EFFECT OF NUMERICAL INTEGRATION FOR ELLIPTIC OBSTACLE PROBLEMS*

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Abstract. An elliptic obstacle problem is approximated by piecewise linear finite elements with numerical integration on the penalty and forcing terms. This leads to diagonal nonlinearities and thereby to a practical scheme. Optimal error estimates in the maximum norm are derived. The proof is based on constructing suitable super and subsolutions that exploit the special structure of the penalization, and using quite precise pointwise error estimates for an associated linear elliptic problem with quadrature via the discrete maximum principle.

Key words. obstacle problem, penalty, finite elements, numerical integration

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0. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ $(n > 1)$ be a bounded domain with boundary Γ of class $C^{1,1}$. The following penalty formulation is known to be equivalent to an elliptic obstacle problem [4,6]:

$$\begin{aligned} -\Delta u + \beta(u-\psi) \ni f, \qquad &\text{in} \quad \Omega, \\ u = 0, \qquad &\text{on} \quad \Gamma. \end{aligned} \tag{0.1}$$

Hereafter $f \in L^\infty(\Omega)$ is a forcing term, ψ is the obstacle which satisfies $\Delta\psi \in L^\infty(\Omega)$, $\psi \leq 0$ on Γ, and β is the maximal monotone graph in $\mathbb{R}^2$

$$\beta(s) := \begin{cases} 0, & s > 0, \\ [-\infty, 0], & s = 0, \\ -\infty, & s < 0. \end{cases}$$

Since $\|\beta(u)\|_{L^\infty(\Omega)} \leq \|f + \Delta\psi\|_{L^\infty(\Omega)} =: \lambda$ [4,6,9], we can replace β by $\max\{\beta(s), -\lambda\}$ without changing the problem. The following regularized version of (0.1) is combined in [9] with continuous piecewise linear finite elements over a quasi-uniform mesh $\mathcal{T}_h$ of size h:

$$\begin{aligned} -\Delta u_\varepsilon + \beta_\varepsilon(u_\varepsilon-\psi) = f, \qquad &\text{in} \quad \Omega, \\ u_\varepsilon = 0, \qquad &\text{on} \quad \Gamma, \end{aligned} \tag{0.2}$$

where $0 < \varepsilon \ll 1$ is the regularization parameter and

$$\beta_\varepsilon(s) := \begin{cases} 0, & s > 0, \\ \max\{s/\varepsilon, -\lambda\}, & s \leq 0. \end{cases}$$

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It is well known that $u_\varepsilon - \psi \geq -\lambda\varepsilon$ and $u \notin W^{2,\infty}(\Omega)$, but both u and u_ε satisfy [4,6]

$$\|u\|_{W^{2,p}(\Omega)},\ \|u_\varepsilon\|_{W^{2,p}(\Omega)} \leq Cp\lambda. \tag{0.3}$$

Maximum norm error estimates of order $O(h^2|\log h|^2)$, thereby *optimal* according to (0.3), are derived in [9] provided $\varepsilon = C^* h^2$ for an arbitrary $C^* > 0$. The resulting discrete scheme presumes exact integration and thus is difficult to implement. It is therefore computationally desirable to incorporate quadrature in that it would lead to a practical method. However its effect on the penalty term β_ε is very unclear because β_ε becomes discontinuous in the limit $\varepsilon \downarrow 0$. So the question arises as to whether there might be any loss of accuracy. This issue is examined in [1] for a simple combustion problem: the (theoretical) rate of convergence does in fact deteriorate as the underlying problem gets closer to the more singular case (0.1). The reason is the insufficient regularity of a crucial nonlinear term related to β.

The purpose of this paper is to demonstrate *optimal* pointwise error estimates for a piecewise linear finite element approximation of the obstacle problem (0.2) with numerical integration. They are sharp in both order of convergence and underlying regularity. The proof exploits the special structure of β_ε, along with precise (and presumably new) error estimates for linear problems, to construct suitable super and subsolutions. They are eventually compared with the discrete solution via the Discrete Maximum Principle. For the latter to hold, an extra condition on the triangulation must be required, the so-called weak acuteness, that at the present time even automatic mesh generation can enforce [11].

Optimal error estimates for free boundaries follow immediately from our results as in [8,9]. In fact, if f and ψ satisfy the *nondegeneracy* condition $f + \Delta\psi \geq C > 0$, then we can prove an $O(h|\log h|)$ interface error bound in distance and in measure by simply using the ideas in [8,9]. We will not pursue this important topic any further.

The outline of the paper is as follows. We start with assumptions and notations in §1. We then set up continuous and approximate problems and finally state our main result (Theorem 1). In §2 we discuss some preliminary results and in §3 we examine an associated linear elliptic problem with quadrature (Theorems 2 and 3). We finally conclude in §4 with the proof of our main result.

1. Formulation of the problem

We partition Ω into n-simplices T and denote by $\mathcal{T}_h$ the resulting mesh and h the meshsize. As usual we ignore the effect of the layer $(\Omega\backslash\Omega_h) \cup (\Omega_h\backslash\Omega)$ which is known not to alter the rate of convergence [14], and instead concentrate on the effect of numerical integration. We thus assume $\Omega = \Omega_h$ and [2,3,7]:

(1.1) the family $\{\mathcal{T}_h\}$ is regular, quasi-uniform and weakly acute.

A triangulation is *acute* if for all $T \in \mathcal{T}_h$ and for all vertices p of T, the projection p_T of p onto the opposite supporting hyperplane $\pi_T(p)$ belongs to ∂T [3,7,8]. This yields

$$\int_\Omega \nabla\phi_i \nabla\phi_j \le 0, \qquad \forall\ 1 \le i \ne j \le J, \tag{1.2}$$

where ϕ_i is the canonical piecewise linear function with value 1 at the node x_i and 0 at the remaining nodes, and J is the total number of nodes. If $\mathcal{T}_h$ satisfies (1.2), then $\mathcal{T}_h$ is called *weakly acute*. A sufficient condition for (1.2) in 2D reads: for all interelement sides S and corresponding pair of adjacent triangles, the sum of the angles opposite to S does not exceed π. Such a geometric constraint can be enforced by automatic mesh generators in 2D [11]. The Discrete Maximum Principle is a consequence of (1.2) (see Lemma 1).

Let $H_0^1(\Omega)$ be the standard Sobolev space and $\mathbb{V}_h^1$ its discrete counterpart:

$$\mathbb{V}_h^1 = \{\chi \in C(\bar\Omega) :\ \chi|_\Gamma = 0,\ \chi|_T \text{ is linear for all } T \in \mathcal{T}_h\}.$$

Let Π_h be the corresponding Lagrange interpolation operator. We now state the continuous and discrete weak formulations. The regularized penalty problem reads:

$$u_\varepsilon \in H_0^1(\Omega) : \qquad \langle \nabla u_\varepsilon, \nabla\phi\rangle + \langle \beta_\varepsilon(u_\varepsilon - \psi), \phi\rangle = \langle f, \phi\rangle, \qquad \forall\ \phi \in H_0^1(\Omega), \tag{1.3}$$

and its discrete counterpart with numerical integration:

$$u_{\varepsilon,h} \in \mathbb{V}_h^1 : \qquad \langle \nabla u_{\varepsilon,h}, \nabla\chi\rangle + \langle \beta_\varepsilon(u_{\varepsilon,h} - \psi), \chi\rangle_h = \langle f, \chi\rangle_h, \qquad \forall\ \chi \in \mathbb{V}_h^1. \tag{1.4}$$

Hereafter $\langle\cdot,\cdot\rangle$ is the standard inner product in $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle_h$ is the following discrete inner product that uses the vertex quadrature rule [2, p.182]:

$$\langle \chi, \phi\rangle_h := \int_\Omega \Pi_h(\chi\phi)dx.$$

Let I indicate the number of interior nodes x_i of $\mathcal{T}_h$. Then (1.4) leads to the following nonlinear system:

$$U \in \mathbb{R}^I : \qquad F(U) := \mathbf{A}U + \mathbf{M}\beta_\varepsilon(U - \Psi) = B. \tag{1.5}$$

Here $\mathbf{A} = (\int_\Omega \nabla\phi_i\nabla\phi_j)_{i,j=1}^I$, the stiffness matrix, is an $I \times I$ symmetric positive definite M-matrix, the (lumped) mass matrix $\mathbf{M} = (\int_\Omega \Pi_h[\phi_i\phi_j])_{i,j=1}^I$ is an $I \times I$ positive diagonal matrix, and B, Ψ are I vectors. The nonlinear map $F : \mathbb{R}^I \to \mathbb{R}^I$ is coercive and strictly monotone because β_ε is nondecreasing. Therefore existence and uniqueness follow immediately [6,10]. Since the nonlinearity is diagonal, (1.5) can be effectively solved by classical techniques [10].

Our main concern here is to prove the following pointwise error estimate. This result is *optimal* in both order of convergence and underlying regularity, as compared with linear problems without numerical integration [2,7,12,13,14].

Theorem 1. *Suppose* $f \in W^{2,p}(\Omega)$ $(p > n/2)$, $\psi \in W^{2,\infty}(\Omega)$, *and* $\varepsilon = C^* h^2$ *with* $C^* > 0$ *arbitrary. Then there exists a* $C > 0$ *independent of* ε *and* h *such that*

$$\|u - u_{\varepsilon,h}\|_{L^\infty(\Omega)} \le Ch^2 |\log h|^2. \tag{1.6}$$

Hereafter C denotes a positive constant which may vary at the various occurrences but is always independent of ε and h. The constant C in (1.6) depends on f, ψ, Ω, C^*, and p, and in fact blows up as $p \to n/2$ as expected even for linear problems. Since f may fail to be continuous for $p = n/2$ [5], (1.4) may also fail to be well defined.

2. Preliminaries

Let $\mathbf{R}_h : H_0^1(\Omega) \to \mathbb{V}_h^1$ be the Riesz projection operator, *i.e.* for $v \in H_0^1(\Omega)$,

$$\langle \nabla v, \nabla \chi \rangle = \langle \nabla \mathbf{R}_h v, \nabla \chi \rangle \qquad \forall\, \chi \in \mathbb{V}_h^1.$$

We recall the following well known optimal pointwise error estimate [12,13,14]:

$$\|\mathbf{R}_h v - v\|_{L^\infty(\Omega)} \le C(h \log h)^2 \|\Delta v\|_{L^\infty(\Omega)}. \tag{2.1}$$

Since $-\Delta u_\varepsilon = f - \beta_\varepsilon(u_\varepsilon - \psi)$ and $\|\beta_\varepsilon(u_\varepsilon - \psi)\|_{L^\infty(\Omega)} \le \|f + \Delta\psi\|_{L^\infty(\Omega)}$, we have

$$\|\mathbf{R}_h u_\varepsilon - u_\varepsilon\|_{L^\infty(\Omega)} \le C(h \log h)^2 \Big(\|f\|_{L^\infty(\Omega)} + \|\psi\|_{W^{2,\infty}(\Omega)} \Big), \tag{2.2}$$

where C is independent of ε and h. The following version of the *Discrete Maximum Principle* is a consequence of (1.2). Set $\mathbf{P}_h \chi = \Pi_h[\max(0,\chi)] \in \mathbb{V}_h^1$ for all $\chi \in \mathbb{V}_h^1$.

Lemma 1. $\int_\Omega \nabla\chi \nabla \mathbf{P}_h \chi \ge \int_\Omega |\nabla \mathbf{P}_h \chi|^2 \qquad \forall\, \chi \in \mathbb{V}_h^1.$

Proof. Let $\chi = \sum_{i=1}^I \chi_i \phi_i$ be any function in $\mathbb{V}_h^1$. Then

$$\begin{aligned} \int_\Omega \nabla(\chi - \mathbf{P}_h\chi) \nabla \mathbf{P}_h \chi &= \sum_{i,j=1}^I \min(0,\chi_i) \int_\Omega \nabla\phi_i \nabla\phi_j \max(0,\chi_j) \\ &= \sum_{i \ne j} \min(0,\chi_i) \int_\Omega \nabla\phi_i \nabla\phi_j \max(0,\chi_j) \ge 0, \end{aligned}$$

where we have used (1.2). The proof is thus complete. □

Our first goal is to prove that the inverse of F in (1.5) has the following positivity property. A map $F : \mathbb{R}^I \to \mathbb{R}^I$ is called *inverse isotone* if for any $x, y \in \mathbb{R}^I$ such that $F(x) \le F(y)$, then $x \le y$ [10, p.466]; inequalities are understood componentwise. Each

$\chi \in \mathbb{V}_h^1$ can be identified with a unique point $(\chi(x_1), \cdots, \chi(x_I))$ in $\mathbb{R}^I$. Such an identification will be used in the sequel. For an arbitrary $\zeta \in \mathbb{V}_h^1$ and all $\chi \in \mathbb{V}_h^1$ set

$$F(\chi)_i := \langle \nabla\chi, \nabla\phi_i\rangle + \langle \beta_\varepsilon(\chi+\zeta), \phi_i\rangle_h, \qquad 1 \le i \le I.$$

LEMMA 2. *F is inverse isotone.*

Proof. Suppose $\phi, \chi \in \mathbb{V}_h^1$ satisfy $F(\phi) \le F(\chi)$. It follows that

$$\int_\Omega \nabla(\phi-\chi)\nabla\varphi + \Pi_h\Big[(\beta_\varepsilon(\phi+\zeta) - \beta_\varepsilon(\chi+\zeta))\varphi\Big] \le 0 \qquad \forall\ \varphi \in \mathbb{V}_h^1, \varphi \ge 0.$$

Take $\varphi = \mathbf{P}_h(\phi-\chi) \in \mathbb{V}_h^1$ and apply Lemma 1 to deduce $\int_\Omega \nabla(\phi-\chi)\nabla\varphi \ge \int_\Omega |\nabla\varphi|^2$. But monotonicity of β_ε implies

$$\int_\Omega \Pi_h\Big[(\beta_\varepsilon(\phi+\zeta) - \beta_\varepsilon(\chi+\zeta))\mathbf{P}_h(\phi-\chi)\Big]$$
$$= \int_\Omega \Pi_h\Big[(\beta_\varepsilon(\phi+\zeta) - \beta_\varepsilon(\chi+\zeta))\max(0, \phi-\chi)\Big] \ge 0.$$

Hence

$$\int_\Omega |\nabla\varphi|^2 \le \int_\Omega \nabla(\phi+\chi)\varphi \le 0.$$

This yields $\nabla\varphi = 0$ and, since $\varphi = 0$ on Γ, $\varphi = 0$. Therefore $\phi \le \chi$, as desired. □

We will also need the following two technical results. The first one is known in the theory of PDEs, but the present proof is simpler.

LEMMA 3. *If $v \in \mathbf{W}^{1,p}(\Omega)$ for all $1 \le p < \infty$ and $\|v\|_{W^{1,p}(\Omega)} \le M_1 p$, then there exists a constant $M_2 > 0$ depending on M_1 such that the modulus of continuity $\sigma_v(t)$ of v satisfies*

$$\sigma_v(t) \le M_2 t |\log t|.$$

Proof. Without loss of generality we may assume that $0 < t < \mathrm{e}^{-n}$. By the Sobolev imbedding theorem, there is $C > 0$ independent of p for $p > n$ large such that

$$\|v\|_{C^{0,\alpha}(\bar\Omega)} \le C\|v\|_{W^{1,p}(\Omega)}, \qquad \text{for } \alpha = 1 - n/p > 0.$$

Let $x, y \in \Omega$ be so that $|x-y| = t$. Since

$$|v(x) - v(y)| \le Ct^{1-n/p}\|v\|_{W^{1,p}(\Omega)} \le Ct\frac{p}{t^{n/p}}M_1,$$

the assertion follows from taking $p = |\log t|$ $(> n)$. □

Let now $z \in W_0^{1,\infty}(\Omega) \cap W^{2,p}(\Omega)$ satisfy $\|z\|_{W^{2,p}(\Omega)} \leq M_1 p$ as $p \uparrow \infty$. In view of Lemma 3, we get $\sigma_{\nabla z}(t) \leq M_2 t|\log t|$. In addition, suppose that there is $t_0 > 0$ such that z satisfies the unilateral constraint:

$$z(x) \geq -t_0^2, \qquad \forall\, x \in \Omega. \tag{2.3}$$

Lemma 4. *If $z(x_0) < 0$, then $|\nabla z(x_0)| < (1 + M_2)t_0|\log t_0|$.*

Proof. Suppose by contradiction that $|\nabla z(x_0)| \geq (1 + M_2)t_0|\log t_0|$, and let

$$x_1 := x_0 - \frac{\nabla z(x_0)}{|\nabla z(x_0)|} t_0.$$

We claim that x_1 must lie in Ω. In fact, let $\xi = -\nabla z(x_0)/|\nabla z(x_0)|$ be the unit vector along $x_1 x_0$ and x^* be the intersection of the line segment $x_1 x_0$ with Γ; thus $z(x^*) = 0$. We have

$$\frac{\partial z(x_0)}{\partial \xi} = -|\nabla z(x_0)| \leq -(1 + M_2)t_0|\log t_0|.$$

Hence for all x between x_0 and x^*, we can write

$$\frac{\partial z(x)}{\partial \xi} = \frac{\partial z(x)}{\partial \xi} - \frac{\partial z(x_0)}{\partial \xi} + \frac{\partial z(x_0)}{\partial \xi} \leq -t_0|\log t_0| < 0.$$

This implies $z(x^*) < 0$ because $z(x_0) < 0$, which is a contradiction. This shows that $x_1 \in \Omega$. Consequently

$$\begin{aligned} z(x_1) =& z(x_0) + (x_1 - x_0)^T \cdot \int_0^1 \nabla z(tx_1 + (1-t)x_0)dt \\ =& z(x_0) + (x_1 - x_0)^T \cdot \nabla z(x_0) + (x_1 - x_0)^T \cdot \int_0^1 \big[\nabla z(tx_1 + (1-t)x_0) - \nabla z(x_0)\big]\, dt. \end{aligned}$$

It follows that

$$-t_0^2 < -|\nabla z(x_0)| t_0 + t_0 \sigma_{\nabla z}(t_0),$$

whence

$$|\nabla z(x_0)| < (1 + M_2)t_0|\log t_0|.$$

Since this is a contradiction, the proof is complete. □

3. Linear Elliptic Problems with Numerical Integration

In this section we demonstrate that the use of the vertex quadrature rule for linear elliptic problems does preserve pointwise accuracy. Even though there is a vast literature on numerical quadrature, the result below does not seem to be available. The midpoint rule for quadratic finite elements is instead examined in L^∞ in [15], but it is not appropriate in the present context because it leads to nondiagonal nonlinearities and requires more regularity than it is reasonable to expect.

The following results are valid even without weak acuteness because their proofs do not use the Discrete Maximum Principle. We start with a simple interpolation result. Let $f \in W^{2,p}$ for $p > n/2$. Then there exists $C = C_p > 0$ such that [2]

$$\|f - \Pi_h f\|_{L^p(\Omega)} \le C h^2 \|f\|_{W^{2,p}(\Omega)}. \tag{3.1}$$

If instead $f \in W^{1,r}(\Omega)$ for $r > n$, then there exists $C = C_r > 0$ such that

$$\|\nabla \Pi_h f\|_{L^r(\Omega)} \le C \|\nabla f\|_{L^r(\Omega)}. \tag{3.2}$$

To see this, consider an element $T \in \mathcal{T}_h$ and denote by f_T the mean value of f in T. Then

$$\begin{aligned} \|\nabla \Pi_h f\|_{L^r(T)} &\le \|\nabla(\Pi_h f - f_T)\|_{L^r(T)} \le C h^{-1} \|\Pi_h f - f_T\|_{L^r(T)} \\ &\le C h^{-1}\big(\|\Pi_h f - f\|_{L^r(T)} + \|f - f_T\|_{L^r(T)}\big) \le C\|\nabla f\|_{L^r(T)}, \end{aligned}$$

as a consequence of an inverse estimate and standard interpolation theory. We also need a technical a priori estimate for $w \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ and $p > n/2$ [5, p.191]:

$$\|w\|_{L^\infty(\Omega)} \le C_p \|\Delta w\|_{L^p(\Omega)}. \tag{3.3}$$

Our next goal is to recall a key result for regularized Green's functions. Let $x_0 \in \Omega$ and $\delta \in C_0^\infty(\Omega)$ satisfy

$$\delta \ge 0, \qquad \int_\Omega \delta = 1,$$

$$\operatorname{supp}\, \delta \subset B(x_0, \gamma h) := \{x \in \Omega : |x - x_0| < \gamma h\},$$

where $\gamma > 0$ is to be determined later. Let $G \in H_0^1(\Omega)$ be the solution to $-\Delta G = \delta$, and $G_h := \mathbf{R}_h G$ be the H_0^1 projection of G. The following result is well known [14, Lemma 5.3]:

$$\|G - G_h\|_{L^1(\Omega)} + h|\log h| \|\nabla(G - G_h)\|_{L^1(\Omega)} \le C h^2 |\log h|^2. \tag{3.4}$$

In light of (3.4) and the usual energy estimate $\|\nabla(G - G_h)\|_{L^2(\Omega)} \le C h \|\delta\|_{L^2(\Omega)}$, a straightforward application of operator interpolation theory yields for $s \in (1,2)$

$$\|\nabla(G - G_h)\|_{L^s(\Omega)} \le C h |\log h|^{2-s} \|\delta\|_{L^s(\Omega)} = C |\log h|^{2-s} h^{1-n+n/s}. \tag{3.5}$$

Let $f \in W^{2,p}(\Omega)$ for $p > n/2$. Let $v \in H_0^1(\Omega)$ and $v_h \in \mathbb{V}_h^1$ be defined by:

$$\langle \nabla v, \nabla \varphi \rangle = \langle f, \varphi \rangle, \qquad \forall\, \varphi \in H_0^1(\Omega), \tag{3.6}$$

$$\langle \nabla v_h, \nabla \chi \rangle = \langle f, \chi \rangle_h, \qquad \forall\, \chi \in \mathbb{V}_h^1. \tag{3.7}$$

We then have the following sharp pointwise error estimate that coincides with that without numerical integration [7,14].

THEOREM 2. *Let* $\Gamma \in C^{2,\alpha}$ *for* $\alpha > 0$. *Then* $\|v - v_h\|_{L^\infty(\Omega)} \le Ch^2 |\log h| \|f\|_{W^{2,p}(\Omega)}$.

Proof. Let $\tilde{v} \in H_0^1(\Omega)$ be the solution of

$$\langle \nabla \tilde{v}, \nabla \varphi \rangle = \langle \Pi_h f, \varphi \rangle, \qquad \forall\, \varphi \in H_0^1(\Omega).$$

In view of (3.1) and (3.3) we can write

$$\|v - \tilde{v}\|_{L^\infty(\Omega)} \le C \|f - \Pi_h f\|_{L^p(\Omega)} \le Ch^2 \|f\|_{W^{2,p}(\Omega)}.$$

Therefore it suffices to show

$$\|\tilde{v} - v_h\|_{L^\infty(\Omega)} \le Ch^2 |\log h| \|f\|_{W^{2,p}(\Omega)}.$$

Set $e_h(x) := \tilde{v} - v_h$ and $x_0 \in \Omega$ be such that $e_h(x_0) = \|e_h\|_{L^\infty(\Omega)}$. By Mean Value Theorem, there is $x_1 \in B(x_0, \gamma h)$ such that $e_h(x_1) = \int_\Omega e_h \delta$. Then by using the definition of H_0^1 projection and well known interpolation properties

$$\begin{aligned}
e_h(x_1) &= \int_\Omega e_h \delta = \int_\Omega \nabla e_h \nabla G = \int_\Omega \nabla e_h \nabla (G - G_h) + \int_\Omega [\Pi_h f G_h - \Pi_h (f G_h)] \\
&\le \int_\Omega \nabla(\tilde{v} - \Pi_h \tilde{v}) \nabla (G - G_h) + Ch^2 \int_\Omega |\nabla \Pi_h f| |\nabla G_h| \\
&\le \int_\Omega |\nabla(\tilde{v} - \Pi_h \tilde{v})| |\nabla (G - G_h)| + Ch^2 \int_\Omega |\nabla \Pi_h f| |\nabla G| \\
&\quad + Ch^2 \int_\Omega |\nabla \Pi_h f| |\nabla (G_h - G)| =: I_1 + I_2 + I_3,
\end{aligned}$$

Since $W^{2,p}(\Omega) \subset C^\xi(\bar{\Omega})$ for $\xi = 2 - n/p > 0$ [5, p.155], $\|\Pi_h f\|_{C^\xi(\bar{\Omega})} \le \|f\|_{C^\xi(\bar{\Omega})}$ and $\Gamma \in C^{2,\alpha}$ for $\alpha > 0$, Schauder regularity theory yields [5, p.98]

$$\|v\|_{W^{2,\infty}(\Omega)},\ \|\tilde{v}\|_{W^{2,\infty}(\Omega)} \le C_p \|f\|_{W^{2,p}(\Omega)}, \qquad p > n/2. \tag{3.8}$$

In light of (3.4) we get

$$I_1 \le Ch^2 |\log h| \|\tilde{v}\|_{W^{2,\infty}(\Omega)} \le Ch^2 |\log h| \|f\|_{W^{2,p}(\Omega)}.$$

In order to estimate I_2 we observe that [5]

$$\|G\|_{W^{2,t}(\Omega)} \le \frac{C}{t-1} \|\delta\|_{L^t(\Omega)} \le \frac{C}{t-1} h^{-n+n/t} \qquad \text{for } 1 < t \le 2.$$

Taking now $t = 1 + |\log h|^{-1}$, we end up with

$$\|G\|_{W^{2,t}(\Omega)} \le C |\log h|.$$

Since $1/p - 1/n < 1/n$, we can choose r such that $1/p - 1/n < 1/r < 1/n$. Let t be as above and s be the conjugate of r. It is clear that $1/s = 1 - 1/r > 1 - 1/n > 1/t - 1/n$. By Sobolev embedding [5, p.155], Hölder's inequality, and (3.2), we have

$$\begin{aligned} I_2 &\leq Ch^2 \|\nabla \Pi_h f\|_{L^r(\Omega)} \|\nabla G\|_{L^s(\Omega)} \leq Ch^2 \|\nabla f\|_{L^r(\Omega)} \|\nabla G\|_{L^s(\Omega)} \\ &\leq Ch^2 \|f\|_{W^{2,p}(\Omega)} \|G\|_{W^{2,t}(\Omega)} \leq Ch^2 |\log h| \|f\|_{W^{2,p}(\Omega)}. \end{aligned}$$

Similarly, with r and s as above, we have $1 - n + n/s = 1 - n/r > 0$. With the aid of (3.5) we obtain

$$\begin{aligned} I_3 &\leq Ch^2 \|\nabla \Pi_h f\|_{L^r(\Omega)} \|\nabla (G - G_h)\|_{L^s(\Omega)} \\ &\leq Ch^2 |\log h|^{2-s} h^{1-n/r} \|\nabla f\|_{L^r(\Omega)} \leq Ch^2 \|f\|_{W^{2,p}(\Omega)}. \end{aligned}$$

Combining the estimates for I_1, I_2 and I_3, we find $e_h(x_1) \leq Ch^2 |\log h| \|f\|_{W^{2,p}(\Omega)}$. Therefore, using an inverse inequality and basic interpolation estimates we get

$$\begin{aligned} \|e_h\|_{L^\infty(\Omega)} = e_h(x_0) &\leq e_h(x_1) + \gamma h \|\nabla e_h\|_{L^\infty(\Omega)} \\ &\leq e_h(x_1) + \gamma h \|\nabla \Pi_h e_h\|_{L^\infty(\Omega)} + \gamma h \|\nabla (\Pi_h e_h - e_h)\|_{L^\infty(\Omega)} \\ &\leq e_h(x_1) + C\gamma \|\Pi_h e_h\|_{L^\infty(\Omega)} + \gamma h \|\nabla (\tilde{v} - \Pi_h \tilde{v})\|_{L^\infty(\Omega)} \\ &\leq Ch^2 |\log h| \cdot \|f\|_{W^{2,p}(\Omega)} + C\gamma \|e_h\|_{L^\infty(\Omega)} + Ch^2 \|\tilde{v}\|_{W^{2,\infty}(\Omega)}. \end{aligned}$$

Now take γ sufficiently small to absorb the second term on the right hand side into the left. Upon using (3.8) again we obtain the asserted estimate. □

If $\Gamma \in C^{1,1} \backslash C^{2,\alpha}$, then (3.8) is no longer valid. We have instead [5, §9]:

$$\|v\|_{W^{2,p}(\Omega)},\ \|\tilde{v}\|_{W^{2,p}(\Omega)} \leq Cp \|f\|_{L^\infty(\Omega)}. \tag{3.9}$$

The explicit dependence on p corresponds to that in the Calderon-Zygmund inequality [5, p.230].

Theorem 3. *Let* $\Gamma \in C^{1,1}$. *Then* $\|v - v_h\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2 \|f\|_{W^{2,p}(\Omega)}$.

Proof. We only have to modify the pointwise estimate of $\tilde{v} - \Pi_h \tilde{v}$ above. By virtue of (3.4), (3.9), and interpolation theory, we get $\|\tilde{v} - \Pi_h \tilde{v}\|_{L^\infty(\Omega)} \leq Ch^2 |\log h| \frac{p}{h^{n/p}} \|f\|_{L^\infty(\Omega)}$. The assertion follows upon taking $p = |\log h|$. □

4. Proof of Theorem 1

Let $w_\varepsilon := u_\varepsilon - v$ and $w_{\varepsilon,h} := u_{\varepsilon,h} - v_h$ where v and v_h are defined in (3.6) and (3.7). Then (1.3) and (1.4) become, respectively,

$$\begin{aligned} \langle \nabla w_\varepsilon, \nabla \varphi \rangle + \langle \beta_\varepsilon (w_\varepsilon + v - \psi), \varphi \rangle &= 0, \qquad \varphi \in H_0^1(\Omega), \\ \langle \nabla w_{\varepsilon,h}, \nabla \chi \rangle + \langle \beta_\varepsilon (w_{\varepsilon,h} + v_h - \Pi_h \psi), \chi \rangle_h &= 0, \qquad \chi \in \mathbb{V}_h^1. \end{aligned}$$

For each $\chi \in \mathbb{V}_h^1 \approx \mathbb{R}^I$, let F be defined by

$$F(\chi)_i := \langle \nabla\chi, \nabla\phi_i\rangle + \langle \beta_\varepsilon(\chi + v_h - \Pi_h\psi), \phi_i\rangle_h, \qquad 1 \le i \le I.$$

By virtue of Lemma 2, F is inverse isotone. Set $\eta(h) := h^2|\log h|^2 \ge C\varepsilon$ and note that (0.3), (2.1), (3.1), Theorem 3, and (3.9) yield the existence of a constant $N > 0$ such that

$$\|\mathbf{R}_h w_\varepsilon - w_\varepsilon\|_{L^\infty(\Omega)}, \|v - v_h\|_{L^\infty(\Omega)}, \|\psi - \Pi_h\psi\|_{L^\infty(\Omega)} \le N\eta(h). \tag{4.1}$$

We introduce a piecewise linear function w_h^+ defined by $w_h^+(x_i) := \mathbf{R}_h w_\varepsilon(x_i) + C_1\eta(h)$ at all interior nodes x_i $(1 \le i \le I)$, and $w_h^+(x_j) := 0$ at boundary nodes x_j $(I < j \le J)$. Our candidate for supersolution reads

$$u_h^+ := w_h^+ + v_h \in \mathbb{V}_h^1.$$

Before we proceed any further, we recall a crucial error estimate from [9]:

$$\|u - u_\varepsilon\|_{L^\infty(\Omega)} \le \lambda\varepsilon. \tag{4.2}$$

We start by choosing $C_1 > 0$ in such a way that $u_h^+ \ge \Pi_h\psi$ in Ω. In fact, in view of (4.1), Lemma 4, and the unilateral constraint $u \ge \psi$, we can write

$$\begin{aligned} u_h^+ - \Pi_h\psi &= (\mathbf{R}_h w_\varepsilon - w_\varepsilon) + (v_h - v) + (\psi - \Pi_h\psi) + (u_\varepsilon - u) + (u - \psi) + C_1\eta(h) \\ &\ge (-3N + C_1)\eta(h) - \lambda\varepsilon \ge 0, \end{aligned}$$

at interior nodes. Moreover $\Pi_h\psi \le 0 = u_h^+$ on Γ, so the above claim follows. Since $\beta_\varepsilon \le 0$, we have for $1 \le i \le I$

$$\int_\Omega \Pi_h\left[\beta_\varepsilon(u_h^+ - \Pi_h\psi)\phi_i\right] = 0 \ \ge \int_\Omega \beta_\varepsilon(u_\varepsilon - \psi)\phi_i,$$

whence

$$\begin{aligned} F(w_h^+)_i &= \int_\Omega \nabla\mathbf{R}_h w_\varepsilon\nabla\phi_i + \int_\Omega \Pi_h\left[\beta_\varepsilon(w_h^+ + v_h - \Pi_h\psi)\phi_i\right] - C_1\eta(h)\sum_{j=I+1}^{J}\int_\Omega \nabla\phi_i\nabla\phi_j \\ &\ge \int_\Omega \nabla w_\varepsilon\nabla\phi_i + \int_\Omega \beta_\varepsilon(w_\varepsilon + v - \psi)\phi_i = 0 \\ &= \int_\Omega \nabla w_{\varepsilon,h}\nabla\phi_i + \int_\Omega \Pi_h\left[\beta_\varepsilon(w_{\varepsilon,h} + v_h - \Pi_h\psi)\phi_i\right] = F(w_{\varepsilon,h})_i, \end{aligned}$$

because of (1.2). Therefore, in view of Lemma 2, we infer that $w_h^+ \ge w_{\varepsilon,h}$ and as a consequence that

$$u_\varepsilon - u_{\varepsilon,h} \ge -C\eta(h). \tag{4.3}$$

Next we take $w_h^-(x_i) := \mathbf{R}_h w_\varepsilon(x_i) - C_2\eta(h)$ for $1 \leq i \leq I$, and $w_h^-(x_j) := 0$ for $I < j \leq J$. We propose the subsolution u_h^- defined by

$$u_h^- := w_h^- + v_h \in \mathbb{V}_h^1.$$

Suppose for a moment that a suitable choice of $C_2 > 0$ leads to the following inequality:

$$\int \Pi_h\left[\beta_\varepsilon(w_h^- + v_h - \Pi_h\psi)\phi_i\right] \leq \int \beta_\varepsilon(w_\varepsilon + v - \psi)\phi_i \qquad 1 \leq i \leq I. \tag{4.4}$$

Then, for $i = 1, \cdots, I$, we see that

$$\begin{aligned} F(w_h^-)_i &= \int_\Omega \nabla \mathbf{R}_h w_\varepsilon \nabla\phi_i + \int_\Omega \Pi_h\left[\beta_\varepsilon(w_h^- + v_h - \Pi_h\psi)\phi_i\right] + C_2\eta(h)\sum_{j=I+1}^{J}\int_\Omega \nabla\phi_i\nabla\phi_j \\ &\leq \int_\Omega \nabla w_\varepsilon\nabla\phi_i + \int_\Omega \beta_\varepsilon(w_\varepsilon + v - \psi)\phi_i = 0 \\ &= \int_\Omega \nabla w_{\varepsilon,h}\nabla\phi_i + \int_\Omega \Pi_h\left[\beta_\varepsilon(w_{\varepsilon,h} + v_h - \Pi_h\psi)\phi_i\right] = F(w_{\varepsilon,h})_i, \end{aligned}$$

because of (1.2). Again from Lemma 2, we deduce that $w_h^- \leq w_{\varepsilon,h}$ and therefore

$$u_\varepsilon - u_{\varepsilon,h} \leq C\eta(h). \tag{4.5}$$

On combining inequalities (4.3) and (4.5) with (4.2), we obtain the asserted error estimate (1.6). To conclude the proof, it only remains to demonstrate (4.4), which reads equivalently

$$\int_T \Pi_h\left[\beta_\varepsilon(z_h)\phi_i\right] \leq \int_T \beta_\varepsilon(z)\phi_i, \qquad \forall\ 1 \leq i \leq I,\ T \in \mathcal{T}_h, \tag{4.6}$$

with $z_h := u_h^- - \Pi_h\psi$ and $z := u_\varepsilon - \psi$. If $z \geq 0$ in T, then

$$\int_T \beta_\varepsilon(z)\phi_i = 0 \geq \int_T \Pi_h[\beta_\varepsilon(z_h)\phi_i], \qquad \forall\ 1 \leq i \leq I.$$

So we can assume that $z(x_0) < 0$ at some point $x_0 \in T$. Since $z \geq -t_0^2 := -\lambda\varepsilon \geq -Ch^2$, we make use of Lemma 4 to conclude $|\nabla z(x_0)| \leq C\sqrt{\eta(h)}$. Hence, in view of Lemma 3, we get

$$|\nabla z(x)| \leq |\nabla z(x_0)| + \sigma_{\nabla z}(|x - x_0|) \leq C\sqrt{\eta(h)}, \qquad \forall\ x \in T.$$

Consequently, for all $x \in T$ we have

$$z(x) = z(x_0) + (x - x_0)^T \cdot \int_0^1 \nabla z(tx + (1-t)x_0)dt \leq C\eta(h),$$

which, for $C_2 > 0$ sufficiently large and all $x, y \in T$, implies

$$\begin{aligned} z_h(x) &= \mathbf{R}_h w_\varepsilon(x) + v_h - \Pi_h \psi - C_2 \eta(h) \\ &\leq w_\varepsilon(x) + v(x) - \psi + (3N - C_2)\eta(h) = z(x) + (3N - C_2)\eta(h) \\ &\leq (C + 3N - C_2)\eta(h) \leq -C\eta(h) \leq -\lambda\varepsilon \leq z(y). \end{aligned}$$

It clearly follows that

$$\int_T \beta_\varepsilon(z)\phi_i \geq \int_T \beta_\varepsilon(-C\eta(h))\phi_i = \int_T \Pi_h[\beta_\varepsilon(-C\eta(h))\phi_i] \geq \int_T \Pi_h[\beta_\varepsilon(z_h)\phi_i].$$

This yields (4.6) and completes the proof of Theorem 1.

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