COMBINED EFFECT OF EXPLICIT TIME-STEPPING AND QUADRATURE FOR CURVATURE DRIVEN FLOWS

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ABSTRACT. The flow of a closed surface of codimension 1 in $\mathbb{R}^n$ driven by curvature is first approximated by a singularly perturbed parabolic double obstacle problem with small parameter $\varepsilon > 0$. Conforming piecewise linear finite elements, with mass lumping, over a quasi-uniform and weakly acute mesh of size h are further used for space discretization, and combined with forward differences for time discretization with uniform time-step τ. The resulting explicit schemes are the basis for an efficient algorithm, the so-called dynamic mesh algorithm, and exhibit finite speed of propagation and discrete nondegeneracy. No iteration is required, not even to handle the obstacle constraints. The zero level set of the fully discrete solution is shown to converge past singularities to the true interface, provided $\tau, h^2 \approx o(\varepsilon^4)$ and no fattening occurs. If the more stringent relations $\tau, h^2 \approx \mathcal{O}(\varepsilon^6)$ are enforced, then an interface rate of convergence $\mathcal{O}(\varepsilon)$ is derived in the vicinity of regular points, along with a companion $\mathcal{O}(\varepsilon^{1/2})$ for type I singularities. For smooth flows, an interface rate of convergence of $\mathcal{O}(\varepsilon^2)$ is proven, provided $\tau, h^2 \approx \mathcal{O}(\varepsilon^5)$ and exact integration is used for the potential term. The analysis is based on constructing fully discrete barriers via an explicit parabolic projection with quadrature, which bears some intrinsic interest, Lipschitz properties of viscosity solutions of the level set approach, and discrete nondegeneracy. These basic ingredients are also discussed.

1. INTRODUCTION

Let $\Sigma(t)$ be the flow that emanates from a smooth surface Σ_0 of codimension 1 in $\mathbb{R}^n$ and propagates in its normal direction with velocity V satisfying the following geometric law

$$V = \kappa + g, \tag{1.1}$$

in the viscosity sense [3], [10], [16], [18]. Hereafter κ denotes the sum of the principal curvatures of $\Sigma(t)$ and g is a given function. This purely geometric evolution may exhibit singularities, topological changes, and even nonuniqueness [2], [4], [38]. Besides its mathematical interest, (1.1) may be viewed as a scaled version of the interface law for various physical processes, where additional PDEs for the bulk

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couple with (1.1) and give rise to g. Singularity formation and continuation of the flow past singularities is then a relevant physical issue, and convergence *past singularities* of fully discrete methods is an important companion scientific issue. The first convergence result for (1.1), along with interface error estimates, was obtained in [31]. We extend here these ideas to a fully practical method which incorporates explicit time-stepping and quadrature.

We first diffuse the sharp interface $\Sigma(t)$ by a singularly perturbed parabolic problem with double obstacle potential [9], [24]–[31], [34]

$$\varepsilon\partial_t u_\varepsilon - \varepsilon\Delta u_\varepsilon - \tfrac{1}{\varepsilon}u_\varepsilon + \mathrm{sign}^{-1}(u_\varepsilon) \ni \tfrac{\pi}{4}g. \tag{1.2}$$

The solution $u_\varepsilon(\cdot,t)$ varies between ± 1, the obstacles, within a narrow transition region of thickness $\mathcal{O}(\varepsilon)$, the noncoincidence set, and attains the values ± 1 elsewhere irrespective of g. Conforming piecewise linear finite elements with mass lumping and forward differences are further used to discretize (1.2) in space and time. We prove that full discretizations exhibit the same local structure of (1.2).

Let Ω be a convex bounded polyhedron of $\mathbb{R}^n$ and $T > 0$ be given such that $\Sigma(t) \subset \Omega$ for all $t \le T$, and set $Q := \Omega \times (0,T)$. The finite element mesh is supposed to be quasi-uniform and *weakly acute*, which in $\mathbb{R}^2$ means that for all pair of triangles sharing a side the sum of the opposite angles to that side does not exceed π; this can be achieved automatically. Let $h > 0$ denote the mesh-size and $\tau > 0$ be the time-step. Under the usual stability constraint $\tau \le C^* h^2$ for forward differences, the resulting *explicit* scheme satisfies the *discrete maximum principle* (DMP), which as in [31] is the key tool for our convergence analysis. However, in contrast with [31] we study here the effect of explicit time-stepping and quadrature, which combined lead to a noniterative scheme with finite speed of propagation and subsequent implementation via the efficient *dynamic mesh algorithm* of [24], [29]. The results of [31] for backward differences and exact integration thus represent the best scenario for accuracy considerations only.

In the level set approach $\Sigma(t)$ is defined, even past singularities, as the zero level set of the continuous solution to the auxiliary degenerate nonlinear PDE

$$\tfrac{\partial_t w}{|\nabla w|} - \mathrm{div}\big(\tfrac{\nabla w}{|\nabla w|}\big) - g = 0, \quad w(\cdot,0) = w_0(\cdot), \tag{1.3}$$

with w_0 satisfying $\Sigma_0 = \{\mathbf{x} \in \mathbb{R}^n : w_0(\mathbf{x}) = 0\}$ [32]. Equation (1.3) corresponds to the motion of all level sets of $w(\cdot,t)$ according to (1.1), and must be interpreted in the viscosity sense [3], [10], [13], [16], [18]. The dimension has thus been increased by one in replacing $\Sigma(t)$ by $w(\cdot,t)$. Our method being *local* thus represents a valid alternative to the *global* level set approach.

Let $U_{\varepsilon,h,\tau}$ and $\Sigma_{\varepsilon,h,\tau}$ indicate the fully discrete solution and its zero level set

$$\Sigma_{\varepsilon,h,\tau}(t) := \{\mathbf{x} \in \Omega : U_{\varepsilon,h,\tau}(\mathbf{x},t) = 0\}.$$

We demonstrate convergence of $\Sigma_{\varepsilon,h,\tau}(t)$ to $\Sigma(t)$ past singularities, provided $\Sigma(t)$ does not develop interior. We now discuss briefly the main ingredients of the construction of fully discrete barriers and their use, and compare with [31]. From now on we will denote by C a positive constant independent of the relevant parameters

ε, h, and τ, that may change at the various occurrences, and use the symbols $\approx, \lessapprox$, and $\gtrapprox$ to indicate the relations $=, \leq$, and $\geq$ up to factors $|\log h|$.

(I) CONTINUOUS BARRIERS. Let g in (1.2) and (1.3) be replaced by $g + \alpha$, and w_0 in (1.3) be replaced by $w_0 + C\alpha$, with $0 < \alpha = o(1)$ to be chosen. Let u^+ be the corresponding solution of (1.2) and w^+ that of (1.3). In §4.2 we use w^+ to construct a viscosity supersolution v^+ of (1.2), whence

$$u_\varepsilon \leq u^+ \leq v^+.$$

We refer to [30] for a slightly different but rigorous construction.

(II) PARABOLIC PROJECTION. We indicate with $H_{h,\tau}(u^+)$ the approximation of u^+ obtained via the discrete heat operator with continuous piecewise linear finite elements and mass lumping in space and forward differences in time. We prove in §9 the 2D pointwise error estimate

$$\|u^+ - H_{h,\tau}(u^+)\|_{L^\infty(Q)} \lessapprox Ch\|u^+\|_{C^{0,1;0,1/2}(\bar{Q})},$$

which is quasi-optimal and deals with minimal regularity requirements. This result extends those in [8], [14], [31], [36] by allowing numerical integration and constitutes our sole restriction to 2D. Its validity in higher dimensions is likely but requires further investigation (see [33]).

(III) DISCRETE NONDEGENERACY. We demonstrate in §5.3 that $U_{\varepsilon,h,\tau}$ has a prescribed quadratic growth away from the fully discrete free boundary, which for all $(\mathbf{x}, t) \in \{-1 < U_{\varepsilon,h,\tau} < \frac{1}{2}\}$ means

$$\max_{|\mathbf{y}-\mathbf{x}|^2+(t-s)\leq r^2} U_{\varepsilon,h,\tau}(\mathbf{y}, s) \geq U_{\varepsilon,h,\tau}(\mathbf{x}, t) + C\tfrac{r^2}{\varepsilon^2}. \tag{1.4}$$

We then proceed as follows, after noticing that (I) and (III) are slightly different from those in [31] but (II) is totally new. We first project u^+ using (II) and next lift $H_{h,\tau}(u^+)$ by the small amount $Ch\varepsilon^{-1} \gtrapprox \|u^+ - H_{h,\tau}(u^+)\|_{L^\infty(Q)}$, for the resulting function $V^+_{\varepsilon,h,\tau}$ to be a candidate for discrete supersolution. However we must compensate for the antimonotone nature of the potential $-u_\varepsilon/\varepsilon$ in (1.2), which gives rise to an error $\approx Ch\varepsilon^{-2}$. This is accomplished by perturbing g by the amount $\alpha \approx Ch\varepsilon^{-2}$, which has the implicit effect of a horizontal shift of the discrete solution without geometric constraints on the mesh. We take $\alpha = o(1)$ and conclude that the parabolic relations

$$h \approx o(\varepsilon^2), \quad \tau \approx o(\varepsilon^4)$$

imply that $V^+_{\varepsilon,h,\tau}$ is a discrete supersolution, and thus $V^+_{\varepsilon,h,\tau} \geq U_{\varepsilon,h,\tau}$. Exploiting the continuous dependence of w with respect to g, this is next used to infer convergence of level sets, a substantially more subtle issue than convergence of solutions [6]. In fact, if $\mathbf{x} \in I(t) = \{w(\cdot, t) < 0\}$, then $w^+(\mathbf{x}, t) < 0$ for α small, and thus $v^+ = -1$ in a vicinity of $(\mathbf{x}, t)$ results from the explicit construction (I). This in turn yields

$$U_{\varepsilon,h,\tau} \leq V^+_{\varepsilon,h,\tau} \lessapprox u^+ + C\tfrac{h}{\varepsilon} \leq v^+ + C\tfrac{h}{\varepsilon} = -1 + C\tfrac{h}{\varepsilon} < -\tfrac{1}{2}. \tag{1.5}$$

We finally deduce $U_{\varepsilon,h,\tau}(\mathbf{x},t) = -1$ for otherwise (1.4) and (1.5) would lead to a contradiction for $r^2 \gtrapprox Ch\varepsilon$. This is in essence our convergence result past singularities of §7, which has to be compared with that of [31] for implicit time stepping without quadrature and $\tau, h^2 \approx o(\varepsilon^3)$. The extra power of ε accounts for the degrading effect of quadrature, because explicit time stepping does not affect accuracy. We would like to point out the importance of dealing with maximum norm estimates which lead to convergence of interfaces, the only objects of interest for the geometric problem at hand.

Invoking the following properties of solutions of (1.3), proven in [31], in §7 we derive interface error estimates for regular and singular points.

(IV) Lipschitz properties: $\|w^+ - w\|_{L^\infty(Q)} \le C\alpha, \quad \|w^+\|_{W^{1,1}_\infty(Q)} \le C.$

We say that $\mathbf{x} \in \Sigma(t)$ is a regular point if w is C^1 in a neighborhood of $(\mathbf{x},t)$ and $\nabla w(\mathbf{x},t) \ne 0$; this may occur even past singularities [1], [38]. In this case, we get $w(\mathbf{x} - \delta\mathbf{n}, t) \le -C\varepsilon$ with $\mathbf{n} = \nabla w(\mathbf{x},t)/|\nabla w(\mathbf{x},t)|$ being a unit normal to $\Sigma(t)$ at $\mathbf{x}$ and $\delta = C\varepsilon/|\nabla w(\mathbf{x},t)|$, whence (IV) yields

$$w^+(\mathbf{x} - \delta\mathbf{n}, t) \le w(\mathbf{x} - \delta\mathbf{n}, t) + C\alpha \le -\varepsilon,$$

provided $\alpha = C\varepsilon$. We use the Lipschitz regularity of (IV) to show that $w^+ < 0$, and also $v^+ = -1$ by the explicit construction (I), in an ε-vicinity of $(\mathbf{x} - \delta\mathbf{n}, t)$. Upon arguing as above but with $\alpha = C\varepsilon$, we infer that $U_{\varepsilon,h,\tau}(\mathbf{x} - \delta\mathbf{n}, t) = -1$ and

$$\operatorname{dist}(\mathbf{x}, \Sigma_{\varepsilon,h,\tau}(t)) \le C\varepsilon|\nabla w(\mathbf{x},t)|^{-1},$$

under the more severe (parabolic) constraints

$$h \approx \mathcal{O}(\varepsilon^3), \quad \tau \approx \mathcal{O}(\varepsilon^6).$$

An order $\mathcal{O}(\varepsilon)$ estimate for the discrete transition layer thickness emerges as a by-product. These ideas are also used to explore how discrete solutions behave near certain singularities, where discrete transition layer thickness becomes $\mathcal{O}(\varepsilon^{1/2})$. This quantifies a smearing effect observed in computations [24], [29], [34].

In §8 we show an optimal $\mathcal{O}(\varepsilon^2)$ interface rate of convergence for smooth flows, provided $\tau, h \approx \mathcal{O}(\varepsilon^5)$ and exact integration is used for the (linear) potential $-u_\varepsilon/\varepsilon$. The prerequisite quasi-optimal $\mathcal{O}(h^2)$ estimates for the parabolic projection with mass lumping are discussed in §§5.4 and 9.

This paper is organized as follows. In §2 we introduce continuous and fully discrete variational inequalities. In §3 we recall the level set approach for curvature driven evolution and its perturbation analysis. In §4 we investigate regularity of (1.2) and review the construction of continuous barriers for (1.2). We discuss qualitative properties in §5; they are finite speed of propagation, DMP, discrete nondegeneracy, and pointwise error estimates of order $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$ for the explicit parabolic projection with quadrature. We construct fully discrete barriers in §6 via the parabolic projection and DMP. They are next used to prove convergence and error estimates for interfaces in §§7 and 8. The auxiliary pointwise error estimates for the parabolic projection are finally proven in §9.

2. Singularly perturbed double obstacle problem

Let $\varepsilon > 0$ be the relaxation parameter and

$$\mathcal{K} := \{\varphi \in H^1(\Omega) : |\varphi| \leq 1, \varphi = 1 \text{ on } \partial\Omega\}$$

be the convex set. Let $\gamma \in C^{1,1}(\mathbb{R})$ be the function given by $\gamma(x) := \sin x$ if $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and constant outside. Set $\langle \varphi_1, \varphi_2 \rangle := \int_\Omega \varphi_1 \varphi_2$ for any $\varphi_1, \varphi_2 \in L^2(\Omega)$. The double obstacle problem (1.2) reads as follows and admits a unique solution: *seek $u_\varepsilon \in L^\infty(0,T;\mathcal{K}) \cap H^1(0,T;L^2(\Omega))$ such that $u_\varepsilon(\cdot, 0) = \gamma(d_0(\cdot)/\varepsilon)$ in Ω, and, for a.e. $t \in (0,T)$ and all $\varphi \in \mathcal{K}$,*

$$\varepsilon\langle \partial_t u_\varepsilon, \varphi - u_\varepsilon \rangle + \varepsilon \langle \nabla u_\varepsilon, \nabla(\varphi - u_\varepsilon)\rangle - \tfrac{1}{\varepsilon}\langle u_\varepsilon, \varphi - u_\varepsilon\rangle \geq \tfrac{\pi}{4}\langle g, \varphi - u_\varepsilon \rangle. \tag{2.1}$$

Let $\mathcal{D}_h$ denote a quasi-uniform and regular decomposition of $\Omega \subset \mathbb{R}^n$ into simplices of size comparable with $h > 0$ [11]. Let $\mathcal{V}_h$ be the finite element space of continuous piecewise linear functions over $\mathcal{D}_h$, $\mathcal{V}_h(a) := \{\chi \in \mathcal{V}_h : \chi = a \text{ on } \partial\Omega\}$, and set $\mathcal{K}_h := \mathcal{K} \cap \mathcal{V}_h(1)$. Let Π_h be the usual Lagrange interpolation operator and $\langle \cdot, \cdot \rangle_h$ be the discrete inner product

$$\langle \chi, \phi \rangle_h := \int_\Omega \Pi_h(\chi\phi) d\mathbf{x} \quad \forall\, \chi, \phi \in C^0(\bar{\Omega}),$$

which corresponds to the vertex quadrature rule [11]. It satisfies [35, p.260]

$$\|\chi\|^2_{L^2(\Omega)} \leq \|\chi\|^2_h := \langle \chi, \chi \rangle_h \leq C\|\chi\|^2_{L^2(\Omega)} \quad \forall\, \chi \in \mathcal{V}_h, \tag{2.2}$$

and the discrepancy between $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_h$ can be bounded, for $1 \leq p \leq \infty$, as

$$|\langle \chi, \phi \rangle - \langle \chi, \phi \rangle_h| \leq Ch^2 \langle |\nabla \chi|, |\nabla \phi| \rangle \leq Ch \|\nabla \chi\|_{L^p(\Omega)} \|\phi\|_{L^{p/(p-1)}(\Omega)}. \tag{2.3}$$

Let $\{\phi_i\}_{i=1}^J$ be the canonical basis of $\mathcal{V}_h$ and $\{\mathbf{x}_i\}_{i=1}^J$ be the nodes of $\mathcal{D}_h$ (let $\{\mathbf{x}_i\}_{i=1}^I$ be the internal nodes). Let $\mathbf{K} := \{k_{ij}\}_{i,j=1}^J$ be the stiffness matrix, $\mathbf{M} := \{m_{ij}\}_{i,j=1}^J$ and $\mathbf{M}_h := \{m_{ij}^h\}_{i,j=1}^J$ be the mass and lumped mass matrices defined by

$$k_{ij} := \langle \nabla \phi_i, \nabla \phi_j \rangle, \quad m_{ij} := \langle \phi_i, \phi_j \rangle, \quad m_{ij}^h := \langle \phi_i, \phi_j \rangle_h.$$

We further assume that $\mathbf{K}$ is an M-matrix, which for $n = 2$ is guaranteed provided $\mathcal{D}_h$ is *weakly acute* [12], namely,

(2.4) *the sum of opposite angles to the common side of any two triangles is $\leq \pi$.*

This rather mild condition can be enforced with automatic mesh generation [29].

Let $\tau := T/N > 0$ be the uniform time-step, which is subject to the stability constraint $\tau \leq C^* h^2$, and $t_n := n\tau$. For all $1 \leq n \leq N$, set

$$K^n := K \times [t_{n-1}, t_n] \quad \forall\, K \in \mathcal{D}_h, \qquad \Omega_i^n := \text{supp}\phi_i \times [t_{n-1}, t_n] \quad \forall\, 1 \leq i \leq I,$$

and for any $\{\chi^n\}_{n=0}^N \subset \mathcal{V}_h$ and χ continuous, set

$$\partial\chi^n := \tfrac{1}{\tau}(\chi^n - \chi^{n-1}), \quad \chi^n := \chi(t_n), \quad \bar{\chi}^n := \tfrac{1}{\tau}\int_{t_{n-1}}^{t_n} \chi(\cdot,t)dt \quad (\bar{\chi}^{N+1} := \chi^N).$$

The fully discrete approximation of (2.1) reads: *for* $U^0 := \Pi_h\gamma(d_0/\varepsilon) \in \mathcal{K}_h$ *seek* $U^n \in \mathcal{K}_h$, $1 \le n \le N$, *such that, for all* $\chi \in \mathcal{K}_h$,

$$\varepsilon\langle\partial U^n, \chi - U^n\rangle_h + \varepsilon\langle\nabla U^{n-1}, \nabla(\chi - U^n)\rangle - \tfrac{1}{\varepsilon}\langle U^{n-1}, \chi - U^n\rangle_h \ge \tfrac{\pi}{4}\langle g^n, \chi - U^n\rangle_h. \tag{2.5}$$

The use of mass lumping for the discrete time derivative, which diagonalizes the corresponding matrix $\mathbf{M}_h$, is crucial to eliminate any iteration in solving (2.5). To see this, we identify any function $\chi \in \mathcal{V}_h$ with the vector of its nodal values $\{\chi_i := \chi(\mathbf{x}_i)\}_{i=1}^J$, and note that (2.5) reads, for all $1 \le i \le I$,

$$\begin{aligned} U_i^{n-1/2} &= (1 + \tfrac{\tau}{\varepsilon^2})U_i^{n-1} - (\mathbf{M}_h^{-1}\mathbf{K}U^{n-1})_i + \tfrac{\tau}{\varepsilon}\tfrac{\pi}{4}g^n(\mathbf{x}_i), \\ U_i^n &= \max\big(-1, \min(1, U_i^{n-1/2})\big). \end{aligned} \tag{2.6}$$

The actual computation of U^n is thus a trivial *two-step* process: we first have to perform the matrix-vector products involved in (2.6) to define $U^{n-1/2}$, and then project each component of $U^{n-1/2}$ onto $[-1,1]$ to compute U^n.

We stress that the stability constraint $\tau \le C^* h^2$ due to the explicit time discretization is tailored here to avoid oscillations: a node value of U^{n-1} cannot jump from 1 to -1 (resp. from -1 to 1) even if all the adjacent values are -1 (resp. 1).

An alternative scheme to (2.5) that preserves all the properties discussed above and below consists of doing exact integration in the potential, namely

$$\varepsilon\langle\partial U^n, \chi - U^n\rangle_h + \varepsilon\langle\nabla U^{n-1}, \nabla(\chi - U^n)\rangle - \tfrac{1}{\varepsilon}\langle U^{n-1}, \chi - U^n\rangle \ge \tfrac{\pi}{4}\langle g^n, \chi - U^n\rangle_h. \tag{2.7}$$

In §8 we will show that this method is more accurate for smooth flows.

3. Generalized curvature driven motion and perturbation analysis

The evolution of the zero level set $\Sigma(t)$ of $w(\cdot,t)$, viscosity solution of (1.3), is independent of the specific form of w_0 for as long as $I_0 := \{\mathbf{x} \in \mathbb{R}^n : w_0(\mathbf{x}) < 0\}$ be the inside of Σ_0 [3], [10], [16], [18]. Then

$$\Sigma := \{(\mathbf{x},t) \in \mathbb{R}^n \times [0,T] : \mathbf{x} \in \Sigma(t)\} \tag{3.1}$$

is called *generalized* geometric evolution and coincides with the classical motion (1.1) before the onset of singularities. The notions of inside $I(t)$ and outside $O(t)$ of $\Sigma(t)$ are thus meaningful in terms of w

$$I(t) := \{\mathbf{x} \in \mathbb{R}^n : w(\mathbf{x},t) < 0\}, \quad O(t) := \{\mathbf{x} \in \mathbb{R}^n : w(\mathbf{x},t) > 0\}.$$

The flow (3.1) is unique and is well defined even past singularities. Let $d(\cdot,t)$ be the signed distance function to $\Sigma(t)$ that is positive in $O(t)$. It is known that d satisfies the following viscosity inequality, provided $\Sigma(t)$ has empty interior [3], [15]

$$\mathcal{H}_g d(\mathbf{x},t) := \partial_t d(\mathbf{x},t) - \Delta d(\mathbf{x},t) - g\big(\mathbf{x} - d(\mathbf{x},t)\nabla d(\mathbf{x},t), t\big) \ge 0 \quad \text{in } \{d > 0\}. \tag{3.2}$$

In case $\Sigma(t)$ fattens up, then $d(\mathbf{x},t)$ must be replaced by $\mathrm{dist}(\mathbf{x},I(t))$ for (3.2) to hold [3, p.446]; (3.2) is also valid with ≤ 0 for $-\mathrm{dist}(x,O(t))$ in $\{d<0\}$. In the sequel we will use the notation $d(\mathbf{x},t)$ instead of $\mathrm{dist}(\mathbf{x},I(t))$ or $-\mathrm{dist}(\mathbf{x},O(t))$.

Singularities are typically isolated points $(\mathbf{x},t)$ that correspond to $\nabla w(\mathbf{x},t)=0$ [1], [38], but $\Sigma(t)$ may also develop interior (fattening) [2], [4]. We say that $(\mathbf{x},t)\in\Sigma$ is a *regular* point if w is C^1 in a space-time neighborhood of $(\mathbf{x},t)$ and

$$\nabla w(\mathbf{x},t)\neq 0. \tag{3.3}$$

Such points do exist even past singularities for flows of interest [1], [2], [4], [38]. We then denote by Σ^* the subset of Σ of all regular points, and note that Σ^* extends to $t=0$ because Σ_0 is smooth. On the other hand a *generic singular* point satisfies the following asymptotic behavior [17], [31]: there exists a set of orthonormal vectors $\{\mathbf{e}_i\}_{i=1}^n$ and $1\leq k\leq n$ such that

$$w(\mathbf{x}+\mathbf{y},t+s)=\beta s+\lambda|\mathbf{y}|^2+o\big(|s|+|\mathbf{y}|^2\big)\quad\forall\ |\mathbf{y}|,|s|\ll 1, \tag{3.4}$$

for $\mathbf{y}:=\sum_{i=1}^k y_i\mathbf{e}_i$, $\beta:=2(k-1)\lambda+g(\mathbf{x},t)$, and $\lambda\neq 0$. This is certainly true if $w\in C^2$ in the vicinity of $(\mathbf{x},t)$, but there is no characterization of C^2 regularity and w may fail to be C^2 while still exhibiting quadratic growth in the directions $\{\mathbf{e}_i\}_{i=1}^k$ [21]. In turn, (3.4) yields the curvature blow-up rate [20]

$$\kappa\approx|\mathbf{y}|^{-1}\approx\left|\tfrac{\lambda}{\beta s}\right|^{1/2}\quad\text{as } s\uparrow 0, \tag{3.5}$$

of type I singularities, which look asymptotically like spheres ($k=n$) or cylinders ($2\leq k<n$). Type II singularities exhibit a blow-up rate higher than (3.5) and are receiving considerable attention.

Even though the geometric flow (1.1) may lead to merging, splitting, and extinction, the level set formulation provides a stable representation of it. To quantify this statement, we first recall that there exists a Lipschitz constant $L>0$ depending on $\|w_0\|_{W^2_\infty(\Omega)}$ and $\|g\|_{W^{1,1}_\infty(Q)}$ such that [31]

$$\|w\|_{W^{1,1}_\infty(Q)}\leq L. \tag{3.6}$$

Let $\alpha>0$ be a perturbation parameter. In the sequel w and $w^\pm$ will stand for viscosity solutions to (1.3) with forcing terms g and $g^\pm:=g\pm\Theta\alpha$, and initial traces

$$w(\cdot,0)=d_0(\cdot)\quad\text{and}\quad w^\pm(\cdot,0)=d_0(\cdot)\pm\theta\alpha, \tag{3.7}$$

respectively, where d_0 is the signed distance to Σ_0 and $\Theta,\theta>1$ will be chosen later. We will select either $\varepsilon\leq\alpha=o(1)$, or $\alpha=C\varepsilon$ ($C\geq 1$), or $\alpha=\varepsilon^2$ at various occurrences. The following crucial result of [31], based on (3.6), quantifies how w and $w^\pm$ deviate from each other: there exists a constant G depending on L so that

$$w-G\alpha\leq w^-\leq w\leq w^+\leq w+G\alpha\quad\text{in } Q. \tag{3.8}$$

An immediate consequence of (3.8) is convergence of the perturbed fronts

$$\Sigma^\pm(t):=\{\mathbf{x}\in\mathbb{R}^n: w^\pm(\mathbf{x},t)=0\}$$

to $\Sigma(t)$ as $\alpha \downarrow 0$, even past singularities, along with a linear rate of convergence in a vicinity of regular points (3.3) [31]

$$\operatorname{dist}(\mathbf{x}, \Sigma^{\pm}(t)) \leq \frac{2G}{|\nabla w(\mathbf{x},t)|}\alpha. \tag{3.9}$$

This result sheds some light on stability properties of the flow (1.1) that make it computable.

Let $d^{\pm}(\cdot,t)$ be the signed distance to $\Sigma^{\pm}(t)$ that is negative inside $\Sigma^{\pm}(t)$. Upon replacing g by $g^{\pm}$ in (3.2), we deduce the viscosity inequalities

$$\mathcal{H}_{g^+} d^+(\mathbf{x},t) \geq 0 \quad \text{in } \{d^+ > 0\}, \quad \mathcal{H}_{g^-} d^-(\mathbf{x},t) \leq 0 \quad \text{in } \{d^- < 0\}. \tag{3.10}$$

For smooth flows, let $\kappa_s^{\pm}(\mathbf{x},t)$ denote the sum of the square of the principal curvatures of $\Sigma^{\pm}(t)$ at $\mathbf{x} - d^{\pm}(\mathbf{x},t)\nabla d^{\pm}(\mathbf{x},t) \in \Sigma^{\pm}(t)$. Then $d^{\pm}$ satisfies [27]

$$\mathcal{H}_{g^{\pm}} d^{\pm}(\mathbf{x},t) = d^{\pm}(\mathbf{x},t)\kappa_s^{\pm}(\mathbf{x},t) + \mathcal{O}(d^{\pm}(\mathbf{x},t)^2). \tag{3.11}$$

If $W := \min_{(\mathbf{x},t)\in\Sigma} |\nabla w(\mathbf{x},t)| > 0$, then a simple consequence of (3.8), (3.9), and the fact that distance functions are Lipschitz in space with constant 1 reads

$$d - \tfrac{2G}{W}\alpha \leq d^- \leq d \leq d^+ \leq d + \tfrac{2G}{W}\alpha \quad \text{in } Q. \tag{3.12}$$

4. Double obstacle problem: regularity and continuous barriers

In this section we discuss regularity of (2.1) along with the construction of barriers for (2.1).

4.1. Intermediate regularity. The solution u_ε satisfies $\|u_\varepsilon\|_{W_p^{2,1}(Q)} \leq Cp\varepsilon^{-2}$ for all $p < \infty$ [31]. We stress that u_ε is not in $W_\infty^{2,1}(Q)$ with norm of order $\mathcal{O}(\varepsilon^{-2})$ because this would require higher regularity of $\varepsilon^{-2}(u_\varepsilon + \frac{\pi}{4}\varepsilon g) = \mathcal{O}(\varepsilon^{-2})$, and so of u_ε still with norms of order $\mathcal{O}(1)$, which we cannot expect. There exists, however, a modulus of continuity $\omega(s) := C\varepsilon^{-2}s|\log s|$, that precisely quantifies what is missing for $\partial_t u_\varepsilon$ and $D^2_{\mathbf{x}} u_\varepsilon$ to be bounded [31]

$$\begin{aligned} \|\nabla u_\varepsilon(\mathbf{x}_1,\cdot) - \nabla u_\varepsilon(\mathbf{x}_2,\cdot)\|_{L^\infty(0,T)} &\leq \omega(|\mathbf{x}_1 - \mathbf{x}_2|) \quad \forall\ |\mathbf{x}_1 - \mathbf{x}_2| \ll 1, \\ \|u_\varepsilon(\cdot,t_1) - u_\varepsilon(\cdot,t_2)\|_{L^\infty(\Omega)} &\leq \omega(|t_1 - t_2|) \quad \forall\ |t_1 - t_2| \ll 1. \end{aligned}$$

We now exploit this property to derive and intermediate regularity result.

Lemma 4.1. $\|u_\varepsilon\|_{C^{0,1;0,1/2}(\bar{Q})} \leq C\varepsilon^{-1}|\log\varepsilon|$.

Proof. We denote $v(\cdot) = u_\varepsilon(\cdot,t)$ for t fixed. Invoking the Mean Value Theorem we can write for $\mathbf{n} := \nabla v(\mathbf{x})/|\nabla v(\mathbf{x})|$ and $\delta > 0$,

$$v(\mathbf{x} + \delta\mathbf{n}) = v(\mathbf{x}) + \delta\nabla v(\mathbf{x})\cdot\mathbf{n} + \delta\int_0^1 \left(\nabla v(\mathbf{x} + s\delta\mathbf{n}) - \nabla v(\mathbf{x})\right)\cdot\mathbf{n}\ ds.$$

This, in conjunction with $\|v\|_{L^\infty(\Omega)} \leq 1$ and the above expression for ω, yields

$$|\nabla v(\mathbf{x})| \leq \tfrac{2}{\delta} + C\tfrac{\delta|\log\delta|}{\varepsilon^2} \quad \forall\ 0 < \delta \ll 1,$$

and the desired bound $|\nabla v(\mathbf{x})| \leq C\varepsilon^{-1}|\log\varepsilon|$ follows for $\delta = \varepsilon$. Similarly, we estimate the Hölder constant in time. We set $z(\cdot) = u_\varepsilon(\mathbf{x},\cdot)$ for $\mathbf{x}$ fixed and get

$$\delta^{-1/2}|z(t+\delta) - z(t)| \leq \min\left(\tfrac{2}{\delta^{1/2}}, C\tfrac{\delta^{1/2}|\log\delta|}{\varepsilon^2}\right) \quad \forall\ \delta > 0.$$

If $\delta \geq \varepsilon^2$, then the right hand side is bounded by $2\varepsilon^{-1}$. Otherwise the fact that $s^{1/2}|\log s|$ is increasing for s small yields the desired bound $C\varepsilon^{-1}|\log\varepsilon|$. $\square$

4.2. Viscosity barriers. We now review the construction of continuous barriers for u_ε, which are valid after the onset of singularities. There exist ε-approximate traveling waves $\Gamma^\pm(\cdot;\mathbf{x},t) \in C^{1,1}(\mathbb{R})$ of (2.1) with forcing terms $g^\pm = g \pm \Theta\alpha$, which are strictly increasing in $(-\frac{\pi}{2}, \frac{\pi}{2})$ from -1 to 1 and constant outside, and satisfy

$$\partial_x^2\Gamma^\pm + \Gamma^\pm - \varepsilon g^\pm(\partial_x\Gamma^\pm - \tfrac{\pi}{4}) = \partial_x\Gamma^\pm \mathcal{O}(\varepsilon^2). \tag{4.1}$$

These functions can be written explicitly by using formal asymptotics as

$$\Gamma^\pm(x;\mathbf{x},t) := \gamma(x) + \varepsilon g^\pm(\mathbf{x},t)\ \eta(x) \quad \forall\ x \in \mathbb{R},\ (\mathbf{x},t) \in Q, \tag{4.2}$$

where $\eta(x) := \frac{1}{2}\big(x\gamma(x) - \frac{\pi}{2} + \gamma'(x)\big)$ for $|x| \le \frac{\pi}{2}$ and zero elsewhere [30]. Convergence of $\Sigma_\varepsilon(t)$ to $\Sigma(t)$ as $\varepsilon \downarrow 0$ even past singularities is based on the following result. Here we give a formal argument slightly different from that in [30], [31], and refer to [30] for a rigorous proof.

Lemma 4.2. *Let $g \in W^{2,1}_\infty(Q)$. Let $\alpha \ge \varepsilon$ and let $u^\pm$ be the solutions to (2.1) with forcing terms $g \pm \alpha$ and initial trace $\gamma(d_0/\varepsilon)$. Let $v^\pm$ be*

$$v^\pm(\mathbf{x},t) := \Gamma^\pm\big(\tfrac{d^\pm(\mathbf{x},t)}{\varepsilon} \mp \tfrac{\pi}{2};\mathbf{x},t\big), \tag{4.3}$$

with $d^\pm(\cdot,t)$ the signed distance to $\Sigma^\pm(t)$. If $\theta = \pi$ and $\Theta \ge 1$ defined in (3.7) is sufficiently large, then $v^\pm$ are viscosity barriers of (2.1) with forcing terms $g \pm \alpha$ and satisfy

$$v^-(\mathbf{x},t) \le u^-(\mathbf{x},t) \le u_\varepsilon(\mathbf{x},t) \le u^+(\mathbf{x},t) \le v^+(\mathbf{x},t) \quad \forall\ (\mathbf{x},t) \in Q.$$

Proof (formal). We first set $u = u^+, v = v^+, d = d^+, \Gamma = \Gamma^+, \Gamma^{(k)} = \partial_x^k\Gamma$, and recall that $g^+ = g + \Theta\alpha$. A standard comparison argument for variational inequalities implies $u \ge u_\varepsilon$, because they satisfy the same problem with right hand sides $g+\alpha \ge g$. Since $v \ge -1$, in order to show that $u \le v$ we have to prove that

$$\mathcal{J}v := \varepsilon\partial_t v - \varepsilon\Delta v - \tfrac{1}{\varepsilon}v - \tfrac{\pi}{4}(g+\alpha) \ge 0 \quad \text{in } \{v < 1\},$$

that is in $\{d < \pi\varepsilon\}$. If $d < 0$, then $\Gamma = -1$ by construction, whence $\mathcal{J}v = \frac{1}{\varepsilon} - \frac{\pi}{4}(g+\alpha) > 0$ for ε sufficiently small. If $0 < d < \pi\varepsilon$, instead, from (4.2) we have

$$\varepsilon\partial_t v = \Gamma'\partial_t d + \varepsilon^2\eta\partial_t g, \quad \varepsilon\Delta v = \tfrac{1}{\varepsilon}\Gamma'' + \Gamma'\Delta d + 2\varepsilon\eta'\nabla g\cdot\nabla d + \varepsilon^2\eta\Delta g.$$

Since $|g^+(\mathbf{x},t) - g^+(\mathbf{x} - d(\mathbf{x},t)\nabla d(\mathbf{x},t),t)| \le \|\nabla g\|_{L^\infty(Q)}d(\mathbf{x},t)$, (3.10) and property $\Gamma' \ge 0$ yield

$$\Gamma'(\partial_t d - \Delta d - g^+) \ge -\pi\varepsilon\Gamma'\|\nabla g\|_{L^\infty(Q)}.$$

Using (4.1) and $g \in W^{2,1}_\infty(Q)$, we thus obtain

$$\begin{aligned}\mathcal{J}v &= -\tfrac{1}{\varepsilon}\big(\Gamma'' + \Gamma - \varepsilon g^+(\Gamma' - \tfrac{\pi}{4})\big) + \Gamma'(\partial_t d - \Delta d - g^+) + \tfrac{\pi}{4}(\Theta-1)\alpha + \mathcal{O}(\varepsilon)\\ &\ge \tfrac{\pi}{4}(\Theta-1)\alpha + \mathcal{O}(\varepsilon) \ge 0,\end{aligned}$$

provided $\Theta > 1$ is sufficiently large, because $\alpha \geq \varepsilon$. Since $\Gamma''(\frac{d(\mathbf{x},t)}{\varepsilon} - \frac{\pi}{2};\mathbf{x},t)$ is a distribution without mass concentrated on $\{d = 0\}$, this shows that v is a (formal) viscosity supersolution. It only remains to check the values of v on the parabolic boundary of Q. Since the transition region $\{|v(\cdot,t)| < 1\}$ does not touch $\partial\Omega$, we have $v = 1$ on $\partial\Omega \times (0,T)$. In addition, for $\theta = \pi$ we have $d(\cdot,0) = d_0(\cdot) + \pi\alpha$; hence $\Gamma(x + \frac{\pi}{2};\cdot,0) \geq \gamma(x)$ and $\alpha \geq \varepsilon$ yield

$$v(\mathbf{x},0) = \Gamma\big(\tfrac{d(\mathbf{x},0)}{\varepsilon} - \tfrac{\pi}{2};\mathbf{x},0\big) \geq \Gamma\big(\tfrac{d_0(\mathbf{x})}{\varepsilon} + \tfrac{\pi}{2};\mathbf{x},0\big) \geq \gamma\big(\tfrac{d_0(\mathbf{x})}{\varepsilon}\big) = u(\mathbf{x},0) \quad \forall\ \mathbf{x} \in \Omega.$$

Therefore we conclude that $v \geq u$ as a consequence of a viscosity comparison result for variational inequalities [30]. The proof is thus complete. $\square$

4.3. Classical barriers. We now discuss a new construction of classical barriers based on the regularity of $\Sigma^\pm(t)$ before the onset of singularities of $\Sigma(t)$. If $\Sigma(t)$ is smooth, then w is nondegenerate and d is smooth, and thus the question arises as to what regularity can be expected for $d^\pm$. Since $w^\pm$ satisfies (1.3), with $g^\pm$ in place of g, and the distance between $\Sigma(t)$ and $\Sigma^\pm(t)$ is small in view of (3.9), we would expect $w^\pm$ and $d^\pm$ to be as smooth as w and d. This is customary in regularity theory [7], but does not seem to be available in the existing literature on viscosity solutions.

Inspired by formal asymptotics, let $\tilde\Gamma^\pm(\cdot;\mathbf{x},t) \in C^{0,1}(\mathbb{R})$ be ε^2-approximate traveling waves of (2.1) with forcing term $g^\pm = g \pm \Theta\varepsilon^2$ given by

$$\begin{aligned}\tilde\Gamma^\pm(x;\mathbf{x},t) := \min\big[1, \max\big[-1, \Gamma^\pm(x;\mathbf{x},t) + \varepsilon^2\big(\kappa_s^\pm(\mathbf{x},t) + \tfrac{1}{2}g^\pm(\mathbf{x},t)^2 \\ - 2\nabla g(\mathbf{x},t)\cdot\nabla d^\pm(\mathbf{x},t)\big)\mu^\pm(x)\big]\big] \quad \forall\ x \in \mathbb{R},\ (\mathbf{x},t) \in Q,\end{aligned} \tag{4.4}$$

where $\mu^\pm(x) := \frac{1}{16}\big(4x\gamma'(x) + (4x^2 - \pi^2)\gamma(x)\big) \mp \frac{\pi}{8}\gamma'(x)$ [25]–[27], [34]. Function $\tilde\Gamma^+(\cdot;\mathbf{x},t)$ (resp. $\tilde\Gamma^-$) is strictly increasing and satisfies $|\tilde\Gamma^\pm| < 1$ in $(-\frac{\pi}{2}, x_\varepsilon)$ (resp. $(-x_\varepsilon, \frac{\pi}{2})$) for a proper $0 \leq \frac{\pi}{2} - x_\varepsilon(\mathbf{x},t) = \mathcal{O}(\varepsilon^2)$, $\partial_x\tilde\Gamma^\pm(\pm x_\varepsilon^\mp;\mathbf{x},t) = \mathcal{O}(\varepsilon^2)$, and

$$\partial_x^2\tilde\Gamma^\pm + \tilde\Gamma^\pm - \varepsilon g^\pm(\partial_x\tilde\Gamma^\pm - \tfrac{\pi}{4}) - \varepsilon^2(\kappa_s^\pm - 2\nabla g\cdot\nabla d^\pm)x\partial_x\tilde\Gamma^\pm = \partial_x\Gamma^\pm\mathcal{O}(\varepsilon^3). \tag{4.5}$$

The preceding construction leads to an optimal (quadratic) order of convergence of $\Sigma_\varepsilon(t)$ to $\Sigma(t)$ as $\varepsilon \downarrow 0$ in the smooth regime [25]–[27]. It is simpler than the original one in [27] at the expense of using regularity of $\Sigma^\pm(t)$.

Lemma 4.3. *Let $g \in W^{3,2}_\infty(Q)$. Let $\alpha = \varepsilon^2$ and let $u^\pm$ be the solutions to* (2.1) *with forcing terms $g \pm \varepsilon^2$ and initial trace $\gamma(d_0/\varepsilon)$. Let $\tilde v^\pm$ be*

$$\tilde v^\pm(\mathbf{x},t) := \tilde\Gamma^\pm\big(\tfrac{d^\pm(\mathbf{x},t)}{\varepsilon};\mathbf{x},t\big), \tag{4.6}$$

with $d^\pm(\cdot,t)$ the signed distance to $\Sigma^\pm(t)$. If $\theta,\Theta \geq 1$ in (3.7) *are sufficiently large and $\Sigma^\pm(t)$ are smooth, then $\tilde v^\pm$ are variational barriers of* (2.1) *with forcing terms $g \pm \varepsilon^2$ and satisfy*

$$\tilde v^-(\mathbf{x},t) \leq u^-(\mathbf{x},t) \leq u_\varepsilon(\mathbf{x},t) \leq u^+(\mathbf{x},t) \leq \tilde v^+(\mathbf{x},t) \quad \forall\ (\mathbf{x},t) \in Q.$$

Proof (formal). We set $u = u^+, v = \tilde{v}^+, d = d^+, \kappa_s = \kappa_s^+, \Gamma = \tilde{\Gamma}^+$, $\Gamma^{(k)} = \partial_x^k \Gamma$, and recall that $g^+ = g + \Theta\varepsilon^2$. Since $v \geq -1$ in Q and $\nabla v \cdot \nu \geq 0$ on the boundary of $\{v < 1\}$, to show that $u \leq v$ we must prove that $\mathcal{J} v \geq 0$ in $\{v < 1\} = \{d < x_\varepsilon \varepsilon\}$. If $d < -\frac{\pi}{2}\varepsilon$, then $\mathcal{J} v > 0$ for ε small. Instead, if $-\frac{\pi}{2}\varepsilon < d < x_\varepsilon \varepsilon$, then (3.11) and (4.4) yield

$$\partial_t d - \Delta d = g^+ + d(\kappa_s - \nabla g \cdot \nabla d) + \mathcal{O}(\varepsilon^2),$$
$$\varepsilon \partial_t v = \Gamma' \partial_t d + \mathcal{O}(\varepsilon^2), \quad \varepsilon \Delta v = \tfrac{1}{\varepsilon}\Gamma'' + \Gamma' \Delta d + 2\varepsilon \eta' \nabla g \cdot \nabla d + \mathcal{O}(\varepsilon^2),$$

respectively. From (4.5) we thus obtain

$$\mathcal{J} v = -\tfrac{1}{\varepsilon}\big(\Gamma'' + \Gamma - \varepsilon g^+(\Gamma' - \tfrac{\pi}{4}) - \varepsilon(\kappa_s - 2\nabla g \cdot \nabla d) d\Gamma'\big) + \tfrac{\pi}{4}(\Theta - 1)\varepsilon^2 + \mathcal{O}(\varepsilon^2) \geq 0,$$

for $\Theta > 1$ sufficiently large. Since $\Gamma''(\frac{d(\mathbf{x},t)}{\varepsilon}; \mathbf{x}, t)$ is a distribution without mass concentrated on $\{d = -\frac{\pi}{2}\varepsilon\}$, this shows that v is a variational supersolution. Since $v = 1$ on $\partial\Omega \times (0, T)$ and, for θ sufficiently large [25],

$$v(\mathbf{x}, 0) = \Gamma\big(\tfrac{d(\mathbf{x},0)}{\varepsilon}; \mathbf{x}, 0\big) = \Gamma\big(\tfrac{d_0(\mathbf{x})}{\varepsilon} + \theta\varepsilon; \mathbf{x}, 0\big) \geq \gamma\big(\tfrac{d_0(\mathbf{x})}{\varepsilon}\big) = u(\mathbf{x}, 0) \quad \forall\, \mathbf{x} \in \Omega,$$

a comparison principle for variational inequalities yields $v \geq u$ [25] . $\square$

5. Discrete qualitative properties

In this section we discuss several important properties of the finite element approximation combined with forward differences and quadrature.

5.1. Finite speed of propagation. If $\mathbf{x}_i$ is a node of $\mathcal{D}_h$ such that $U_j^{n-1} = 1$ (resp. $U_j^{n-1} = -1$) for all adjacent nodes $\mathbf{x}_j$ to $\mathbf{x}_i$, including $\mathbf{x}_i$, then (2.6) yields

$$U_i^{n-1/2} > U_i^{n-1} = 1 \quad (\text{resp } U_i^{n-1/2} < U_i^{n-1} = -1),$$

for ε so small that $|g| \leq 4/\pi\varepsilon$. Therefore $U_i^n = U_i^{n-1} = 1$ (resp. $U_i^n = U_i^{n-1} = -1$), which says that the discrete transition layer (or noncoincidence set)

$$\mathcal{T}^{n-1} := \{\mathbf{x} \in \Omega : |U^{n-1}(\mathbf{x})| < 1\}$$

cannot move faster than one triangle per time step; this is reminiscent of hyperbolic conservation laws. Consequently, U^n does not have to be computed at vertices lying outside $\bar{\mathcal{T}}^{n-1}$ and there is no need for the entire mesh $\mathcal{D}_h$ but just for a rather small part of it. That is why we introduced the *dynamic* mesh version of (2.5) in [29].

This property is still valid without mass lumping. To see this we let $U_i^n < 1$, and use (2.5) with exact integration and $\chi = U^n + (1 - U_i^n)\phi_i$, to arrive at

$$\langle U^n, \phi_i \rangle \geq \langle (1 + \tfrac{\tau}{\varepsilon^2}) U^{n-1} + \tfrac{\tau}{\varepsilon}\tfrac{\pi}{4} g^n, \phi_i \rangle - \tau \langle \nabla U^{n-1}, \nabla \phi_i \rangle,$$

whence, since $U^n \leq 1$,

$$\langle 1, \phi_i \rangle > \langle U^n, \phi_i \rangle \geq \langle 1 + \tfrac{\tau}{\varepsilon^2} + \tfrac{\tau}{\varepsilon}\tfrac{\pi}{4} g^n, \phi_i \rangle \geq \langle 1, \phi_i \rangle,$$

which is a contradiction. Therefore mass lumping is not justified in light of finite speed of propagation but rather in terms of solvability. In fact $\mathbf{M}_h$ would be replaced by the nondiagonal $\mathbf{M}$ and thus an iterative procedure would be needed to compute U^n. Another important consequence of mass lumping is discussed next.

5.2. Discrete maximum principle. Note that, in view of (2.4), $k_{ij} \leq 0$ for $i \neq j$ and $\mathbf{A} := \tau^{-1}\mathbf{M}_h - \mathbf{K} \geq 0$ provided $\tau \leq C^* h^2$, because $\mathbf{M}_h$ is diagonal. This positivity property is responsible for the validity of the following parabolic Discrete Maximum Principle (DMP) for *noncylindrical* discrete domains.

Let D be a connected union of stars Ω_i^n, or alternatively the sets of its space-time nodes. For internal nodes $(\mathbf{x}_i, t_n)$ to Q, the (explicit) stencil Ξ_i^n is given by

$$\Xi_i^n := \{(\mathbf{x}_i, t_n)\} \cup \{(\mathbf{x}_j, t_{n-1}) : \mathbf{x}_j \in \operatorname{supp}\phi_i\} \quad \forall\, 1 \leq i \leq I,\ 1 \leq n \leq N,$$

and so comprises all nodes that dictate the value U_i^n in (2.5). The *parabolic boundary* $\partial_p D$ of D is thus defined by

$$\partial_p D := \{(\mathbf{x}_i, t_n) \in \partial D : \Xi_i^n \not\subset D \text{ or } (\mathbf{x}_i, t_n) \in \partial_p Q\}.$$

Note that $\{U^n\}_{n=0}^N$ is completely determined in D from its values on $\partial_p D$, and that $\partial_p D$ is the *minimal* set where data must be prescribed.

Lemma 5.1. *Let* $\{\chi^n\}_{n=0}^N \subset \mathcal{V}_h$ *satisfy*

$$\langle \partial\chi^n, \phi_i\rangle_h + \langle \nabla\chi^{n-1}, \nabla\phi_i\rangle \leq 0 \quad \forall\, (\mathbf{x}_i, t_n) \in D \backslash \partial_p D. \tag{5.1}$$

If $\tau \leq C^* h^2$, *then* $\max_{(\mathbf{x}_i, t_n) \in D} \chi_i^n \leq \max_{(\mathbf{x}_i, t_n) \in \partial_p D} \chi_i^n$.

Proof. We first point out that (5.1) just relates values χ_i^n and χ_j^{n-1} for $(\mathbf{x}_j, t_{n-1}) \in \Xi_i^n \subset D$. On subctracting $\mu := \max\{\chi_i^n : (\mathbf{x}_i, t_n) \in \partial_p D\}$ from $\{\chi^n\}_{n=0}^N$, we can assume $\mu = 0$. We argue by induction and assume $\chi_i^{n-1} \leq 0$ for all $(\mathbf{x}_i, t_{n-1}) \in D$; we can actually start because $(\mathbf{x}_i, t_n) \in \partial_p D$ for the smallest possible n. For each $(\mathbf{x}_i, t_n) \in D \backslash \partial_p D$, we observe that (5.1) becomes

$$\chi_i^n \langle \tfrac{1}{\tau}, \phi_i\rangle_h = \tfrac{1}{\tau}\langle \chi^n, \phi_i\rangle_h \leq \tfrac{1}{\tau}\langle \chi^{n-1}, \phi_i\rangle_h - \langle \nabla\chi^{n-1}, \nabla\phi_i\rangle = (\phi_i)^T \mathbf{A}\chi^{n-1} \leq 0,$$

because $\chi_j^{n-1} \leq 0$ for all $(\mathbf{x}_j, t_{n-1}) \in \Xi_i^n \subset D$; hence $\chi_i^n \leq 0$ for all $(\mathbf{x}_i, t_n) \in D$. $\square$

This result still holds if $\partial_p D$ is replaced by a larger subset of D, e.g., by

$$\partial_p^e D := \{(\mathbf{x}_i, t_n) \in \partial D : \Omega_i^n \not\subset D \text{ or } (\mathbf{x}_i, t_n) \in \partial_p Q\}.$$

5.3. Discrete nondegeneracy. The discrete solution $U_{\varepsilon,h,\tau}$ detaches from the obstacles -1 and 1 with a prescribed quadratic growth. To make this precise, we need some further notation. For any $(\mathbf{x}, t) \in Q$ and $r > 0$, we consider the cylinder

$$Q_r(\mathbf{x}, t) := \{(\mathbf{y}, s) \in Q : |\mathbf{y} - \mathbf{x}| < r, 0 < t - s < r^2\}$$

and its discrete counterpart $Q_r^{h,\tau}(\mathbf{x}, t)$ of all sets Ω_i^n contained in $\bar{Q}_r(\mathbf{x}, t)$. For $r \geq Ah$ with $A \geq 1$ sufficiently large and h small, $Q_r^{h,\tau}(\mathbf{x}, t)$ is nonempty and either $|\mathbf{x} - \mathbf{y}|$ or $(t - s)^{1/2} \geq Cr$ for all $(\mathbf{y}, s) \in \partial_p Q_r^{h,\tau}(\mathbf{x}, t) \backslash \partial_p Q$. The next result is a consequence of Lemma 5.1 and is similar to [31, Lemma 6.1]. We give a proof for completeness.

Lemma 5.2. *There exist $\varepsilon^*, h^* > 0$ such that for all $h \le h^*$, $\tau \le C^* h^2$, $\varepsilon^* \ge \varepsilon \ge r \ge Ah$, and any node $(\mathbf{x}_j, t_m)$ satisfying $-1 < U_j^m \le -1/2$ (resp. $1/2 \le U_j^m < 1$) and either $t_m \ge r^2$ or $U_i^0 = -1$ (resp. $U_i^0 = 1$) for all nodes $\mathbf{x}_i$ such that $|\mathbf{x}_i - \mathbf{x}_j| < r$, there exists a node $(\mathbf{x}_i, t_n) \in Q_r^{h,\tau}(\mathbf{x}_j, t_m)$ such that*

$$U_i^n \ge U_j^m + C\tfrac{r^2}{\varepsilon^2} \quad (\textit{resp. } U_i^n \le U_j^m - C\tfrac{r^2}{\varepsilon^2}). \tag{5.2}$$

Proof. We only consider the case $-1 < U_j^m \le -1/2$, and assume $U_i^n \le -1/4$ for all nodes $(\mathbf{x}_i, t_n) \in Q_j^m := Q_r^{h,\tau}(\mathbf{x}_j, t_m)$ because otherwise (5.2) is trivial. We can take $\chi = U^n - (1 + U_i^n)\phi_i$ in (2.5) to obtain, for ε sufficiently small,

$$\langle \partial U^n, \phi_i\rangle_h + \langle \nabla U^{n-1}, \nabla \phi_i\rangle \le \tfrac{1}{\varepsilon^2}\langle U^{n-1} + \varepsilon\tfrac{\pi}{4} g^n, \phi_i\rangle_h \le -\tfrac{1}{8\varepsilon^2}\langle 1, \phi_i\rangle, \tag{5.3}$$

provided $U_i^n > -1$. Let $D \subset Q_j^m$ be the (closed) set containing $(\mathbf{x}_j, t_m)$ which is the connected union of all those Ω_i^n's of Q_j^m such that $U_i^n > -1$. The enlarged parabolic boundary $\partial_p^e D$ of D is made of those nodes $(\mathbf{x}_i, t_n) \in D$ such that $\Omega_i^n \not\subset D$ or $(\mathbf{x}_i, t_n) \in \partial_p Q$, the latter reducing to $t_n = 0$ because $\operatorname{dist}(\mathbf{x}_j, \partial\Omega) > r$. Consequently $(\mathbf{x}_i, t_n) \in \partial_p^e D$ satisfies either (a) $U_i^n > -1$ and $(\mathbf{x}_i, t_n) \in \partial_p Q_j^m \backslash \partial_p Q$ or (b) $U_i^n = -1$, which means that $(\mathbf{x}_i, t_n)$ belongs either to the discrete free boundary $\partial\{U^n > -1\}$ or the initial coincidence set $\{U^0 = -1\}$.

Let $\sigma(\cdot, t) \in \mathcal{V}_h$ be given by $\sigma(\mathbf{x}, t) := \Pi_h\big(|\mathbf{x} - \mathbf{x}_j|^2\big) + t_m - t$, and note that σ satisfies, for a constant $\beta > 0$ independent of h, τ, and $(\mathbf{x}_j, t_m)$ [6],

$$\langle \partial\sigma^n, \phi_i\rangle + \langle \nabla\sigma^{n-1}, \nabla\phi_i\rangle \ge -\beta\langle 1, \phi_i\rangle \quad \forall\ 1 \le i \le I,\ 1 \le n \le N. \tag{5.4}$$

Since $U_i^n > -1$ for all $(\mathbf{x}_i, t_n) \in D\backslash\partial_p^e D$, combining (5.3) with (5.4), the functions $\{S^n := U^n - \sigma^n/8\beta\varepsilon^2\}_{n=0}^N$ satisfy

$$\langle \partial S^n, \phi_i\rangle_h + \langle \nabla S^{n-1}, \nabla\phi_i\rangle \le 0 \quad \forall\ (\mathbf{x}_i, t_n) \in D\backslash\partial_p^e D.$$

Lemma 5.1 yields the existence of $(\mathbf{x}_i, t_n) \in \partial_p^e D$ such that $S_i^n \ge S_j^m = U_j^m > -1$. Since condition (b) above, namely $U_i^n = -1$, leads to the contradiction $S_i^n < -1$, then (a) applies, whence $\sigma_i^n \ge Cr^2$. This, combined with $U_i^n - \sigma_i^n/8\beta\varepsilon^2 = S_i^n \ge S_j^m = U_j^m$, yields (5.2), as desired. The case $1/2 \le U_j^m < 1$ is similar except that we cannot longer use $\operatorname{dist}(\mathbf{x}_j, \partial\Omega) > r$ but instead $U^n = 1$ on $\partial\Omega$. $\square$

5.4. Explicit parabolic projection with quadrature. The following *explicit* parabolic projection with *quadrature* in the time derivative will be essential in constructing discrete barriers in §6, but also bears some intrinsic interest. Given $y \in C^{0,1;0,1/2}(\bar{Q})$, $y = 0$ on $\partial\Omega \times (0, T)$, let $\{Y^n\}_{n=0}^N \subset \mathcal{V}_h(0)$ be defined by

$$\langle \partial Y^n, \chi\rangle_h + \langle \nabla Y^{n-1}, \nabla\chi\rangle = \langle \partial y^n, \chi\rangle + \langle \nabla\bar{y}^n, \nabla\chi\rangle \quad \forall\chi \in \mathcal{V}_h(0), \tag{5.5}$$

with $Y^0 := \Pi_h y^0$. Let $H_{h,\tau}^0$ be the parabolic projection operator defined by

$$H_{h,\tau}^0(y; \cdot, t) := \tfrac{t - t_{n-1}}{\tau} Y^n(\cdot) + \tfrac{t_n - t}{\tau} Y^{n-1}(\cdot) \quad \forall\ t_{n-1} < t \le t_n,\ 1 \le n \le N.$$

Continuity in time is immaterial except for Corollary 7.3.

The following error estimates extend the quasi-optimal L^∞ error estimates shown in [14], [31], [36] in 2D by allowing numerical integration. We also insist in minimal regularity requirements. It is worth stressing that this constitutes the sole restriction of our analysis to 2D. These estimates seem plausible in any space dimensions, as a recent result proven in [33] for caloric functions and without numerical integration suggests. Their extension is not immediate and thus requires further thought.

Theorem 5.1. *There exists $C^* > 0$ such that, if $\tau \le C^* h^2$ is of order h^2, then*

$$\|y - H^0_{h,\tau}(y)\|_{L^\infty(Q)} \le Ch|\log h|^3 \|y\|_{C^{0,1;0,1/2}(\bar{Q})}.$$

This estimate is quasi-optimal according to the required regularity on y, but does not take full advantage of polynomial interpolation. To derive an *optimal order* error estimate we have to demand stronger regularity on y than expected without numerical integration, thereby revealing the possible degrading effect of quadrature. In §9 we will show that

$$\begin{aligned} \|y - H_{h,\tau}(y)\|_{L^\infty(Q)} \le C|\log h|^3 \Big(\tau \|\partial_t y\|_{L^\infty(Q)} + \max_{0\le n\le N} \|y^n - \Pi_h y^n\|_{L^\infty(\Omega)} \\ + \max_{1\le n\le N} \|\partial y^n - \Pi_h \partial y^n\|_{L^1(\Omega)} + h^2 \max_{1\le n\le N} \|\nabla \Pi_h \partial y^n\|_{L^\infty(\Omega)}\Big). \end{aligned} \tag{5.6}$$

If we set

$$\mathcal{Y}(y) := \|\partial_t \partial_{\mathbf{x}} y\|_{L^2(0,T;L^\infty(\Omega))} + \|\partial_t \partial^2_{\mathbf{x}} y\|_{L^\infty(0,T;L^1(\Omega))} + \|y\|_{C^{1,1;0,1}(\bar{Q})},$$

then the following result improves upon [8] in terms of regularity.

Theorem 5.2. *There exists $C^* > 0$ such that, if $\tau \le C^* h^2$ is of order h^2, then*

$$\|y - H^0_{h,\tau}(y)\|_{L^\infty(Q)} \le Ch^2|\log h|^3 \mathcal{Y}(y).$$

Both Theorems 5.1 and 5.2 will be proven in §9. Here we derive two projection estimates for both solutions $u^\pm$ of (2.1) and classical barriers $\tilde{v}^\pm$, that will be instrumental in our interface error analyses in §§7 and 8. They are proven here in 2D, but will be assumed valid in the sequel for dimension $n \ge 2$.

Let $H_{h,\tau}$ denote the parabolic projection associated with the heat operator (5.5) and Dirichlet boundary condition 1: $H_{h,\tau}(1 + y) := 1 + H^0_{h,\tau}(y)$.

Corollary 5.1. *If u is a solution to* (2.1) *with forcing term of order at most ε^{-1}, and $\tau \le C^* h^2$ is of order h^2, then*

$$\|u - H_{h,\tau}(u)\|_{L^\infty(Q)} \le C\tfrac{h}{\varepsilon}|\log h|^3 |\log \varepsilon|.$$

Proof. Take $y = u - 1$ in Theorem 5.1 and use Lemma 4.1. □

Corollary 5.2. *Let $\tilde{v}^\pm$ be the classical barriers for* (2.1) *defined in* (4.6)*, and let $\tau \le C^* h^2$ be of order h^2 and $h \ge C\varepsilon^3$. Then, for as long as $\Sigma^\pm(t)$ are smooth,*

$$\|\tilde{v}^\pm - H_{h,\tau}(\tilde{v}^\pm)\|_{L^\infty(Q)} \le C\tfrac{h^2}{\varepsilon^2}|\log h|^3.$$

Proof. Since $\mathcal{Y}(\tilde{v}^\pm) = \infty$ because $\tilde{v}^\pm$ is only globally Lipschitz, and thus Theorem 5.2 does not apply directly, we argue directly with (5.6). Set $v = \tilde{v}^+$ and let $\mathcal{T}_\varepsilon := \{(\mathbf{x}, t) \in Q : \mathbf{x} \in \mathcal{T}_\varepsilon(t)\}$ be the space-time transition region, where $\mathcal{T}_\varepsilon(t) := \{|v(\cdot, t)| < 1\} = \{-\frac{\pi}{2}\varepsilon < d^+(\cdot, t) < x_\varepsilon \varepsilon\}$. We replace $y = v - 1$ into (5.6) and note that $\|\partial_t v\|_{L^\infty(Q)} \le C\varepsilon^{-1}$, but the remaining terms cannot be handled directly because the first derivatives of v may jump across the boundary of $\mathcal{T}_\varepsilon$. We then proceed elementwise and consider three cases according to whether or not a generic element K^n crosses $\partial\mathcal{T}_\varepsilon$. If K^n belongs to the complement of $\mathcal{T}_\varepsilon$, then it gives no contribution because $v = \pm 1$ is constant. If, instead, K^n belongs to $\bar{\mathcal{T}}_\varepsilon$, then v is smooth thereby leading to

$$\begin{aligned} \|v^n - \Pi_h v^n\|_{L^\infty(K)} &\le Ch^2 \|\partial^2_{\mathbf{x}} v\|_{L^\infty(K^n)} \le C\tfrac{h^2}{\varepsilon^2}, \\ \|\partial v^n - \Pi_h \partial v^n\|_{L^1(K)} &\le Ch^2 |K| \|\partial^2_{\mathbf{x}} \partial_t v\|_{L^\infty(K^n)} \le C\tfrac{h^2}{\varepsilon^3}|K|, \\ \|\nabla \Pi_h \partial v^n\|_{L^\infty(K)} &\le C\|\partial_{\mathbf{x}}\partial_t v\|_{L^\infty(K^n)} \le \tfrac{C}{\varepsilon^2}. \end{aligned}$$

Upon adding the second estimate on K, we derive the asserted upper bound because $|\mathcal{T}_\varepsilon(t)| = \mathcal{O}(\varepsilon)$ in the smooth regime. It remains to consider the elements K^n crossing $\partial\mathcal{T}_\varepsilon$, for which v may only be Lipschitz. For any point $(\mathbf{x}, t) \in K^n \cap \mathcal{T}_\varepsilon$, let $(\mathbf{y}, s) \in K^n \cap \partial\mathcal{T}_\varepsilon$ be a closest point to $(\mathbf{x}, t)$; hence $(\delta\mathbf{x} + (1-\delta)\mathbf{y}, t)$ and $(\mathbf{y}, \delta t + (1-\delta)s)$ belong to $K^n \cap \mathcal{T}_\varepsilon$ for all $0 < \delta \le 1$. Since v is smooth in $\mathcal{T}_\varepsilon$ and $\partial_x \tilde{\Gamma}^+(x_\varepsilon^-; \mathbf{y}, s) = \mathcal{O}(\varepsilon^2)$ (see right before (4.5)), we deduce

$$\begin{aligned} |\varphi(\mathbf{x}, t)| &\le |\varphi(\mathbf{y}, s)| + |\varphi(\mathbf{y}, t) - \varphi(\mathbf{y}, s)| + |\varphi(\mathbf{x}, t) - \varphi(\mathbf{y}, t)| \\ &\le C\varepsilon + C\tfrac{\tau}{\varepsilon^2} + C\tfrac{h}{\varepsilon^2} \le C\tfrac{h}{\varepsilon^2}, \end{aligned} \tag{5.7}$$

for both $\varphi := \partial_{\mathbf{x}} v$ and $\varphi := \partial_t v$, because $h \ge C\varepsilon^3$. Here $\varphi(\mathbf{y}, s)$ indicates the limit value of $\varphi(\mathbf{y}, \delta t + (1-\delta)s)$ as $\delta \downarrow 0$. Consequently,

$$\begin{aligned} \|v^n - \Pi_h v^n\|_{L^\infty(K)} &\le Ch \|\partial_{\mathbf{x}} v\|_{L^\infty(K^n)} \le C\tfrac{h^2}{\varepsilon^2}, \\ \|\partial v^n - \Pi_h \partial v^n\|_{L^1(K)} &\le 2|K| \|\partial_t v\|_{L^\infty(K^n)} \le C|K|\tfrac{h}{\varepsilon^2}, \\ \|\nabla \Pi_h \partial v^n\|_{L^\infty(K)} &\le \tfrac{C}{h}\|\partial_t v\|_{L^\infty(K^n)} \le \tfrac{C}{\varepsilon^2}. \end{aligned}$$

Adding the second estimate over K, and exploiting the fact that the resulting strip has measure $\mathcal{O}(h)$, yields the desired estimates, thereby concluding the proof. □

6. Fully discrete barriers

We now construct fully discrete supersolutions $V^+_{\varepsilon,h,\tau}$ and $\tilde{V}^+_{\varepsilon,h,\tau}$ to (2.5) and (2.7), respectively. Let u^+ be the solution to (2.1) with perturbed forcing term $g + \alpha$ and initial datum $\gamma(d_0/\varepsilon)$, and let $\tilde{v}^+$ be the supersolution defined in (4.6). Let $V^+_{\varepsilon,h,\tau}$ and $\tilde{V}^+_{\varepsilon,h,\tau}$ be defined as follows

$$V^+_{\varepsilon,h,\tau} := H_{h,\tau}(u^+) + \Lambda\, \xi(\varepsilon, h, \tau), \quad \tilde{V}^+_{\varepsilon,h,\tau} := H_{h,\tau}(\tilde{v}^+) + \Lambda\, \tilde{\xi}(\varepsilon, h, \tau), \tag{6.1}$$

with $\Lambda \ge 1$ to be chosen later. The expressions

$$\begin{aligned} \xi(\varepsilon, h, \tau) &:= \|u^+ - H_{h,\tau}(u^+)\|_{L^\infty(Q)} \le C\tfrac{h}{\varepsilon}|\log h|^3 |\log \varepsilon|, \\ \tilde{\xi}(\varepsilon, h, \tau) &:= \|\tilde{v}^+ - H_{h,\tau}(\tilde{v}^+)\|_{L^\infty(Q)} \le C\tfrac{h^2}{\varepsilon^2}|\log h|^3, \end{aligned}$$

valid for $\tau \le C^* h^2$ of order h^2, result from Corollaries 5.1 and 5.2.

Lemma 6.1. *If Λ is sufficiently large and $\alpha \geq 8\Lambda\xi/\pi\varepsilon$, then $V^+_{\varepsilon,h,\tau}$ is a supersolution of* (2.5)*, namely $V^+_{\varepsilon,h,\tau} \geq -1$ and, for all nodes $\mathbf{x}_i$ such that $V^{+,n}_{\varepsilon,h,\tau}(\mathbf{x}_i) < 1$,*

$$\varepsilon\langle \partial V^{+,n}_{\varepsilon,h,\tau}, \phi_i\rangle_h + \varepsilon\langle \nabla V^{+,n-1}_{\varepsilon,h,\tau}, \nabla\phi_i\rangle - \tfrac{1}{\varepsilon}\langle V^{+,n-1}_{\varepsilon,h,\tau}, \phi_i\rangle_h \geq \tfrac{\pi}{4}\langle g^n, \phi_i\rangle_h. \tag{6.2}$$

Proof. Since no confusion is possible we set $u = u^+$, $V = V^+_{\varepsilon,h,\tau}$, $H = H_{h,\tau}(u)$, and $\xi = \xi(\varepsilon, h, \tau)$. In view of Corollary 5.1 and (6.1), we see that, for all $(\mathbf{x}, t) \in Q$,

$$-1 \leq u \leq H + \xi = V - (\Lambda - 1)\xi \leq V \quad \forall\ \Lambda \geq 1. \tag{6.3}$$

If $V_i^n < 1$, there exists $\mu_i^n < 1$ such that $u(\mathbf{x}, t) \leq \mu_i^n$ for all $(\mathbf{x}, t) \in \Omega_i^n$. In fact, for Λ sufficiently large, (6.3) and Lemma 4.1 yield

$$u(\mathbf{x}, t) = u(\mathbf{x}_i, t_n) + \big(u(\mathbf{x}, t_n) - u(\mathbf{x}_i, t_n)\big) + \big(u(\mathbf{x}, t)) - u(\mathbf{x}, t_n)\big) < 1 - (\Lambda - 1 - C)\xi < 1.$$

Taking $\varphi = u + (1 - \mu_i^n)\phi_i \in \mathcal{K}$ in (2.1), with g replaced by $g + \alpha$, leads to

$$\varepsilon\langle \partial_t u, \phi_i\rangle + \varepsilon\langle \nabla u, \nabla\phi_i\rangle - \tfrac{1}{\varepsilon}\langle u, \phi_i\rangle \geq \tfrac{\pi}{4}\langle g + \alpha, \phi_i\rangle \quad \text{for a.e. } t_{n-1} < t < t_n,$$

whence, integrating on (t_{n-1}, t_n) and making use of (5.5) with $y = u - 1$, we get

$$\varepsilon\langle \partial H^n, \phi_i\rangle_h + \varepsilon\langle \nabla H^{n-1}, \nabla\phi_i\rangle \geq \tfrac{1}{\varepsilon}\langle \bar{u}^n, \phi_i\rangle + \tfrac{\pi}{4}\langle \bar{g}^n + \alpha, \phi_i\rangle.$$

Applying Corollary 5.1, (2.3) for $p = 1$, and Lemma 4.1 we find that

$$\begin{aligned}
&\varepsilon\langle \partial V^n, \phi_i\rangle_h + \varepsilon\langle \nabla V^{n-1}, \nabla\phi_i\rangle - \tfrac{1}{\varepsilon}\langle V^{n-1}, \phi_i\rangle_h - \tfrac{\pi}{4}\langle g^n, \phi_i\rangle_h \\
&\quad \geq \tfrac{1}{\varepsilon}\langle \bar{u}^n - V^{n-1}, \phi_i\rangle_h + \tfrac{1}{\varepsilon}\big(\langle \Pi_h\bar{u}^n, \phi_i\rangle - \langle \Pi_h\bar{u}^n, \phi_i\rangle_h\big) + \tfrac{1}{\varepsilon}\langle \bar{u}^n - \Pi_h\bar{u}^n, \phi_i\rangle \\
&\qquad + \tfrac{\pi}{4}\big(\langle \Pi_h g^n, \phi_i\rangle - \langle \Pi_h g^n, \phi_i\rangle_h\big) + \tfrac{\pi}{4}\langle \bar{g}^n - g^n, \phi_i\rangle + \tfrac{\pi}{4}\langle g^n - \Pi_h g^n, \phi_i\rangle + \tfrac{\pi}{4}\langle \alpha, \phi_i\rangle \\
&\quad \geq \langle -(\Lambda + 2)\tfrac{\xi}{\varepsilon} - C(\tfrac{h}{\varepsilon^2}|\log\varepsilon| + h + \tau) + \tfrac{\pi}{4}\alpha, \phi_i\rangle \geq \langle -2\Lambda\tfrac{\xi}{\varepsilon} + \tfrac{\pi}{4}\alpha, \phi_i\rangle \geq 0,
\end{aligned}$$

provided $\alpha \geq 8\Lambda\xi/\pi\varepsilon = Ch\varepsilon^{-2}|\log h|^3|\log\varepsilon|$. This implies the assertion. □

Lemma 6.2. *Let $\Sigma^+(t)$ be smooth for $0 \leq t \leq T$. If $\Lambda \gg 1$, $\alpha \geq 8\Lambda\tilde{\xi}/\pi\varepsilon$, and $h \geq C\varepsilon^3$, then $\tilde{V}^+_{\varepsilon,h,\tau}$ is a supersolution of* (2.7)*, that is $\tilde{V}^+_{\varepsilon,h,\tau} \geq -1$ and, for all nodes $\mathbf{x}_i$ such that $\tilde{V}^{+,n}_{\varepsilon,h,\tau}(\mathbf{x}_i) < 1$,*

$$\varepsilon\langle \partial\tilde{V}^{+,n}_{\varepsilon,h,\tau}, \phi_i\rangle_h + \varepsilon\langle \nabla\tilde{V}^{+,n-1}_{\varepsilon,h,\tau}, \nabla\phi_i\rangle - \tfrac{1}{\varepsilon}\langle \tilde{V}^{+,n-1}_{\varepsilon,h,\tau}, \phi_i\rangle \geq \tfrac{\pi}{4}\langle g^n, \phi_i\rangle_h. \tag{6.4}$$

Proof. We set $v = \tilde{v}^+$ and $V = \tilde{V}^+_{\varepsilon,h,\tau}$ and argue as in Lemma 6.1. We claim that if $V_i^n < 1$, then $v(\mathbf{x}, t) < 1$ for all points $(\mathbf{x}, t) \in \Omega_i^n$. To see this, assume by contradiction that Ω_i^n intersects $\partial\mathcal{T}_\varepsilon$ and let $(\mathbf{y}, s) \in \Omega_i^n \cap \partial\mathcal{T}_\varepsilon$ be a closest point to $(\mathbf{x}_i, t_n)$ where $v(\mathbf{y}, s) = 1$. From (5.7) we thus have $|\partial_t v|, |\partial_{\mathbf{x}} v| \leq Ch\varepsilon^{-2}$ in Ω_i^n and

$$v(\mathbf{x}_i, t_n) = v(\mathbf{y}, s) + \big(v(\mathbf{y}, t_n) - v(\mathbf{y}, s)\big) + \big(v(\mathbf{x}_i, t_n) - v(\mathbf{y}, t_n)\big) \geq 1 - C\tfrac{h}{\varepsilon^2}(\tau + h),$$

which contradicts $v(\mathbf{x}_i, t_n) < 1 - (\Lambda - 1)\tilde{\xi}$ for Λ sufficiently large. The rest of the proof is the same as that of Lemma 6.1, with the only difference that the quadrature error $h\varepsilon^{-2}|\log\varepsilon|$ for the potential is no longer present. □

We denote by $U_{\varepsilon,h,\tau}$ the continuous piecewise linear function in space and time determined from the discrete solution $\{U^n\}_{n=0}^N$ of (2.5) or (2.7) by

$$U_{\varepsilon,h,\tau}(\cdot, t) := \tfrac{t - t_{n-1}}{\tau}U^n(\cdot) + \tfrac{t_n - t}{\tau}U^{n-1}(\cdot) \quad \forall\ t_{n-1} < t \leq t_n,\ 1 \leq n \leq N.$$

Lemma 6.3. *Under the assumptions of* Lemma 6.1 *(resp.* Lemma 6.2*) there holds*

$$U_{\varepsilon,h,\tau} \le V^+_{\varepsilon,h,\tau} \quad (\textit{resp. } U_{\varepsilon,h,\tau} \le \tilde V^+_{\varepsilon,h,\tau}) \quad \textit{in } Q.$$

Proof. For simplicity we set $U = U_{\varepsilon,h,\tau}$ and $V = V^+_{\varepsilon,h,\tau}$. The first inequality is certainly true on $\partial\Omega$, where $V^n = 1 + \Lambda\xi$, and for $n = 0$, because $V^0 = U^0 + \Lambda\xi$. We assume $U^{n-1} \le V^{n-1}$ and argue by induction. If $U^n_i = -1$ or $V^n_i = 1$, then obviously $U^n_i \le V^n_i$. Instead, if $U^n_i > -1$ and $V^n_i < 1$, then from (2.5) we obtain

$$\varepsilon\langle \partial U^n, \phi_i\rangle_h + \varepsilon\langle \nabla U^{n-1}, \nabla\phi_i\rangle - \tfrac{1}{\varepsilon}\langle U^{n-1}, \phi_i\rangle_h \le \tfrac{\pi}{4}\langle g^n, \phi_i\rangle_h, \tag{6.5}$$

along with (6.2). Subtracting (6.2) from (6.5), and reordering, we get

$$\langle \partial(U^n - V^n), \phi_i\rangle_h + \langle \nabla(U^{n-1} - V^{n-1}), \nabla\phi_i\rangle \le \tfrac{1}{\varepsilon^2}\langle U^{n-1} - V^{n-1}, \phi_i\rangle_h \le 0.$$

On invoking Lemma 5.1 we thus deduce $U^n \le V^n$, as desired. Since numerical integration on the potentail plays no role in this argument, the result for $\tilde V^+_{\varepsilon,h,\tau}$ follows from $\tilde V^+_{\varepsilon,h,\tau}(\cdot,0) \ge U^0(\cdot) + \Lambda\tilde\xi$ and the use of (2.7) and (6.4). $\square$

The fully discrete subsolutions $V^-_{\varepsilon,h,\tau} \le U_{\varepsilon,h,\tau}$ and $\tilde V^-_{\varepsilon,h,\tau} \le U_{\varepsilon,h,\tau}$ are defined as

$$V^-_{\varepsilon,h,\tau} := H_{h,\tau}(u^-) - \Lambda\ \xi(\varepsilon,h,\tau), \quad \tilde V^-_{\varepsilon,h,\tau} := H_{h,\tau}(\tilde v^-) - \Lambda\ \tilde\xi(\varepsilon,h,\tau).$$

7. Error analysis past singularities

Let $\Sigma_{\varepsilon,h,\tau}$ indicate the discrete interface and $\mathcal{T}_{\varepsilon,h,\tau}$ the discrete transition region (or noncoincidence set)

$$\Sigma_{\varepsilon,h,\tau}(t) := \{\mathbf{x} \in \Omega : U_{\varepsilon,h,\tau}(\mathbf{x},t) = 0\}, \quad \mathcal{T}_{\varepsilon,h,\tau}(t) := \big\{\mathbf{x} \in \Omega : |U_{\varepsilon,h,\tau}(\mathbf{x},t)| < 1\big\}.$$

Let $\mathrm{thick}(\mathcal{T}_{\varepsilon,h,\tau}(t); \mathbf{x}, \mathbf{n})$ denote the thickness of $\mathcal{T}_{\varepsilon,h,\tau}(t)$ in the direction $\mathbf{n}$ across $\mathbf{x} \in \mathcal{T}_{\varepsilon,h,\tau}(t)$. In this section we establish that both $\Sigma_{\varepsilon,h,\tau}(t)$ and $\mathcal{T}_{\varepsilon,h,\tau}(t)$ converge to $\Sigma(t)$ as $\varepsilon, h, \tau \downarrow 0$, provided $\Sigma(t)$ does not develops interior. We also prove interface and layer thickness estimates for both regular and type I singular points.

Corollary 7.1 (*convergence past singularities*)**.** *Let* $h = o(\varepsilon^2|\log\varepsilon|^{-4})$ *and* $\tau \le C^* h^2$ *be of order* h^2. *Let* $\mathbf{x} \in I(t)$ *(resp.* $O(t)$*). There exists* ε^* *depending on* $(\mathbf{x}, t)$ *such that*

$$U_{\varepsilon,h,\tau}(\mathbf{x},t) = -1 \quad (\textit{resp. } U_{\varepsilon,h,\tau}(\mathbf{x},t) = 1) \quad \forall\ \varepsilon \le \varepsilon^*.$$

Proof. If $\mathbf{x} \in I(t)$, then $w(\mathbf{x},t) < 0$. If also $\alpha = o(1)$, then $w^+(\mathbf{x},t) < 0$ because of the continuous dependence of viscosity solutions with respect to perturbations of data. Since w^+ is continuous, there exists $r > 0$ such that

$$w^+(\mathbf{y},s) < 0 \quad \forall\ |\mathbf{y} - \mathbf{x}|, |t - s|^{1/2} < r;$$

this parabolic ball is denoted by $B_r(\mathbf{x},t)$. We recall that $d^+(\mathbf{y},s)$ stands for $\mathrm{dist}(\mathbf{y}, \partial\{w^+(\cdot,s) < 0\})$ and coincides with the signed distance to $\Sigma^+(s)$, provided

$\Sigma^+(s)$ does not develop interior. We infer that $d^+(\mathbf{y},s)<0$, and from Lemma 4.2 and properties of Γ^+ that

$$-1 \le u^+(\mathbf{y},s) \le v^+(\mathbf{y},s) = \Gamma^+\big(\tfrac{d^+(\mathbf{y},s)}{\varepsilon} - \tfrac{\pi}{2}; \mathbf{y}, s\big) = -1 \quad \forall\ (\mathbf{y},s)\in B_r(\mathbf{x},t).$$

If $\alpha = 8\Lambda\xi/\pi\varepsilon = o(1)$ for Λ large, Lemma 6.3 and Corollary 5.1 yield

$$U_{\varepsilon,h,\tau} \le V^+_{\varepsilon,h,\tau} \le u^+ + (\Lambda+1)\xi \le -1 + C\tfrac{h}{\varepsilon}|\log h|^3|\log\varepsilon| \le -\tfrac12$$

in $B_r(\mathbf{x},t)$. If $U_{\varepsilon,h,\tau}(\mathbf{x},t) > -1$, then Lemma 5.2 yields $U_{\varepsilon,h,\tau}(\mathbf{y},s) > -1 + Cr^2\varepsilon^{-2}$ for some $(\mathbf{y},s)\in B_r(\mathbf{x},t)$. We arrive at a contradiction on choosing h and ε satisfying $r \ge R(h\varepsilon)^{1/2}|\log h|^{3/2}|\log\varepsilon|^{1/2}$ for R large. This shows that $U_{\varepsilon,h,\tau}(\mathbf{x},t) = -1$, provided $h = o(\varepsilon^2|\log\varepsilon|^{-4})$. $\square$

Corollary 7.1 shows convergence of $\Sigma_\varepsilon(t)$ to $\Sigma(t)$ whenever the latter does not develop interior. Without further assumptions on the flow, convergence of interfaces is the best we can expect. The next two improvements rely on the precise behavior of $w(\mathbf{y},s)$ in a vicinity of $(\mathbf{x},t)\in\Sigma$.

Corollary 7.2 (*error estimates for regular points*)**.** *Let $h = \mathcal{O}(\varepsilon^3|\log\varepsilon|^{-4})$ and $\tau \le C^*h^2$ be of order h^2. Let $\Sigma^\sharp$ be a compact subset of Σ^*. There exists ε^* depending on the smallest value of $|\nabla w|$ in $\Sigma^\sharp$ but not on $(\mathbf{x},t)$ such that*

$$\operatorname{dist}\big(\mathbf{x},\Sigma_{\varepsilon,h,\tau}(t)\big),\ \operatorname{thick}\big(\mathcal{T}_{\varepsilon,h,\tau}(t);\mathbf{x},\tfrac{\nabla w(\mathbf{x},t)}{|\nabla w(\mathbf{x},t)|}\big) \le \tfrac{C}{|\nabla w(\mathbf{x},t)|}\varepsilon \quad \forall\ (\mathbf{x},t)\in\Sigma^\sharp,\ \varepsilon\le\varepsilon^*.$$

Proof. We simply use the fact that w is C^1 and expand it in the direction $\mathbf{n} := \nabla w(\mathbf{x},t)/|\nabla w(\mathbf{x},t)|$ to obtain

$$w(\mathbf{x}-\delta\mathbf{n},t) = w(\mathbf{x},t) - \delta\nabla w(\mathbf{x},t)\cdot\mathbf{n} + o(\delta) \le -\tfrac12\delta|\nabla w(\mathbf{x},t))| \quad \forall\ 0<\delta\ll 1,$$

because $w(\mathbf{x},t)=0$. We take $\alpha = C\varepsilon \ge 8\Lambda\xi/\pi\varepsilon$ with $C\ge1$ and use (3.8) to get

$$w^+(\mathbf{x}-\delta\mathbf{n},t) \le w(\mathbf{x}-\delta\mathbf{n},t) + CG\varepsilon \le -\tfrac12\delta|\nabla w(\mathbf{x},t)| + CG\varepsilon = -\varepsilon,$$

provided $\delta = 2(CG+1)\varepsilon/|\nabla w(\mathbf{x},t)| \le 2(CG+1)\varepsilon/W$, where W is a lower bound for $|\nabla w|$ in $\Sigma^\sharp$. With the aid of (3.6), we see that $w^+<0$ in $B_r(\mathbf{x}-\delta\mathbf{n},t)$ for all $r\le C\varepsilon$. We then make use of Lemma 5.2, as in Corollary 7.1, to prove $U_{\varepsilon,h,\tau}(\mathbf{x}-\delta\mathbf{n},t) = -1$. This yields the assertion. $\square$

We infer that both the distance between $\mathbf{x}$ and $\Sigma_{\varepsilon,h,\tau}(t)$ and the local thickness of $\mathcal{T}_{\varepsilon,h,\tau}(t)$ are $\mathcal{O}(\varepsilon)$ for regular points. The local structure of the double obstacle formulation (2.1) is thus retained by the fully discrete problem (2.5), a key computational goal. The occurrence of singularities, and related loss of error control, does not affect this linear rate of convergence, which happens to be valid irrespective of whether $(\mathbf{x},t)$ lies before or in between singularities. This assertion is rather unexpected but in the same spirit of (3.9).

Corollary 7.3 (*error estimates for type I singular points*). *Let* $h = \mathcal{O}(\varepsilon^3|\log\varepsilon|^{-4})$ *and* $\tau \le C^* h^2$ *be of order* h^2. *Let* $\mathbf{x} \in \Sigma(t)$ *satisfy* (3.4) *with* $\beta, \lambda > 0$. *There exists* ε^* *depending on* $(\mathbf{x}, t)$ *such that, for all* $\varepsilon \le \varepsilon^*$,

$$U_{\varepsilon,h,\tau}(\mathbf{x}+\mathbf{y}, t \pm s) = \pm 1 \quad \textit{for all } C\tfrac{\varepsilon}{\beta} \le s \le C\tfrac{\varepsilon^*}{\beta},\ |\mathbf{y}|^2 < C\tfrac{\beta}{\lambda}s;$$
$$\textit{there exists } |s| \le C\tfrac{\varepsilon}{\beta} \textit{ for which } \mathbf{x} \in \Sigma_{\varepsilon,h,\tau}(t+s);$$
$$\mathrm{thick}\big(\mathcal{T}_{\varepsilon,h,\tau}(t); \mathbf{x}, \mathbf{e}_i\big) \le C\big(\tfrac{\varepsilon}{\lambda}\big)^{1/2} \quad \textit{for all } 1 \le i \le k.$$

Proof. From (3.4) and (3.8) we can write for $\alpha = C\varepsilon$ and $C \ge 1$ large,

$$w^+(\mathbf{x}+\mathbf{y}, t-s) \le -\beta s + \lambda|\mathbf{y}|^2 + o(s) + CG\varepsilon \le -\tfrac{\beta}{4}s + CG\varepsilon \le -\varepsilon,$$

provided $s \ge 4(CG+1)\varepsilon/\beta$ and $|\mathbf{y}|^2 \le \beta s/2\lambda$. The argument continues as in Corollary 7.2 to derive $U_{\varepsilon,h,\tau}(\mathbf{x}+\mathbf{y}, t-s) = -1$ and $U_{\varepsilon,h,\tau}(\mathbf{x}+\mathbf{y}, t+s) = 1$. Since

$$w^-(\mathbf{x}+\mathbf{y}, t) \ge w(\mathbf{x}+\mathbf{y}, t) - CG\varepsilon = \lambda|\mathbf{y}|^2 + o(|\mathbf{y}|^2) - CG\varepsilon \ge \tfrac{\lambda}{2}|\mathbf{y}|^2 - CG\varepsilon = \varepsilon,$$

for $|\mathbf{y}|^2 = 2(CG+1)\varepsilon/\lambda$, use Lemma 5.2 once more to get $U_{\varepsilon,h,\tau}(\mathbf{x}+\mathbf{y}, t) = 1$. $\square$

This result partially explains why it is feasible to approximate the time at which type I singularities occur, and in particular the extinction time. It also quantifies a smearing effect observed in practice in the vicinity of singularities, which gives rise to a much more diffuse transition layer. To enhance resolution we may think of reducing locally the size of ε, along with that of h and τ, which would then become space-time dependent. This simple idea turns out to be extremely successful computationally [24], [28], [29], [34], but the ensuing fully discrete method is yet to be studied.

8. Error analysis for smooth flow

In view of extra regularity of the continuous barriers $\tilde{v}^\pm$ of §4, not necessarily valid for general solutions of (2.1), refined results for the smooth regime can be derived. Let $U_{\varepsilon,h,\tau}$ be the discrete solution of scheme (2.7). We show that the Hausdorff distance between $\Sigma(t)$ and $\Sigma_{\varepsilon,h,\tau}(t)$ is $\mathcal{O}(\varepsilon^2)$, which is a fully discrete version of the quadratic rate of convergence of [25]–[27].

Corollary 8.1 (*quadratic estimates for smooth flow*). *Let* $h = \mathcal{O}(\varepsilon^{5/2}|\log\varepsilon|^{-2})$ *and* $\tau \le C^* h^2$ *be of order* h^2. *Let* $\Sigma(t)$ *and* $\Sigma^\pm(t)$ *be smooth for all* $t \in [0, T]$. *There exists* ε^* *such that*

$$\mathrm{dist}_H\big(\Sigma(t), \Sigma_{\varepsilon,h,\tau}(t)\big) \le C\varepsilon^2 \quad \forall\ t \in [0,T],\ \varepsilon \le \varepsilon^*.$$

Proof. Given $\varepsilon > 0$, let $\alpha = \varepsilon^2 \ge 8\Lambda\tilde{\xi}/\pi\varepsilon$, $\tau \le C^* h^2$ be of order h^2, and $h \ge C\varepsilon^3$, be enforced for Lemmas 4.3 and 6.3 and Corollary 5.2 to hold. Given $\mathbf{x} \in \Sigma(t)$ and $\mathbf{n} := \nabla d(\mathbf{x}, t)$, by virtue of (3.12) we infer that

$$d^+(\mathbf{x} - \delta\mathbf{n}, t) \le d(\mathbf{x} - \delta\mathbf{n}, t) + \tfrac{2G}{W}\varepsilon^2 \le -\tfrac{1}{2}\delta + \tfrac{2G}{W}\varepsilon^2 = -\varepsilon^2,$$

provided $\delta = (2+4G/W)\varepsilon^2$. In light of definitions (4.2) and (4.6), and property $\partial_x\Gamma^{\pm}(0;\mathbf{x},t) > 1/2$, we deduce that

$$\tilde{v}^+(\mathbf{x}-\delta\mathbf{n},t) = \Gamma^{\pm}\big(\tfrac{d^+(\mathbf{x}-\delta\mathbf{n},t)}{\varepsilon};\mathbf{x}-\delta\mathbf{n},t\big) \le -\tfrac{\varepsilon}{4}.$$

Invoking Corollary 5.2 and Lemma 6.3 we obtain the asymptotic relation

$$U_{\varepsilon,h,\tau}(\mathbf{x}-\delta\mathbf{n},t) \le \tilde{V}^+_{\varepsilon,h,\tau}(\mathbf{x}-\delta\mathbf{n},t) \le -\tfrac{\varepsilon}{4} + C\tfrac{h^2}{\varepsilon^2}|\log h|^3 < 0.$$

The assertion is thus proven because $\delta = \mathcal{O}(\varepsilon^2)$. $\square$

9. Proof of Theorems 5.1 and 5.2

Both Theorems 5.1 and 5.2 are based on a parabolic duality argument [14], [23], [36]; see (9.11) below. It is worth stressing that all proofs below apply, and in fact are even simpler, for *implicit* time-stepping. Therefore Theorems 5.1 and 5.2 are also valid in such a case.

We start by recalling an interpolation estimate valid for $s=1$ and $2<r\le\infty$ or $s=2$ and $1\le r\le\infty$ [11, p.122], [5, p.104]

$$\|\varphi-\Pi_h\varphi\|_{L^r(\Omega)} + h\|\nabla(\varphi-\Pi_h\varphi)\|_{L^r(\Omega)} \le Ch^s|\varphi|_{W^s_r(\Omega)}. \tag{9.1}$$

Let R_h be the $H^1_0(\Omega)$ projection onto $\mathcal{V}_h(0)$, namely

$$\varphi\in H^1_0(\Omega), R_h\varphi\in\mathcal{V}_h(0): \quad \langle\nabla(R_h\varphi-\varphi),\nabla\chi\rangle = 0 \quad \forall\,\chi\in\mathcal{V}_h(0). \tag{9.2}$$

The following quasi-optimal pointwise error estimate is valid for R_h [37]

$$\|\varphi-R_h\varphi\|_{L^\infty(\Omega)} \le C|\log h|\|\varphi-\Pi_h\varphi\|_{L^\infty(\Omega)}. \tag{9.3}$$

With the notation

$$\rho^n := Y^n - R_h\bar{y}^{n+1}, \quad \eta^n := y^n - R_h\bar{y}^{n+1},$$

and the by-product of (9.3) and (9.1) for $r=\infty$ and $s=1,2$, respectively,

$$\|\eta^n\|_{L^\infty(\Omega)} \le C\begin{cases} (h+\tau^{1/2})|\log h|\|y\|_{C^{0,1;0,1/2}(\bar{Q})} \\ (h^2+\tau)|\log h|\|y\|_{C^{1,1;0,1}(\bar{Q})}, \end{cases} \tag{9.4}$$

we realize that it only remains to estimate the error ρ^n. We use weighted L^2-based estimates as an intermediate step in deriving pointwise bounds. Nitsche's weight function for a generic $\mathbf{x}_0\in\Omega$ reads [5], [11], [22]

$$\sigma(\mathbf{x}) := \big(|\mathbf{x}-\mathbf{x}_0|^2+h^2\big)^{1/2} \quad \forall\,\mathbf{x}\in\Omega, \tag{9.5}$$

and satisfies, independently of $\mathbf{x}_0$,

$$\begin{gathered} \int_\Omega \sigma^{-s}(\mathbf{x})d\mathbf{x} \le \tfrac{C}{s-2}h^{2-s} \text{ as } s\downarrow 2, \quad \int_\Omega \sigma^{-2}(\mathbf{x})d\mathbf{x} \le C|\log h|, \\ |D^s\sigma^2(\mathbf{x})| \le 2\sigma^{2-s}(\mathbf{x}) \quad \forall\,\mathbf{x}\in\Omega,\ s=1,2, \\ \max_K\sigma \le C\min_K\sigma \quad \forall\,K\in\mathcal{D}_h. \end{gathered} \tag{9.6}$$

This, together with the elementwise inverse inequality $\|\chi\|_{L^\infty(K)} \le Ch^{-1}\|\chi\|_{L^2(K)}$ for all $\chi \in \mathcal{V}_h$, yields the weighted inverse estimate $\|\sigma^s\nabla\chi\|_{L^2(\Omega)} \le Ch^{-1}\|\sigma^s\chi\|_{L^2(\Omega)}$ for $s \in \mathbb{R}$, links weighted and maximum norms [5], [11], [22]

$$\|\sigma^{-1}\varphi\|_{L^2(\Omega)} \le C|\log h|^{1/2}\|\varphi\|_{L^\infty(\Omega)} \quad \forall\, \varphi \in L^\infty(\Omega), \tag{9.7}$$

$$\|\chi\|_{L^\infty(\Omega)} = |\chi(\mathbf{x}_0)| \le C\|\sigma^{-1}\chi\|_{L^2(\Omega)} \quad \forall\, \chi \in \mathcal{V}_h, \tag{9.8}$$

and provides the equivalence of discrete and continuous weighted norms

$$\|\sigma^s\chi\|_{L^2(\Omega)} \le \|\sigma^s\chi\|_h \le C\|\sigma^s\chi\|_{L^2(\Omega)} \quad \forall\, \chi \in \mathcal{V}_h,\ |s| = 1. \tag{9.9}$$

In view of (9.2) we are able to rewrite (5.5) as an equation for $\rho^n \in \mathcal{V}_h(0)$,

$$\begin{aligned} \langle \rho^n - \rho^{n-1}, \chi\rangle_h + \tau\langle\nabla\rho^{n-1}, \nabla\chi\rangle &= \langle \eta^n - \eta^{n-1}, \chi\rangle_h \\ &+ \big(\langle y^n - y^{n-1}, \chi\rangle - \langle y^n - y^{n-1}, \chi\rangle_h\big) \quad \forall\, \chi \in \mathcal{V}_h(0). \end{aligned} \tag{9.10}$$

This *error equation* suggests considering the following (backward and explicit) *dual discrete parabolic problem* with mass lumping and terminal datum $Z^m := \zeta \in \mathcal{V}_h(0)$ for $1 \le m \le N$ [14], [23]: for any $m \ge n \ge 1$ find $Z^{n-1} \in \mathcal{V}_h(0)$ satisfying

$$\langle Z^{n-1} - Z^n, \chi\rangle_h + \tau\langle\nabla Z^n, \nabla\chi\rangle = 0 \quad \forall\, \chi \in \mathcal{V}_h(0). \tag{9.11}$$

9.1. Proof of Theorem 5.1. We take $\chi = Z^n \in \mathcal{V}_h(0)$ in (9.10), sum by parts from $n = 1$ to $n = m$, and use (9.11) and the fact that $\rho^n \in \mathcal{V}_h(0)$, to arrive at

$$\begin{aligned} \langle \rho^m, \zeta\rangle_h &= \langle \eta^m, \zeta\rangle_h + \big(\langle y^m, \zeta\rangle - \langle y^m, \zeta\rangle_h\big) \\ &+ \langle Y^0 - y^0, Z^0\rangle_h - \big(\langle y^0, Z^0\rangle - \langle y^0, Z^0\rangle_h\big) \\ &+ \sum_{n=1}^m \langle \eta^{n-1}, Z^{n-1} - Z^n\rangle_h \\ &+ \sum_{n=1}^m \big(\langle y^{n-1}, Z^{n-1} - Z^n\rangle - \langle y^{n-1}, Z^{n-1} - Z^n\rangle_h\big). \end{aligned} \tag{9.12}$$

In order to get an L^∞ bound for ρ^m we thus choose

$$\zeta := \Pi_h(\sigma^{-2}\rho^m)$$

in (9.12) and examine each resulting contribution separately. First, (9.9) yields

$$\langle \rho^m, \zeta\rangle_h = \|\sigma^{-1}\rho^m\|_h^2 \ge \|\sigma^{-1}\rho^m\|_{L^2(\Omega)}^2.$$

Noting that $\|\sigma\zeta\|_h = \|\sigma^{-1}\rho^m\|_h$, in light of (9.4), (9.7) for $\|\cdot\|_h$, and (9.9) we get

$$\begin{aligned} \langle \eta^m, \zeta\rangle_h &\le \tfrac{C}{\delta}\|\sigma^{-1}\eta^m\|_h^2 + \delta\|\sigma^{-1}\rho^m\|_{L^2(\Omega)}^2 \\ &\le \tfrac{C}{\delta}(h^2+\tau)|\log h|^3\|y\|_{C^{0,1;0,1/2}(\bar{Q})}^2 + \delta\|\sigma^{-1}\rho^m\|_{L^2(\Omega)}^2, \end{aligned}$$

for $\delta > 0$ to be selected. For the next term in (9.12) we add and subtract $\langle \Pi_h y^m, \zeta \rangle$. In view of (9.7), (9.9), and (9.1) for $r = \infty$ and $s = 1$, we obtain

$$\begin{aligned}\langle y^m - \Pi_h y^m, \zeta \rangle &\le \tfrac{C}{\delta} \|\sigma^{-1}(y^m - \Pi_h y^m)\|^2_{L^2(\Omega)} + \delta \|\sigma^{-1}\rho^m\|^2_{L^2(\Omega)} \\ &\le \tfrac{C}{\delta} h^2 |\log h| \|\partial_{\mathbf{x}} y\|^2_{L^\infty(Q)} + \delta \|\sigma^{-1}\rho^m\|^2_{L^2(\Omega)},\end{aligned}$$

and, by virtue of $\|\nabla \Pi_h \varphi\|_{L^\infty(\Omega)} \le \|\nabla \varphi\|_{L^\infty(\Omega)}$ and (2.3) for $p = 2$,

$$\begin{aligned}\langle \Pi_h y^m, \zeta \rangle - \langle \Pi_h y^m, \zeta \rangle_h &\le Ch \|\sigma^{-1} \nabla \Pi_h y^m\|_{L^2(\Omega)} \|\sigma^{-1}\rho^m\|_{L^2(\Omega)} \\ &\le \tfrac{C}{\delta} h^2 |\log h| \|\partial_{\mathbf{x}} y\|^2_{L^\infty(Q)} + \delta \|\sigma^{-1}\rho^m\|^2_{L^2(\Omega)}.\end{aligned}$$

A similar argument can be used to bound the terms involving y^0 and Y^0, except that now we get the factor $\delta \|\sigma Z^0\|^2_{L^2(\Omega)}$ instead of $\delta \|\sigma^{-1}\rho^m\|^2_{L^2(\Omega)}$, together with

$$\tfrac{C}{\delta} h^2 |\log h|^2 \|\nabla y^0\|^2_{L^\infty(\Omega)}.$$

This, combined with the next two terms in (9.12), shows the need of a priori bounds for Z^0, $Z^n - Z^{n-1}$, and $\nabla(Z^n - Z^{n-1})$ in weighted L^2-based norms. In this regard we enunciate the following key results, that extend those in [14; (5.3),(5.4)] and [36] to forward differences with quadrature.

Lemma 9.1. *There exist constants C, C^* independent of $1 \le m \le N$ such that, if $\tau \le C^* h^2$ is of order h^2, then*

$$\begin{aligned}\|\sigma Z^0\|^2_{L^2(\Omega)} &+ \max_{1 \le n \le m} t_{m-n+1} \|\sigma \nabla Z^n\|^2_{L^2(\Omega)} + \tau^{-1} \sum_{n=1}^m t_{m-n+1} \|\sigma(Z^n - Z^{n-1})\|^2_{L^2(\Omega)} \\ &+ \sum_{n=1}^m t_{m-n+1} \|\sigma \nabla (Z^n - Z^{n-1})\|^2_{L^2(\Omega)} \le C |\log h|^2 \|\sigma \zeta\|^2_{L^2(\Omega)}.\end{aligned}$$

We postpone the proof of this lemma until §9.3 and instead finish the previous argument. In light of (9.4), (9.7) for $\|\cdot\|_h$, and (9.9) we can write

$$\begin{aligned}\sum_{n=1}^m \langle \eta^{n-1}, Z^{n-1} - Z^n \rangle_h &\le C \Big(\tau \sum_{n=1}^m t^{-1}_{m-n+1} \|\sigma^{-1}\eta^{n-1}\|^2_h \Big)^{1/2} \\ &\quad \times \Big(\tau^{-1} \sum_{n=1}^m t_{m-n+1} \|\sigma(Z^n - Z^{n-1})\|^2_{L^2(\Omega)} \Big)^{1/2} \\ &\le \tfrac{C}{\delta} (h^2 + \tau) |\log h|^5 |\log \tau| \|y\|^2_{C^{0,1;0,1/2}(\bar{Q})} + \delta \|\sigma^{-1}\rho^m\|^2_{L^2(\Omega)},\end{aligned}$$

because $\tau \sum_{n=1}^m t^{-1}_{m-n+1} \le C |\log \tau|$. We finally add and subtract $\langle \Pi_h y^{n-1}, Z^{n-1} - Z^n \rangle$ to last term in (9.12). Then use (9.7) and (9.1) for $r = \infty$ and $s = 1$, to get

$$\begin{aligned}\sum_{n=1}^m \langle y^{n-1} - \Pi_h y^{n-1}, Z^{n-1} - Z^n \rangle &\le \Big(\tau \sum_{n=1}^m t^{-1}_{m-n+1} \|\sigma^{-1}(y^{n-1} - \Pi_h y^{n-1})\|^2_{L^2(\Omega)} \Big)^{1/2} \\ &\quad \times \Big(\tau^{-1} \sum_{n=1}^m t_{m-n+1} \|\sigma(Z^{n-1} - Z^n)\|^2_{L^2(\Omega)} \Big)^{1/2} \\ &\le \tfrac{C}{\delta} h^2 |\log h|^3 |\log \tau| \|\partial_{\mathbf{x}} y\|^2_{L^\infty(Q)} + \delta \|\sigma^{-1}\rho^m\|^2_{L^2(\Omega)},\end{aligned}$$

and, in conjunction with the first bound in (2.3), to arrive at

$$\begin{aligned}
\sum_{n=1}^{m} \big(\langle \Pi_h y^{n-1}, Z^{n-1}-Z^n\rangle - \langle \Pi_h y^{n-1}, Z^{n-1}-Z^n\rangle_h\big)& \\
\le Ch^2 \sum_{n=1}^{m} \|\sigma^{-1}\nabla \Pi_h y^{n-1}\|_{L^2(\Omega)} \|\sigma \nabla (Z^n - Z^{n-1})\|_{L^2(\Omega)}& \\
\le C\tfrac{h^2}{\tau^{1/2}} |\log h|^{1/2} \Big(\tau \sum_{n=1}^{m} t_{m-n+1}^{-1} \|\nabla y^{n-1}\|^2_{L^\infty(\Omega)}\Big)^{1/2}& \\
\times \Big(\sum_{n=1}^{m} t_{m-n+1} \|\sigma \nabla (Z^n - Z^{n-1})\|^2_{L^2(\Omega)}\Big)^{1/2}& \\
\le \tfrac{C}{\delta}\tfrac{h^4}{\tau} |\log h|^3 |\log \tau| \|\partial_{\mathbf{x}} y\|^2_{L^\infty(Q)} + \delta \|\sigma^{-1}\rho^m\|^2_{L^2(\Omega)}.&
\end{aligned}$$

Inserting the above estimates back into (9.12) and choosing δ appropriately, we readily obtain the key estimate

$$\|\sigma^{-1}\rho^m\|^2_{L^2(\Omega)} \le C\big(h^2 + \tau + \tfrac{h^4}{\tau}\big) |\log h|^5 |\log \tau| \|y\|^2_{C^{0,1;0,1/2}(\bar{Q})}.$$

To complete the proof, we note that, for all $t_{m-1} < t \le t_m$, $1 \le m \le N$,

$$y(\cdot,t) - H^0_{h,\tau}(y;\cdot,t) = \tfrac{t - t_{m-1}}{\tau}\big(y(\cdot,t) - Y^m(\cdot)\big) + \tfrac{t_m - t}{\tau}\big(y(\cdot,t) - Y^{m-1}(\cdot)\big).$$

Since $\rho^m = Y^m - R_h \bar{y}^{m+1}$ is piecewise linear, (9.8) yields

$$\|y(t) - Y^m\|_{L^\infty(\Omega)} \le \|y(t) - R_h \bar{y}^{m+1}\|_{L^\infty(\Omega)} + C\|\sigma^{-1}\rho^m\|_{L^2(\Omega)},$$

for $\mathbf{x}_0$ in (9.5) being the point at which ρ^m attains its maximum. We then resort to (9.4) to deduce

$$\|y(t) - R_h \bar{y}^{m+1}\|_{L^\infty(\Omega)} \le C\big(h + \tau^{1/2}\big) |\log h| \|y\|_{C^{0,1;0,1/2}(\bar{Q})},$$

whence

$$\|y(t) - Y^m\|_{L^\infty(\Omega)} \le C\big(h + \tau^{1/2} + \tfrac{h^2}{\tau^{1/2}}\big) |\log h|^{5/2} |\log \tau|^{1/2} \|y\|_{C^{0,1;0,1/2}(\bar{Q})}.$$

A similar estimate for $\|y(t) - Y^{m-1}\|_{L^\infty(\Omega)}$ completes the proof of Theorem 5.1. □

9.2. Proof of Theorem 5.2. We simply rewrite (9.12), this time without summation by parts for the last term in (9.10), that is

$$\begin{aligned}
\langle \rho^m, \zeta\rangle_h = \langle \eta^m, \zeta\rangle_h + \langle Y^0 - y^0, Z^0\rangle_h + \sum_{n=1}^{m} \langle \eta^{n-1}, Z^{n-1} - Z^n\rangle_h& \\
+ \sum_{n=1}^{m} \big(\langle y^n - y^{n-1}, Z^n\rangle - \langle y^n - y^{n-1}, Z^n\rangle_h\big).&
\end{aligned} \tag{9.13}$$

The first three terms at the right hand side can be handled as in the proof of Theorem 5.1 with the only precaution of using full $C^{1,1;0,1}(\bar{Q})$ regularity in the interpolation estimates (9.1) and (9.4). The remaining term, instead, requires an additional a priori bound, which will be shown in §9.4; see [36, p.284] for a semidiscrete version.

Lemma 9.2. *There exist constants C, C^* independent of $1 \le m \le N$ such that, if $\tau \le C^* h^2$, then*

$$\max_{1\le n\le m} t_{m-n+1}\|Z^n\|_{L^\infty(\Omega)} \le C|\log h|^2\|\sigma\zeta\|_{L^2(\Omega)}.$$

We then add and subtract $\langle \Pi_h(y^n - y^{n-1}), Z^n\rangle$ from the last term in (9.13) and make use of (9.1), with $r = 1$ and $s = 2$, to deduce

$$\begin{aligned}\sum_{n=1}^m \langle (y^n - y^{n-1}) - \Pi_h(y^n - y^{n-1}), Z^n\rangle &\le \Big(\max_{1\le n\le m} t_{m-n+1}\|Z^n\|_{L^\infty(\Omega)}\Big)\\ &\quad\times\Big(\sum_{n=1}^m t_{m-n+1}^{-1}\|(y^n - y^{n-1}) - \Pi_h(y^n - y^{n-1})\|_{L^1(\Omega)}\Big)\\ &\le \tfrac{C}{\delta}h^4|\log h|^4|\log\tau|^2\|\partial_t\partial_{\mathbf{x}}^2 y\|^2_{L^\infty(0,T;L^1(\Omega))} + \delta\|\sigma^{-1}\rho^m\|^2_{L^2(\Omega)}.\end{aligned}$$

Using the first bound in (2.3), (9.7), and the second bound in Lemma 9.1, we get

$$\begin{aligned}&\sum_{n=1}^m \big(\langle \Pi_h(y^n - y^{n-1}), Z^n\rangle - \langle \Pi_h(y^n - y^{n-1}), Z^n\rangle_h\big)\\ &\qquad\le Ch^2\sum_{n=1}^m \|\sigma^{-1}\nabla\Pi_h(y^n - y^{n-1})\|_{L^2(\Omega)}\|\sigma\nabla Z^n\|_{L^2(\Omega)}\\ &\qquad\le Ch^2|\log h|^{1/2}\Big(\sum_{n=1}^m \tau^{-1}\|\nabla(y^n - y^{n-1})\|^2_{L^\infty(\Omega)}\Big)^{1/2}\\ &\qquad\quad\times\Big(\tau\sum_{n=1}^m t_{m-n+1}^{-1}\Big)^{1/2}\Big(\max_{1\le n\le m} t_{m-n+1}^{1/2}\|\sigma\nabla Z^n\|_{L^2(\Omega)}\Big)\\ &\qquad\le \tfrac{C}{\delta}h^4|\log h|^3|\log\tau|\|\partial_t\partial_{\mathbf{x}} y\|^2_{L^2(0,T;L^\infty(\Omega))} + \delta\|\sigma^{-1}\rho^m\|^2_{L^2(\Omega)}.\end{aligned}$$

Replacing these estimates back into (9.13) and arguing as at the end of Theorem 5.1 yield the asserted estimate. $\square$

9.3. Proof of Lemma 9.1. It will be convenient, but entirely equivalent, to deal with a dual problem with initial condition $\varphi^0 := \zeta \in \mathcal{V}_h(0)$, instead of (9.11),

$$\langle \varphi^n - \varphi^{n-1}, \chi\rangle_h + \tau\langle\nabla\varphi^{n-1}, \nabla\chi\rangle = 0 \quad \forall\, \chi \in \mathcal{V}_h(0). \tag{9.14}$$

We prove various stability bounds for (9.14), which combined yield Lemma 9.1.

Lemma 9.3. *There exists $C^* > 0$ such that, if $\tau \le C^* h^2$, then*

$$\sum_{n=1}^m \tau\|\varphi^n\|^2_{L^2(\Omega)} \le C|\log h|^2\|\sigma\zeta\|^2_{L^2(\Omega)}.$$

Proof. Add (9.14) over n, and set $\Phi^n = \sum_{k=0}^n \varphi^k$, to obtain

$$\langle \varphi^n - \zeta, \chi\rangle_h + \tau\langle\nabla\Phi^n, \nabla\chi\rangle = \tau\langle\nabla\varphi^n, \nabla\chi\rangle.$$

Take $\chi = \tau\varphi^n \in \mathcal{V}_h(0)$ as a test function and add again over n. Then (2.2) yields

$$\tau \sum_{n=1}^{m} \|\varphi^n\|_{L^2(\Omega)}^2 + \tau^2 \sum_{n=1}^{m} \langle \nabla\Phi^n, \nabla(\Phi^n - \Phi^{n-1})\rangle \leq \tau^2 \sum_{n=1}^{m} \|\nabla\varphi^n\|_{L^2(\Omega)}^2 + \tau\langle\zeta, \Phi^m\rangle_h. \tag{9.15}$$

Since $2\mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, the second term in (9.15) telescopes and thus gives rise to

$$\tau^2 \sum_{n=1}^{m} \langle \nabla\Phi^n, \nabla(\Phi^n - \Phi^{n-1})\rangle = \tfrac{\tau^2}{2}\|\nabla\Phi^m\|_{L^2(\Omega)}^2 - \tfrac{\tau^2}{2}\|\nabla\zeta\|_{L^2(\Omega)}^2 + \tfrac{\tau^2}{2}\sum_{n=1}^{m}\|\nabla\varphi^n\|_{L^2(\Omega)}^2.$$

For the first term on the right hand side of (9.15), use an inverse inequality to get

$$\tau^2 \sum_{n=1}^{m} \|\nabla\varphi^n\|_{L^2(\Omega)}^2 \leq \delta\tau \sum_{n=1}^{m} \|\varphi^n\|_{L^2(\Omega)}^2,$$

for $\delta = C\tau h^{-2}$. This term can be absorbed into the left hand side of (9.15) provided $\delta < 1$. We are thus left with the following two terms to be bounded

$$\tau^2\|\nabla\zeta\|_{L^2(\Omega)}^2, \quad \tau\langle\zeta, \Phi^m\rangle_h.$$

Using (9.5) together with an inverse inequality, we get

$$\|\zeta\|_{L^2(\Omega)} \leq h^{-1}\|\sigma\zeta\|_{L^2(\Omega)}, \quad \|\nabla\zeta\|_{L^2(\Omega)} \leq Ch^{-2}\|\sigma\zeta\|_{L^2(\Omega)}. \tag{9.16}$$

On the other hand, since (2.2) yields

$$\tau\langle\zeta, \Phi^m\rangle_h \leq \tfrac{1}{2\delta}|\log h|^2\|\sigma\zeta\|_{L^2(\Omega)}^2 + \delta\tau^2|\log h|^{-2}\|\sigma^{-1}\Phi^m\|_{L^2(\Omega)}^2,$$

it only remains to estimate the rightmost term. We resort to Hölder inequality with p large and Sobolev inequality in 2D [19, p.158], which combine with (9.6) to yield

$$\|\sigma^{-1}\Phi^m\|_{L^2(\Omega)} \leq \|\sigma^{-1}\|_{L^{2p/(p-2)}(\Omega)}\|\Phi^m\|_{L^p(\Omega)} \leq Cph^{-2/p}\|\nabla\Phi^m\|_{L^2(\Omega)}.$$

On choosing $p = |\log h|$ we readily obtain the desired bound, because $\|\nabla\Phi^m\|_{L^2(\Omega)}$ can be hidden into the left hand side of (9.15) for δ small. $\square$

Lemma 9.4. *There exists $C^* > 0$ such that, if $\tau \leq C^*h^2$, then*

$$\max_{0\leq n\leq m} t_{n+1}\|\varphi^n\|_{L^2(\Omega)}^2 + \sum_{n=1}^{m} t_n\|\varphi^n - \varphi^{n-1}\|_{L^2(\Omega)}^2 + \tau\sum_{n=1}^{m} t_n\|\nabla\varphi^n\|_{L^2(\Omega)}^2 \leq C|\log h|^2\|\sigma\zeta\|_{L^2(\Omega)}^2.$$

Proof. Take $\chi = t_n\varphi^n \in \mathcal{V}_h(0)$ in (9.14) and add over n. For $1 \le k \le m$, we get

$$2\sum_{n=1}^{k} t_n\langle \varphi^n - \varphi^{n-1}, \varphi^n\rangle_h = \sum_{n=1}^{k} t_n\langle \varphi^n - \varphi^{n-1}, (\varphi^n + \varphi^{n-1}) + (\varphi^n - \varphi^{n-1})\rangle_h$$
$$= t_{k+1}\|\varphi^k\|_h^2 + \sum_{n=1}^{k} t_n\|\varphi^n - \varphi^{n-1}\|_h^2 - \sum_{n=0}^{k}\tau\|\varphi^n\|_h^2.$$

On the other hand, we can write $\langle\nabla\varphi^{n-1}, \nabla\varphi^n\rangle = \langle\nabla\varphi^n, \nabla\varphi^n\rangle + \langle\nabla(\varphi^{n-1} - \varphi^n), \nabla\varphi^n\rangle$ and, using an inverse inequality, we see that

$$\langle\nabla(\varphi^{n-1} - \varphi^n), \nabla\varphi^n\rangle \ge -\tfrac{C}{h}\|\varphi^n - \varphi^{n-1}\|_{L^2(\Omega)}\|\nabla\varphi^n\|_{L^2(\Omega)}.$$

Then

$$2\tau\sum_{n=1}^{k} t_n\langle\nabla\varphi^{n-1}, \nabla\varphi^n\rangle \ge \tau\sum_{n=1}^{k} t_n\|\nabla\varphi^n\|_{L^2(\Omega)}^2 - C\tfrac{\tau}{h^2}\sum_{n=1}^{k} t_n\|\varphi^n - \varphi^{n-1}\|_{L^2(\Omega)}^2.$$

Combining the previous inequalities with (2.2) yields

$$t_{k+1}\|\varphi^k\|_{L^2(\Omega)}^2 + \left(1 - C\tfrac{\tau}{h^2}\right)\sum_{n=1}^{k} t_n\|\varphi^n - \varphi^{n-1}\|_{L^2(\Omega)}^2 + \tau\sum_{n=1}^{k} t_n\|\nabla\varphi^n\|_{L^2(\Omega)}^2$$
$$\le C\sum_{n=0}^{k}\tau\|\varphi^n\|_{L^2(\Omega)}^2,$$

and the asserted estimate follows from Lemma 9.3 and (9.16), for $\tau \le C^* h^2$. □

Lemma 9.5. *There exists $C^* > 0$ such that, if $\tau \le C^* h^2$, then*

$$\max_{0\le n\le m}\|\sigma\varphi^n\|_{L^2(\Omega)}^2 + \sum_{n=1}^{m}\|\sigma(\varphi^n - \varphi^{n-1})\|_{L^2(\Omega)}^2$$
$$+ \tau\sum_{n=1}^{m}\|\sigma\nabla\varphi^n\|_{L^2(\Omega)}^2 \le C|\log h|^2\|\sigma\zeta\|_{L^2(\Omega)}^2.$$

Proof. Take $\chi = \Pi_h(\sigma^2\varphi^n) \in \mathcal{V}_h(0)$ in (9.14) and add over n. For $k \le m$, we get

$$2\sum_{n=1}^{k}\langle\varphi^n - \varphi^{n-1}, \sigma^2\varphi^n\rangle_h = \|\sigma\varphi^k\|_h^2 - \|\sigma\zeta\|_h^2 + \sum_{n=1}^{k}\|\sigma(\varphi^n - \varphi^{n-1})\|_h^2.$$

We next split $\langle\nabla\varphi^{n-1}, \nabla\Pi_h(\sigma^2\varphi^n)\rangle$ and bound each term separately. In view of (9.6) we have

$$\langle\nabla\varphi^n, \nabla(\sigma^2\varphi^n)\rangle = \|\sigma\nabla\varphi^n\|_{L^2(\Omega)}^2 + \langle\nabla\varphi^n, \varphi^n\nabla\sigma^2\rangle \ge \tfrac{1}{2}\|\sigma\nabla\varphi^n\|_{L^2(\Omega)}^2 - 2\|\varphi^n\|_{L^2(\Omega)}^2,$$

and, invoking an inverse inequality, we get

$$\langle\nabla(\varphi^{n-1}-\varphi^n),\nabla(\sigma^2\varphi^n)\rangle \geq -\tfrac{C}{\delta h^2}\|\sigma(\varphi^n-\varphi^{n-1})\|^2_{L^2(\Omega)}-\delta\|\sigma\nabla\varphi^n\|^2_{L^2(\Omega)}-\|\varphi^n\|^2_{L^2(\Omega)}.$$

Finally, using (9.1) for $r=s=2$, it follows that

$$\begin{aligned}\langle\nabla\varphi^{n-1},\nabla\Pi_h(\sigma^2\varphi^n)-\nabla(\sigma^2\varphi^n)\rangle&\\ &\geq -Ch\|\nabla\varphi^{n-1}\|_{L^2(\Omega)}\big(\|\varphi^n\|_{L^2(\Omega)}+\|\sigma\nabla\varphi^n\|_{L^2(\Omega)}\big)\\ &\geq -\delta\|\sigma\nabla\varphi^n\|^2_{L^2(\Omega)}-\tfrac{C}{\delta}h^2\|\nabla\varphi^{n-1}\|^2_{L^2(\Omega)}-\|\varphi^n\|^2_{L^2(\Omega)}.\end{aligned}$$

Collecting the previous bounds, and using an inverse inequality and (9.9), we get

$$\begin{aligned}\|\sigma\varphi^k\|^2_{L^2(\Omega)}+\big(1-C\tfrac{\tau}{h^2}\big)\sum_{n=1}^k\|\sigma(\varphi^n-\varphi^{n-1})\|^2_{L^2(\Omega)}+\tfrac12\sum_{n=1}^k\|\sigma\nabla\varphi^n\|^2_{L^2(\Omega)}&\\ \leq C\|\sigma\zeta\|^2_{L^2(\Omega)}+C\tau\|\zeta\|^2_{L^2(\Omega)}+C\sum_{n=1}^k\tau\|\varphi^n\|^2_{L^2(\Omega)},&\end{aligned}$$

for $\delta=1/8$. Assertion then follows from Lemma 9.3 and (9.16), for $\tau\leq C^*h^2$. □

We see that the estimate for σZ^0 in Lemma 9.1 follows from the first term in Lemma 9.5. To derive the remaining estimates, take $\chi=\tau^{-1}t_n\Pi_h(\sigma^2(\varphi^n-\varphi^{n-1}))\in\mathcal{V}_h(0)$ as a test function in (9.14). Upon summation on $n\leq k\leq m$, we obtain

$$\tau^{-1}\sum_{n=1}^k t_n\|\sigma(\varphi^n-\varphi^{n-1})\|^2_h+\sum_{n=1}^k t_n\langle\nabla\varphi^{n-1},\nabla\Pi_h(\sigma^2(\varphi^n-\varphi^{n-1}))\rangle=0.$$

We split the second term and estimate each part separately. In view of (9.6) we get

$$\begin{aligned}\sum_{n=1}^k t_n\langle\nabla\varphi^{n-1},(\varphi^n-\varphi^{n-1})\nabla\sigma^2\rangle\geq -\tfrac12\tau^{-1}\sum_{n=1}^k t_n\|\sigma(\varphi^n-\varphi^{n-1})\|^2_{L^2(\Omega)}&\\ -2\tau\sum_{n=1}^k t_n\|\nabla\varphi^{n-1}\|^2_{L^2(\Omega)}.&\end{aligned}$$

Upon writing $2\varphi^{n-1}=(\varphi^n+\varphi^{n-1})+(\varphi^{n-1}-\varphi^n)$, we infer that

$$\begin{aligned}\sum_{n=1}^k t_n\langle\nabla\varphi^{n-1},\sigma^2\nabla(\varphi^n-\varphi^{n-1})\rangle=\tfrac12 t_{k+1}\|\sigma\nabla\varphi^k\|^2_{L^2(\Omega)}-\tfrac12\sum_{n=0}^k\tau\|\sigma\nabla\varphi^n\|^2_{L^2(\Omega)}&\\ -\tfrac12\sum_{n=1}^k t_n\|\sigma\nabla(\varphi^n-\varphi^{n-1})\|^2_{L^2(\Omega)}.&\end{aligned}$$

For the remaining term we resort to (9.1), for $r=s=2$, to arrive at

$$\begin{aligned}\sum_{n=1}^k t_n\langle\nabla\varphi^{n-1},\nabla(\Pi_h-I)(\sigma^2(\varphi^n-\varphi^{n-1}))\rangle\geq -Ch^2\sum_{n=1}^k t_n\|\nabla\varphi^{n-1}\|^2_{L^2(\Omega)}&\\ -\sum_{n=1}^k t_n\|\varphi^n-\varphi^{n-1}\|^2_{L^2(\Omega)}-\sum_{n=1}^k t_n\|\sigma\nabla(\varphi^n-\varphi^{n-1})\|^2_{L^2(\Omega)}.&\end{aligned}$$

Since $\|\sigma\nabla(\varphi^n-\varphi^{n-1})\|_{L^2(\Omega)} \le Ch^{-1}\|\sigma(\varphi^n-\varphi^{n-1})\|_{L^2(\Omega)}$, collecting the previous bounds and making use of (9.9) yield

$$\begin{aligned}(1-C\tfrac{\tau}{h^2})\tau^{-1}\sum_{n=1}^k t_n\|\sigma(\varphi^n-\varphi^{n-1})\|^2_{L^2(\Omega)} + t_{k+1}\|\sigma\nabla\varphi^k\|^2_{L^2(\Omega)} \\ \le C(\tau+h^2)\sum_{n=1}^k t_n\|\nabla\varphi^{n-1}\|^2_{L^2(\Omega)} + \sum_{n=0}^k \tau\|\sigma\nabla\varphi^n\|^2_{L^2(\Omega)} \\ +2\sum_{n=1}^k t_n\|\varphi^n-\varphi^{n-1}\|^2_{L^2(\Omega)}.\end{aligned}$$

By virtue of Lemmas 9.4 and 9.5 and (9.16) we get the bound for second and third terms in Lemma 9.1, provided $\tau \le C^* h^2$ is of order h^2. Its proof is thus complete. $\square$

9.4. Proof of Lemma 9.2. We again deal with (9.14) instead of (9.11). We first note that multiplying (9.14) by t_n leads to the equation satisfied by $\psi^n := t_{n+1}\varphi^n$

$$\langle\psi^n-\psi^{n-1},\chi\rangle_h + \tau\langle\nabla\psi^{n-1},\nabla\chi\rangle = \tau\langle\varphi^n,\chi\rangle_h \quad \forall\,\chi\in\mathcal{V}_h(0). \tag{9.17}$$

For any $1\le k\le m$, we introduce a fully *discrete fundamental solution* $\{\Psi^n\}_{n=0}^k$ as follows. Let $\mathbf{x}_i$ be an interior node of $\mathcal{D}_h$, and let $\delta_i := 3\phi_i/|\mathrm{supp}\phi_i| \in \mathcal{V}_h(0)$. Here ϕ_i is the corresponding basis function, and δ_i is a discrete Dirac function, because $\langle 1,\delta_i\rangle_h = 1$. Let $\Psi^k := \delta_i$ and $\Psi^{n-1}\in\mathcal{V}_h(0)$, for $k\ge n\ge 1$, be given by

$$\langle\Psi^{n-1}-\Psi^n,\chi\rangle_h + \tau\langle\nabla\Psi^n,\nabla\chi\rangle = 0 \quad \forall\,\chi\in\mathcal{V}_h(0).$$

Taking $\chi=\Psi^n$ in (9.17), adding over n and summing by parts the first term yield

$$t_{k+1}\varphi_i^k = \psi_i^k = \langle\psi^k,\delta_i\rangle_h = \langle\psi^k,\Psi^k\rangle_h = \tau\sum_{n=0}^k\langle\varphi^n,\Psi^n\rangle_h. \tag{9.18}$$

By virtue of Lemma 9.4 we have

$$t_{n+1}^{1/2}\|\varphi^n\|_{L^2(\Omega)} \le C|\log h|\|\sigma\zeta\|_{L^2(\Omega)} \quad \forall\,0\le n\le k.$$

Invoking again Lemma 9.4, this time applied to (9.14) with $\varphi^0 := \delta_i$ and $\mathbf{x}_0=\mathbf{x}_i$ in the definition (9.5) of $\sigma =: \sigma_i$, since σ_i is of order h in $\mathrm{supp}\phi_i$, we infer that

$$t_{k-n+1}^{1/2}\|\Psi^n\|_{L^2(\Omega)} \le C|\log h|\|\sigma_i\delta_i\|_{L^2(\Omega)} \le C|\log h| \quad \forall\,0\le n\le k.$$

Inserting the last two estimates into (9.18), from (2.2) we realize that

$$|t_{k+1}\varphi_i^k| \le C|\log h|^2\|\sigma\zeta\|_{L^2(\Omega)}\tau\sum_{n=0}^k t_{k-n+1}^{-1/2}t_{n+1}^{-1/2} \le C|\log h|^2\|\sigma\zeta\|_{L^2(\Omega)},$$

because $\tau\sum_{n=0}^k t_{k-n+1}^{-1/2}t_{n+1}^{-1/2} \le \int_0^{t_{k+2}}(t_{k+2}-t)^{-1/2}t^{-1/2}dt \le C$. Since $\mathbf{x}_i$ is arbitrary, this concludes the proof of Lemma 9.2. $\square$

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