{F}(v)-\mathfrak{F}(z), v-z\rangle \quad \forall\, v,w,z\in D(\mathfrak{F}), \tag{14}$$

angle_boundedness_formula

for every selection in $\mathfrak{F}(v), \mathfrak{F}(w), \mathfrak{F}(z)$ [1]. We point out that subdifferentials as well as k cyclically monotone operators, with $k\ge 3$, are 1-angle-bounded. If $\mathfrak{F}$ is *linear*, then (14) is equivalent to both the *strong sector* [1] condition

$$|\langle \mathfrak{F}(v), w\rangle|^2 \le 4\gamma^2 \langle \mathfrak{F}(v), v\rangle \langle \mathfrak{F}(w), w\rangle \quad \forall\, v,w\in\mathcal{H},$$

and the control of the antisymmetric part of $\mathfrak{F}$ in terms of its symmetric part with $4\gamma^2 = \mu^2+1$

$$|\langle \mathfrak{F}(v), w\rangle - \langle \mathfrak{F}(w), v\rangle| \le 2\mu \langle \mathfrak{F}(v), v\rangle^{1/2} \langle \mathfrak{F}(w), w\rangle^{1/2} \quad \forall\, v,w\in\mathcal{H}. \tag{15}$$

Amann_angle_boundedness

main_non_potential

Theorem 5 *Let $U_0\in D(\mathfrak{F})$. If $\mathfrak{F}$ is γ-angle-bounded, then*

$$\max_{0\le t\le T} |u(t)-U(t)| \le \Big(|u_0-U_0|^2 + \gamma^2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n\Big)^{1/2} + \|f-\bar F\|_{L^1(0,T;\mathcal{H})}.$$

Moreover, if $U_0 := u_0$ and $F_n := f(t_n^+)$, with $f\in BV(0,T;\mathcal{H})$, we have

$$\max_{0\le t\le T} |u(t)-U(t)| \le \tau\Big(\tfrac{\gamma}{\sqrt{2}}\big|(f(0^+)-\mathfrak{F}(u_0))^\circ\big| + (1+\gamma)\operatorname{Var} f\Big).$$

We finally note that further coercivity properties of $\mathfrak{F}$, and associated error estimates, can be derived in the spirit of §1, and that Lipschitz perturbations of $\mathfrak{F}$ can be accomodated as well at the expense of the usual exponential stability factor.

appl

4. Applications. The above abstract results yield novel estimates for degenerate parabolic PDEs and parabolic variational inequalities among other strongly nonlinear problems. We present two simple but relevant examples in the cylinder $Q=\Omega\times(0,T)$ with $\Omega\subset\mathbb{R}^d$ Lipschitz.

Stefan Problem. For $r\in\mathbb{R}$ let $\beta(r) = (r-1)^+ - r^-$ and let $j(r) := \int_0^r \beta(s)\,ds$ be its (convex) primitive. In $\mathcal{H} := H^{-1}(\Omega)$ we introduce the l.s.c. convex potential $\phi := \int_\Omega j(w(x))\,dx$ with domain $D(\phi) = L^2(\Omega)$, and its subdifferential $\mathfrak{F}(v) := -\Delta\beta(v)$ with domain $D(\mathfrak{F}) := \{v\in D(\phi) : \beta(v)\in H_0^1(\Omega)\}$. Given $U_0 = u_0\in D(\mathfrak{F}), f\in BV(0,T;H^{-1}(\Omega))$ and $F_n = f(t_n^+)$, u and U_n solve

$$u(t), U_n\in L^2(\Omega), \quad \beta(u(t)), \beta(U_n)\in H_0^1(\Omega): \quad \partial_t u(t) - \Delta\beta(u(t)) = f(t), \quad \delta U_n - \Delta\beta(U_n) = F_n.$$

According to the abstract theory we have $\sigma(v;w)\ge \frac{1}{2}\|\beta(v)-\beta(w)\|_{L^2(\Omega)}^2$ and $\mathcal{D}_n$, $E\le E(u,U;T)$

$$\tau_n\mathcal{D}_n = \int_\Omega (\beta(U_n)-\beta(U_{n-1}))\,(U_n-U_{n-1}), \quad E = \max\big(\|u-U\|_{L^\infty(0,T;H^{-1}(\Omega))}, \|\beta(u)-\beta(U)\|_{L^2(Q)}\big).$$

Consequently, we obtain optimal a posteriori and a priori error estimates which improve upon [3]:

$$E \le \|f-F\|_{L^1(0,T;H^{-1}(\Omega))} + \sum_{n=1}^N \tau_n^2\mathcal{D}_n \le \tau\Big(\tfrac{1}{\sqrt{2}}\|\Delta\beta(u_0)+f(0^+)\|_{H^{-1}(\Omega)} + 2\operatorname{Var} f\Big).$$

Parabolic Obstacle Problem. In $\mathcal{H} := L^2(\Omega)$ we consider the linear (unbounded) operator $A : v \mapsto -\nu^2 \Delta v + \boldsymbol{b} \cdot \nabla v + cv$ defined in $D(A) := \{v \in H_0^1(\Omega) : \Delta v \in L^2(\Omega)\}$; here $\boldsymbol{b}$ is a Lipschitz vector field, $c \in L^\infty(\Omega)$, and they satisfy

$$|\boldsymbol{b}(x)| \le b_0, \quad -\tfrac{1}{2}\operatorname{div}\boldsymbol{b}(x) + c(x) + \nu^2\eta^2 \ge d_0^2 > 0, \quad \eta := \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla v\|_{L^2(\Omega)}}{\|v\|_{L^2(\Omega)}}.$$

Then A satisfies Amann_angle_boundedness (15) with $\mu = b_0(\nu d_0)^{-1}$. Let $\mathcal{K} := \{v \in L^2(\Omega) : v \ge 0\}$ be the convex set of admissible displacements, and let $I_\mathcal{K}$ be its indicator function (i.e. $I_\mathcal{K}(v) = 0$ if $v \in \mathcal{K}$ and ∞ otherwise). Then, the operator $\mathfrak{F}(v) = Av + \partial I_\mathcal{K}(v)$ with domain $D(\mathfrak{F}) = \mathcal{K} \cap D(A)$ is γ-angle-bounded with $\gamma := \max(1, \frac{1}{2}\sqrt{1+\mu^2})$; the size of γ is thus proportional to the ratio advection/diffusion. Given $u_0 \in D(\mathfrak{F})$ and $f \in L^2(Q)$, $u(t)$ satisfies the parabolic variational inequality

$$u(0) = u_0, \quad \langle \partial_t u + Au, v - u\rangle \ge \langle f, v - u\rangle \quad \forall v \in \mathcal{K}.$$

Given $U_0 \in D(\mathfrak{F})$ and $\{F_n\}_{n=1}^N \in L^2(\Omega)$, U_n solves the elliptic variational inequality for $n \ge 1$

$$U_n \in D(\mathfrak{F}) : \quad \langle \delta U_n + AU_n, v - U_n\rangle \ge \langle F_n, v - U_n\rangle \quad \forall\, v \in \mathcal{K}.$$

Since $Z_n = F_n - \delta U_n - AU_n \in \partial I_\mathcal{K}(U_n)$, we have a residual monotone term $\mathcal{D}_n$ and error E

$$\mathcal{D}_n = \tau_n\langle A\delta U_n, \delta U_n\rangle + \tau_n\langle \delta Z_n, \delta U_n\rangle, \quad E = \max\big(\|u - U\|_{L^\infty(0,T;L^2(\Omega))}, \tfrac{\nu}{\sqrt{2}}\|\nabla(u - U)\|_{L^2(Q)}\big).$$

The following optimal error estimates with variable step sizes are the first for variational inequalities

$$E \le \gamma\Big(\sum_{n=1}^N \tau_n^2 \mathcal{D}_n\Big)^{1/2} + \|f - F\|_{L^1(0,T;L^2(\Omega))} \le \tau\Big(\tfrac{\gamma}{\sqrt{2}}\|Au_0 - f(0^+)\|_{L^2(\Omega)} + (1+\gamma)\operatorname{Var}(f)\Big).$$

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