RESIDUAL TYPE A POSTERIORI ERROR ESTIMATES FOR ELLIPTIC OBSTACLE PROBLEMS

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Abstract. A posteriori error estimators of residual type are derived for piecewise linear finite element approximations to elliptic obstacle problems. An instrumental ingredient is a new interpolation operator which requires minimal regularity, exhibits optimal approximation properties and preserves positivity. Both upper and lower bounds are proved and their optimality is explored with several examples. Sharp a priori bounds for the a posteriori estimators are given, and extensions of the results to double obstacle problems are briefly discussed.

Key words. a posteriori error estimates, residual, elliptic obstacle, positivity preserving interpolation.

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1 Introduction

A posteriori error estimates are computable quantities in terms of the discrete solution and data, which are instrumental for adaptive mesh refinement (and coarsening), error control, and equidistribution of the computational effort. Since the seminal paper [3] by Babuška and Rheinboldt there has been an ever increasing interest on the development of reliable and efficient adaptive algorithms for various linear and nonlinear boundary value problems. We refer to Verfürth [23] for an extensive review on residual-type estimators, the main concern of this paper. The theory for nonlinear equations of [23] is based on linearization, and thus applies to differentiable problems.

Variational inequalities have found applications in such diverse fields as elasticity and fluid dynamics to operations research [4] [9], [11], [13], [16], [17], [20]. They lead to non-Lipschitz

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nonlinearities, and so to failure of the basic tool for a posteriori error analysis - linearization. In this paper we derive the first residual-type a posteriori error estimates for the simplest elliptic obstacle problem:

$$u \in \mathcal{K}: \quad (\nabla u, \nabla(v-u)) \geq (f, v-u) \quad \forall v \in \mathcal{K}. \tag{1.1}$$

Hereafter, Ω is a polyhedral domain in $\mathbf{R}^d$ ($d = 2,3$), $f \in L^2(\Omega)$, the obstacle $\psi \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfies $\psi \leq 0$ on $\partial\Omega$, and $\mathcal{K}$ is the convex set of admissible displacements

$$\mathcal{K} := \{v \in H_0^1(\Omega): \; v \geq \psi \text{ a.e. in } \Omega\}. \tag{1.2}$$

We denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$. The unilateral constraint $u \geq \psi$ yields a line singularity, or free boundary, which is the internal boundary of the *contact* (or coincidence) set

$$\mathcal{C} := \{x \in \Omega : u(x) = \psi(x)\}; \tag{1.3}$$

$\mathcal{N} = \Omega \backslash \mathcal{C}$ stands for the noncoincidence set. The singularity is thus intrinsic to the PDE and not merely of geometric nature such as reentrant corners, which could also be present. The singularity location, as well as strength, is a priori unknown and a prime computational goal.

Hoppe and Kornhuber [14], [18] resorted to hierarchical estimators by enriching the shape functions with quadratics. They obtained upper and lower bounds for a semi-local procedure under the rather restrictive *saturation* assumption, which postulates that quadratics approximate better than linears. Even though this is true for solutions u better than H^2, it is false for the important class of Lipschitz obstacles for which u is not H^2; the same comment applies to reentrant corners. In [14], [18] there is also a local procedure which relies on the strong assumption that the discrete convex set $\mathcal{K}_h$ for linears be contained in that for quadratics and may lead to severe error underestimation. Hlaváček et al. [13] derived a posteriori error estimates by combining the primal and dual formulations, but must solve an extra *global* problem; this is thus impractical. Ainsworth et al. [1] also used the dual variational principle and $\mathcal{K}_h \subset \mathcal{K}$, but tried to localize the estimators. Their upper error bound, however, hinges on a sign condition for the element residuals which may fail in practice. Johnson [15] proved H^1 upper a posteriori error estimates for a penalty formulation of (1.1) with variable penalty parameter and $\psi = 0$; an optimal pointwise error analysis for general obstacles is carried out in [10].

Our model (1.1) is deceivingly simple. If $\psi = 0$ and u_h is the discrete solution, then

$$(\nabla e_u, \nabla(e_u - v)) \leq (f, e_u - v) - (\nabla u_h, \nabla(e_u - v)) = \langle R_h, e_u - v\rangle \quad \forall\, v + u_h \in \mathcal{K}, \tag{1.4}$$

is the variational inequality satisfied by the error $e_u = u - u_h$ and $R_h \in H^{-1}(\Omega)$ is the residual. To estimate $\| \nabla e_u \|_{L^2(\Omega)}$, we take $v = 0$ and use Galerkin orthogonality to write

$$-(\nabla u_h, \nabla e_u) \leq (f, u_h - \Pi_h u) + (\nabla u_h, \nabla(\Pi_h u - u)), \tag{1.5}$$

provided $\Pi_h u \geq 0$. Since $e_u + u_h - \Pi_h u = u - \Pi_h u = (e_u - \Pi_h e_u) + (u_h - \Pi_h u)$, we then obtain

$$\| \nabla e_u \|_{L^2(\Omega)}^2 \leq \langle R_h, e_u - \Pi_h e_u \rangle + \langle R_h, u_h - \Pi_h u_h \rangle. \tag{1.6}$$

We thus realize that to evaluate R_h in $H^{-1}(\Omega)$, the interpolation operator Π_h must be H^1-stable, possess optimal approximation properties, and be positivity preserving, namely,

$$\varphi \geq 0 \quad \Rightarrow \quad \Pi_h(\varphi) \geq 0. \tag{1.7}$$

The usual Lagrange interpolation operator is not stable in H^1 and the averaging type operators of Clément [8] and Scott-Zhang [21] do not satisfy (1.7). In §3 we present a novel interpolation operator Π_h with the requisite properties. Rather remarkably, such a Π_h is *not* a projection and cannot be so together with H^1-stable and positivity preserving [19]; thus $\Pi_h u_h - u_h \neq 0$ must be accounted for.

This paper is organized as follows. In §2 we review the variational setting of (1.1) and introduce its finite element discretization. In §3 we define and analyze the positive operator Π_h. In §4 we present a simple explicit residual-type estimator, and comment on the optimality of both upper and lower bounds. The estimator consists of four distinct parts, namely the interior and jump residuals, which are standard, and the obstacle oscillation and jump residual which take the regularity of ψ into account. We point out that none of these four components arise directly from standard techniques for mildly nonlinear elliptic problems. We deal with rough obstacles, namely $\psi \in H^1(\Omega) \cap C^0(\bar{\Omega})$, which includes the important class of Lipschitz obstacles; the saturation assumption is thus not valid in our setting. We prove an upper a posteriori error estimate of residual type in §5, and a lower error estimate in §6. We derive an a priori estimate reminiscent of that in [5] for the a posteriori quantities in §7. We conclude with extensions of our results to double obstacle problems in §8.

2 Setting and Notation

We assume that Ω is a bounded polyhedral domain in $\mathbf{R}^d$ $(d = 2, 3)$, perhaps with reentrant corners; Ω may *not* be convex. Let $f \in L^2(\Omega)$ and $\psi \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfying $\psi \leq 0$ on $\partial\Omega$. It is known that under these conditions (1.1) admits a unique solution $u \in H_0^1(\Omega)$ and there exists a nonnegative Radon measure μ such that [11], [13], [17], [20]

$$\int_\Omega v d\mu = (\nabla u, \nabla v) - (f, v) \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

and

$$\operatorname{supp}(\mu) \subset \mathcal{C} = \{x \in \Omega : \ u(x) = \psi(x)\}. \tag{2.2}$$

If $\psi \in H^2(\Omega)$, then also $u \in H^2(\Omega)$ provided $\partial\Omega$ is $C^{1,1}$ (or Ω is convex), and μ is absolutely continuous with respect to the Lebesgue measure $d\mu = \lambda dx$ with $0 \le \lambda \in L^2(\Omega)$.

Let $\mathcal{P}_h$ be a regular partition of Ω into simplices. Given a simplex $S \in \mathcal{P}_h$, h_S stands for its diameter and γ_S is the minimum angle of S; mesh regularity is equivalent to $\gamma_S \ge \gamma > 0$. We denote by $h = h(x)$ the mesh density function, that is the piecewise constant function $h|_S = h_S$ for all $S \in \mathcal{P}_h$. We also denote by $\mathcal{B}_h$ the collection of all internal interelement boundaries or sides e of the simplicies in $\mathcal{P}_h$; h_e stands for the size of $e \in \mathcal{B}_h$. For each $S \in \mathcal{P}_h$ and each $e \in \mathcal{B}_h$, we set

$$(\phi, \varphi)_S := \int_S \phi\varphi dx, \quad \langle \phi, \varphi \rangle_e := \int_e \phi\varphi ds.$$

Let $\mathcal{W}_h$ indicate the space of continuous piecewise linear finite elements over $\mathcal{P}_h$ and $\mathcal{V}_h = \mathcal{W}_h \cap H_0^1(\Omega)$. Let $I_h : C(\bar{\Omega}) \to \mathcal{W}_h$ denote the usual Lagrange interpolation operator, i.e. $I_h(\phi)(x_i) = \phi(x_i)$ for all nodes x_i of $\mathcal{P}_h$. Let $\mathcal{K}_h$ be the discrete counterpart of (1.2), that is

$$\mathcal{K}_h := \{v_h \in \mathcal{V}_h : \ v_h \ge I_h\psi \ \text{ on } \Omega \ \}. \tag{2.3}$$

The discrete solution u_h satisfies

$$u_h \in \mathcal{K}_h : \quad (\nabla u_h, \nabla(v_h - u_h)) \ge (f, v_h - u_h) \ \ \forall v_h \in \mathcal{K}_h. \tag{2.4}$$

Since the discrete unilateral constraint in (2.3) is only enforced at the nodes of $\mathcal{P}_h$, its implementation is an easy task in practice; alternative constraints such as $v_h \ge \psi$ or $v_h \ge \Pi_h\psi$ simplify the error analysis but complicate the implementation. It is well known that the solution $u_h \in \mathcal{K}_h$ exists and is unique [5], [7], and can be efficiently computed via monotone multigrid methods [16], for instance.

We also assume a rather technical condition about the behavior of ψ on $\partial\Omega$, namely,

$$\psi \le I_h\psi \ \text{ on } \partial\Omega. \tag{2.5}$$

We remark that this assumption is not very restrictive in practice since the free boundary $\partial\mathcal{C} \cap \Omega$ does not touch that part of the boundary $\partial\Omega$ where $\psi < 0$ and thus the local shape of ψ there does not dictate those of u and u_h.

Let $\{x_j\}_{j=1}^J$ be the interior nodes of mesh $\mathcal{P}_h$. For each integer $1 \le j \le J$, let $\phi_j \in \mathcal{V}_h$ denote the j^{th} canonical basis function, i.e. $\phi_j(x_i) = \delta_{ij}$, $1 \le i \le J$. For each $1 \le j \le J$, let

$$S_j := \cup\{S \in \mathcal{P}_h : \operatorname{supp}(\phi_j) \cap S \ne \emptyset\}, \quad \Sigma_j := \{e \in \mathcal{B}_h : e \cap x_j \ne \emptyset\} \tag{2.6}$$

be the star surrounding x_j and the set of its interior edges, respectively. Let Δ_j be the maximal ball centered at x_j such that $\Delta_j \subset S_j$. Let h_j denote the diameter of S_j and ρ_j the radius of

Δ_j. Then mesh regularity implies the existence of a constant C independent of h such that

$$Ch_j \le \rho_j \le h_j.$$

We now introduce the discrete counterpart λ_h of λ, which may be viewed as a Lagrange multiplier associated with the discrete constraint $u_h \ge I_h\psi$ of (2.4). Let $\lambda_h \in \mathcal{V}_h$ be given by

$$(\lambda_h, v_h)_h = (\nabla u_h, \nabla v_h) - (f, v_h) \quad \forall v_h \in \mathcal{V}_h, \tag{2.7}$$

where

$$(\phi, \varphi)_h := \sum_{S\in\mathcal{P}_h} \int_S I_h(\phi\varphi)dx$$

is the usual vertex quadrature rule for any piecewise uniformly continuous functions ϕ and φ. From (2.4) we infer that

$$(\lambda_h, v_h - u_h)_h \ge 0 \quad \forall v_h \in \mathcal{K}_h,$$

whence

$$\lambda_h(x_j) \ge 0, \quad \lambda_h(x_j)(u_h(x_j) - \psi(x_j)) = 0, \qquad \forall\, 1 \le j \le J. \tag{2.8}$$

We indicate with $\mathcal{C}_h$ the discrete coincidence (or contact) set

$$\mathcal{C}_h := \cup\{S_j : u_h(x_j) = \psi(x_j)\}. \tag{2.9}$$

We denote by $\mathcal{P}_h^C$ the collection of elements whose union is $\mathcal{C}_h$. We also set $\mathcal{N}_h := \Omega\backslash\mathcal{C}_h$ and $\mathcal{P}_h^N := \mathcal{P}_h\backslash\mathcal{P}_h^C$, and stress that $\lambda_h = 0$ in $\mathcal{N}_h$.

Given a closed set A, typically a simplex S, an edge e, or a node x_j, $\tilde{A}$ stands for

$$\tilde{A} = \cup\{S \in \mathcal{P}_h : A \cap S \ne \emptyset\};$$

thus $S_j = \tilde{x}_j$. For any $v_h \in \mathcal{W}_h$, we define the jump of ∇v_h across an interelement side $e \in \mathcal{B}_h$ as

$$[\![\nabla v_h]\!] := (\nabla v_h|_{S_1} - \nabla v_h|_{S_2}) \cdot \nu_e$$

with the convention that the unit normal vector ν_e to e points from S_2 to S_1, and observe that

$$(\nabla u_h, \nabla v_h) = -\sum_{e\in\mathcal{B}_h} \int_e [\![\nabla u_h]\!] v \qquad \forall\, v \in H_0^1(\Omega). \tag{2.10}$$

Moreover, if A is a union of elements and $\mathcal{B}_h(A)$ is the set of edges interior to A, we define

$$|\!|\!|\, h^k [\![v_h]\!] \,|\!|\!|_{L^2(A)} := \Big(\sum_{e\in\mathcal{B}_h(A)} \int_e h_e^{2k} [\![v_h]\!]^2 ds \Big)^{1/2} \quad \forall\, v_h \in \mathcal{V}_h,\ k \in \mathbf{R}. \tag{2.11}$$

3 Positivity Preserving Interpolation

In this section we introduce the new interpolation operator Π_h, which requires minimum regularity, exhibits optimal approximation properties and preserves positivity. Such a Π_h will be used in the a posteriori error analysis of §5. For any $\varphi \in L^1(\Omega)$, we define $\Pi_h : L^1(\Omega) \to \mathcal{V}_h$ by

$$(\Pi_h\varphi)(x) := \sum_{j=1}^{J} \Big(\frac{1}{|\Delta_j|}\int_{\Delta_j} \varphi dx\Big)\phi_j(x). \tag{3.1}$$

We note that (3.1) corresponds to using the local L^2 projection over the space of constants $P^0(\Delta_j)$ instead of that of linears $P^1(S_j)$ in the definition of the Clément interpolation operator [8]. In view of the symmetry of Δ_j with respect to x_j, Π_h still satisfies the crucial property

$$\Pi_h\varphi(x_j) = \varphi(x_j) \qquad \forall\ \varphi \in P^1(\Delta_j). \tag{3.2}$$

This in turn yields the following invariance property for all *interior* elements $S \in \mathcal{P}_h$:

$$\Pi_h\varphi|_S = \varphi|_S \qquad \forall\ \varphi \in P^1(\tilde{S}), \tag{3.3}$$

where $\tilde{S}$ is the union of all elements surrounding S. In addition, Π_h preserves the constraint (1.7), namely, $\Pi_h\varphi \geq 0$ if $\varphi \geq 0$. We first discuss the stability of Π_h.

Lemma 3.1 *The following local stability estimates are valid for all $1 \leq p \leq \infty$ and $S \in \mathcal{P}_h$*

$$\|\,\Pi_h\varphi\,\|_{L^p(S)} \leq C\|\,\varphi\,\|_{L^p(\tilde{S})} \quad \forall\varphi \in L^p(\Omega), \qquad \|\,\nabla\Pi_h\varphi\,\|_{L^p(S)} \leq C\|\,\nabla\varphi\,\|_{L^p(\tilde{S})} \quad \forall\varphi \in W_0^{1,p}(\Omega).$$

Proof. The L^p-stability is obvious from (3.1). To show the $W^{1,p}$-stability, we first consider an interior element S for which (3.3) applies. Then

$$\begin{aligned}
\|\,\nabla\Pi_h\varphi\,\|_{L^p(S)} &= \inf_{\alpha\in P^0(\tilde{S})} \|\,\nabla\Pi_h(\varphi-\alpha)\,\|_{L^p(S)} \\
&\leq Ch_S^{-1} \inf_{\alpha\in P^0(\tilde{S})} \|\,\Pi_h(\varphi-\alpha)\,\|_{L^p(S)} \\
&\leq Ch_S^{-1} \inf_{\alpha\in P^0(\tilde{S})} \|\,\varphi-\alpha\,\|_{L^p(\tilde{S})} \\
&\leq C\|\,\nabla\varphi\,\|_{L^p(\tilde{S})},
\end{aligned}$$

where we have used a local inverse estimate and next an interpolation error estimate. We consider now a boundary element S, namely, an element that has at least one node on $\partial\Omega$. Since φ vanishes on a set of $\partial\tilde{S}$ of positive measure, we see that

$$\|\,\nabla\Pi_h\varphi\,\|_{L^p(S)} \leq Ch_S^{-1}\|\,\Pi_h\varphi\,\|_{L^p(S)} \leq Ch_S^{-1}\|\,\varphi\,\|_{L^p(\tilde{S})} \leq C\|\,\nabla\varphi\,\|_{L^p(\tilde{S})},$$

where we have employed the Poincaré inequality. This completes the proof. $\Box$

As a consequence of Lemma 3.1 we conclude that Π_h exhibits optimal approximation properties with minimal regularity requirements.

Lemma 3.2 *The following estimate is valid for $k = 1, 2$ and for all $1 \le p \le \infty$, $S \in \mathcal{P}_h$*

$$\| \varphi - \Pi_h \varphi \|_{L^p(S)} \le C h_S^k \| D^k \varphi \|_{L^p(\tilde{S})}, \quad \forall\, \varphi \in W^{k,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Proof. Let $\zeta \in P^1(\tilde{S})$ be arbitrary. Since $\varphi - \Pi_h\varphi = (\varphi - \zeta) - (\Pi_h\varphi - \zeta)$, it suffices to prove

$$\| \Pi_h\varphi - \zeta \|_{L^p(S)} \le C \| \varphi - \zeta \|_{L^p(\tilde{S})} + C h_S \| \nabla(\varphi - \zeta) \|_{L^p(\tilde{S})},$$

and next use interpolation theory. This follows directly from (3.3) and Lemma 3.1 provided S is an interior element. We examine now the more delicate case of a boundary element S. For all vertices x_j of S lying on $\partial\Omega$ there is a side e_j of $\tilde{S}$ also lying on $\partial\Omega$. Combining the (scaled) trace inequality

$$\| v \|_{L^p(e)} \le C \Big(h_e^{-1/p} \| v \|_{L^p(\tilde{e})} + h_e^{1-1/p} \| \nabla v \|_{L^p(\tilde{e})} \Big) \quad \forall\, v \in W^{1,p}(\Omega),\ e \in \mathcal{B}_h, \tag{3.4}$$

the fact that $\varphi = 0$ on $\partial\Omega$, and then a local inverse inequality, we get

$$\begin{aligned} |\zeta(x_j)| &\le C h_S^{(1-d)/p} \| \zeta \|_{L^p(e_j)} = C h_S^{(1-d)/p} \| \zeta - \varphi \|_{L^p(e_j)} \\ &\le C h_S^{-d/p} \Big(\| \zeta - \varphi \|_{L^p(\tilde{S})} + h_S \| \nabla(\zeta - \varphi) \|_{L^p(\tilde{S})} \Big). \end{aligned}$$

For interior vertices x_j of S we use (3.1), (3.2), and Hölder inequality to arrive at

$$|(\Pi_h\varphi - \zeta)(x_j)| \le |\Delta_j|^{-1} \int_{\Delta_j} |\varphi - \zeta| \le C h_S^{-d/p} \| \zeta - \varphi \|_{L^p(\tilde{S})}.$$

The desired estimate results from $(\Pi_h\varphi - \zeta)|_S = \sum_{x_j \in S} (\Pi_h\varphi - \zeta)(x_j)\phi_j$ and the property $\| \phi_j \|_{L^p(S)} \le C h_S^{d/p}$. $\Box$

Combining Lemmas 3.1 and 3.2 with $k = 1, p = 2$ and using the (scaled) trace inequality (3.4), we readily obtain the following result.

Corollary 3.1 *The following L^2-based error estimates are valid for all $\varphi \in H_0^1(\Omega)$*

$$\| \varphi - \Pi_h\varphi \|_{L^2(S)} \le C h_S \| \nabla\varphi \|_{L^2(\tilde{S})}, \tag{3.5}$$

$$\| \varphi - \Pi_h\varphi \|_{L^2(e)} \le C h_e^{1/2} \| \nabla\varphi \|_{L^2(\tilde{e})}. \tag{3.6}$$

Lemma 3.3 *There exist a constant $C > 0$ depending only on the minimum angle of $\mathcal{P}_h$ such that for all $1 \le j \le J$ we have*

$$|(v_h - \Pi_h v_h)(x_j)| \le C h_j^{\frac{3-d}{2}} ||| \, [\![\nabla v_h]\!] \, |||_{L^2(S_j)} \quad \forall \, v_h \in \mathcal{V}_h. \tag{3.7}$$

Before proving Lemma 3.3 we state a simple by-product which constitutes a discrete version of Lemma 3.2 for $p = 2$. Together with Lemma 3.3, they are instrumental in the derivation of a posteriori error estimates.

Corollary 3.2 *There exist a constant $C > 0$ depending only on the minimum angle of $\mathcal{P}_h$ such that, for all $v_h \in \mathcal{V}_h$, $S \in \mathcal{P}_h$, and $e \in \mathcal{B}_h$*

$$\| \, v_h - \Pi_h v_h \, \|_{L^2(S)} \le C ||| \, h^{3/2} [\![\nabla v_h]\!] \, |||_{L^2(\tilde{S})}, \tag{3.8}$$

$$\| \, v_h - \Pi_h v_h \, \|_{L^2(e)} \le C ||| \, h [\![\nabla v_h]\!] \, |||_{L^2(\tilde{e})}. \tag{3.9}$$

Proof of Lemma 3.3. Since v_h is linear on $S \cap \Delta_j$ for any $S \in \mathcal{P}_h$, we have

$$v_h(x) = v_h(x_j) + \nabla v_h|_S \cdot (x - x_j), \quad \forall x \in S \cap \Delta_j.$$

Thus, by definition of Π_h, we get

$$\begin{aligned}
(v_h - \Pi_h v_h)(x_j) &= v_h(x_j) - \frac{1}{|\Delta_j|} \int_{\Delta_j} v_h(x) dx \\
&= -\frac{1}{|\Delta_j|} \sum_{S \in \mathcal{P}_h} \int_{S \cap \Delta_j} \nabla v_h|_S \cdot (x - x_j) dx \\
&= -\frac{1}{2|\Delta_j|} \sum_{S \in \mathcal{P}_h} \int_{S \cap \Delta_j} \nabla v_h|_S \cdot \nabla (x - x_j)^2 dx \\
&= -\frac{1}{2|\Delta_j|} \sum_{S \in \mathcal{P}_h} \int_{\partial (S \cap \Delta_j)} (\nabla v_h|_S \cdot \nu)(x - x_j)^2 ds \\
&= \frac{1}{2|\Delta_j|} \sum_{e \in \Sigma_j} \int_{e \cap \Delta_j} [\![\nabla v_h]\!] (x - x_j)^2 ds \\
&- \frac{1}{2|\Delta_j|} \sum_{S \in \mathcal{P}_h} \int_{\partial \Delta_j \cap S} (\nabla v_h|_S \cdot \nu)(x - x_j)^2 ds =: \mathrm{I} + \mathrm{II},
\end{aligned} \tag{3.10}$$

where the fourth equality results from integration by parts and so ν denotes the unit outer normal to $\partial(S \cap \Delta_j)$. Using $|x - x_j| \le h_j$, together with the fact that the cardinal of Σ_j is bounded by a constant depending only on the minimum angle of the mesh $\mathcal{P}_h$, we easily obtain

$$\mathrm{I} \le \frac{1}{2|\Delta_j|} \Big(\sum_{e \in \Sigma_j} \| \, [\![\nabla v_h]\!] \, \|_{L^2(e)}^2 \Big)^{1/2} \Big(\sum_{e \in \Sigma_j} h_j^4 |e| \Big)^{1/2} \le C h_j^{\frac{3-d}{2}} ||| \, [\![\nabla v_h]\!] \, |||_{L^2(S_j)}.$$

On the other hand, since $v_h|_S$ is linear, integration by parts yields

$$\int_{\partial(\Delta_j\cap S)} (\nabla v_h|_S\cdot\nu)ds = \int_{\Delta_j\cap S} \Delta v_h dx = 0,$$

which coupled with the property $|x-x_j| = \rho_j$ on $\partial\Delta_j$ implies

$$\mathrm{II} = \frac{-\rho_j^2}{2|\Delta_j|}\sum_{S\in\mathcal{P}_h}\int_{\partial\Delta_j\cap S}(\nabla v_h|_S\cdot\nu)ds = \frac{-\rho_j^2}{2|\Delta_j|}\sum_{e\in\Sigma_j}\int_{e\cap\Delta_j}[\![\nabla v_h]\!]ds \le Ch_j^{\frac{3-d}{2}} |\!|\!|\, [\![\nabla v_h]\!]\, |\!|\!|_{L^2(S_j)}.$$

Substituting the estimates for I and II into (3.10) leads to (3.7) and completes the proof. $\square$

Remark 3.1. It is important to realize that there does *not* exist any higher order interpolation operator which is also positivity preserving [19]. $\square$

Remark 3.2. We stress that Π_h in (3.1) is *not* a projection operator, i.e., $\Pi_h\chi \ne \chi$ for $\chi\in\mathcal{V}_h$ unless χ is locally linear. It can be proved that any projection operator $Q_h : C(\bar\Omega)\to\mathcal{V}_h$ which is positivity preserving reduces to the usual Lagrange interpolation operator, which is known not to be H^1 stable [19]. $\square$

4 The Main Results

We state now the upper and lower a posteriori error estimates but postpone their proof until sections 5 and 6. We discuss variants of these results and assess their optimality.

Theorem 4.1 (Upper Bound) *Let $f\in L^2(\Omega)$ and $\psi\in H^1(\Omega)\cap C(\bar\Omega)$ satisfy (2.5) and $\psi\le 0$ on $\partial\Omega$. Then there exists a constant $C_1>0$ depending on the minimum angle of $\mathcal{P}_h$ such that the following a posteriori error estimate holds*

$$\|\, u-u_h \,\|_{H^1(\Omega)} \le C_1 \sum_{i=1}^4 \mathcal{E}_i, \tag{4.1}$$

where the error indicators $\mathcal{E}_i$ are given by

$\mathcal{E}_1 :=$	$\|\!\|\!\|\, h^{1/2}[\![\nabla u_h]\!]\, \|\!\|\!\|_{L^2(\Omega)}$	*jump residual,*
$\mathcal{E}_2 :=$	$\|\, hf \,\|_{L^2(\Omega)}$	*interior residual,*
$\mathcal{E}_3 :=$	$\big(\|\, \nabla(\psi - I_h\psi) \,\|_{L^2(\Omega)} + \|\, h^{-1}(\psi - I_h\psi) \,\|_{L^2(\Omega)}\big)$	*obstacle oscillation,*
$\mathcal{E}_4 :=$	$\|\!\|\!\|\, h^{1/2}[\![\nabla I_h\psi]\!]\, \|\!\|\!\|_{L^2(\mathcal{C}_h)}$	*obstacle jump residual.*

Remark 4.1. Since $\| [\![\nabla u_h]\!] \|_{L^2(\partial S)} \le \| [\![\nabla (u_h - I_h\psi)]\!] \|_{L^2(\partial S)} + \| [\![\nabla I_h\psi]\!] \|_{L^2(\partial S)}$, and the last term is already present in $\mathcal{E}_4$ for $S \in \mathcal{P}_h^C$, we could replace $\mathcal{E}_1(S)^2 := h_S \| [\![\nabla u_h]\!] \|_{L^2(S)}^2$ by

$$\mathcal{E}_1^{\mathrm{new}}(S)^2 = \begin{cases} h_S \| [\![\nabla u_h]\!] \|_{L^2(\partial S)}^2 & \text{if } S \in \mathcal{P}_h^N, \\ h_S \| [\![\nabla (u_h - I_h\psi)]\!] \|_{L^2(\partial S)}^2 & \text{if } S \in \mathcal{P}_h^C, \end{cases}$$

Note that the new $\mathcal{E}_1^{\mathrm{new}}(S)$ vanishes for S's interior to the discrete coincidence set $\mathcal{C}_h$. □

Remark 4.2. We now discuss the *stability* of $\mathcal{C}_h$ and whether $\mathcal{E}_2$ could be localized to $\mathcal{N}_h$, as may seem reasonable from the fact that f plays a role only in the noncoincidence set. For ε small, we consider the parametrized problem $\Omega = (0,2), \psi = 0$, and

$$f_\varepsilon(x) = \begin{cases} 1+\varepsilon & \text{if } 0 < x < 1, \\ -1+\varepsilon & \text{if } 1 < x < 2, \end{cases} \quad \Rightarrow \quad u_\varepsilon(x) = \begin{cases} \delta_\varepsilon x + \alpha_\varepsilon x(1-x) & \text{if } 0 < x < 1, \\ \delta_\varepsilon x + \beta_\varepsilon (x-1)(x - \frac{1}{\beta_\varepsilon}) & \text{if } 1 < x < \xi_\varepsilon, \\ 0 & \text{if } \xi_\varepsilon < x < 2, \end{cases}$$

where $\alpha_\varepsilon = \frac{1+\varepsilon}{2}, \beta_\varepsilon = \frac{1-\varepsilon}{2}, \delta_\varepsilon = \frac{3-\varepsilon}{2} - \sqrt{2(1-\varepsilon)}$ and $\xi_\varepsilon = \sqrt{\beta_\varepsilon^{-1}}$; thus $\mathcal{C} = [\xi_\varepsilon, 2)$. We see that $\mathcal{C}_\varepsilon$ depends continuously on ε, which is consistent with the stability condition $f \le \nu < 0$ in a vicinity of the free boundary [6], [11, §2.11]. Consider a mesh $\mathcal{P}_h$ with only one interior point, namely $x_1 = 1$. Since $(f, \phi_1) = \varepsilon$ we infer that $u_{\varepsilon,h} = 0$ for $\varepsilon \le 0$ and $u_{\varepsilon,h} > 0$ for $\varepsilon > 0$; thus

$$\mathcal{C}_{\varepsilon,h} = \begin{cases} \Omega & \text{if } \varepsilon \le 0, \\ \emptyset & \text{if } \varepsilon > 0. \end{cases} \tag{4.2}$$

We conclude that $\mathcal{C}_{\varepsilon,h}$ is *unstable*, and note that this is presumably due to the change of sign of f within $\mathrm{supp}(\phi_1)$. We also point out that we *cannot* replace $\mathcal{P}_{\varepsilon,h}$ in $\mathcal{E}_2$ by $\mathcal{P}_{\varepsilon,h}^N = \emptyset$ for $\varepsilon \le 0$, for otherwise $\mathcal{E}_2 = 0$. In fact, $u_h = \psi = 0$ would lead to $\mathcal{E}_1 = \mathcal{E}_3 = 0$ and so to the contradiction $0 < \| u_\varepsilon - u_{\varepsilon,h} \|_{H^1(\Omega)} = 0$. □

Remark 4.3. Let $\Omega = (0,1), \psi = 0, h = \frac{1}{2(n+1)}$ be the mesh-size, and ε be a small parameter. Let f_ε be the periodic function with period $2h$ given by $f_\varepsilon(x) = 1+\varepsilon$ for $0 < x < h$ and $f_\varepsilon(x) = -1+\varepsilon$ for $h < x < 2h$. For $\varepsilon = 0$ the coincidence set $\mathcal{C}_0$ contains a discrete set of points $x_k = (2k+\frac{3}{2})h$ for $1 \le k \le n-1$, but this set disappears for $\varepsilon > 0$ and develops interior for $\varepsilon < 0$; this instability of $\mathcal{C}_\varepsilon$ is consistent with [6], [11]. However, since $(f, \phi_j) = \varepsilon h$, we infer that (4.2) is valid again, and conclude that the instability of $\mathcal{C}_h$ and comments of Remark 4.2 are not related to a particular mesh $\mathcal{P}_h$ but rather to a combination of $\mathcal{P}_h$ and f. $\Box$

Remark 4.4. Let $\Omega = (0,1), \psi = 0, h = \frac{1}{2n}$ be the mesh-size, and $\varepsilon > 0$ be a small parameter. Let F_ε and U_ε be 2-periodic functions given by

$$F_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon} & \text{if } |x| < \varepsilon, \\ \frac{-1}{1-\varepsilon} & \text{if } \varepsilon \leq |x| \leq 1, \end{cases} \qquad U_\varepsilon(x) = \begin{cases} \frac{\xi_\varepsilon}{4} - \frac{x^2}{4\varepsilon} & \text{if } |x| < \varepsilon, \\ \frac{(x-\xi_\varepsilon)^2}{2(1-\varepsilon)} & \text{if } \varepsilon \leq |x| \leq \xi_\varepsilon, \\ 0 & \text{if } \xi_\varepsilon < |x| < 1, \end{cases}$$

where $\xi_\varepsilon = \frac{1+\varepsilon}{2}$. Let f_ε be the $2h$-periodic function defined by $f_\varepsilon(x) = F_\varepsilon(\frac{x-2hk}{h} - 1)$ for $2hk < x < 2(k+1)h$, and let $u_\varepsilon(x) = h^2 U_\varepsilon(\frac{x-2hk}{h} - 1)$ be the solution of the corresponding obstacle problem. We see that $\mathcal{C}_\varepsilon$ is a collection of intervals of size either ξ_ε or $2\xi_\varepsilon$, which is stable with respect to ε; the measure of $\mathcal{C}_\varepsilon$ is about $1/2$. On the other hand if the mesh is also $2h$-periodic with $x_0 = 0, x_1 = h(1-\varepsilon), x_2 = h(1+\varepsilon)$ and $x_3 = 2h$, then $(f, \phi_j) \leq 0$ and $u_h = 0$. We thus conclude that, for all $\varepsilon > 0$ small, $\mathcal{N}_{\varepsilon,h} = \emptyset$ whereas $\mathcal{N}_\varepsilon$ amounts to about half of $(0,1)$ in measure. Since $\mathcal{N}_\varepsilon$ and $\mathcal{N}_{\varepsilon,h}$ can be strikingly different, it is not advisable to localize $\mathcal{E}_2$ to either $\mathcal{N}$ or $\mathcal{N}_h$ unless further structural assumptions are made. $\Box$

Remark 4.5. The third term $\mathcal{E}_3$ in (4.1) disappears when ψ is piecewise linear, and is thus a measure of the elementwise oscillation of ψ. Consider $\Omega = (-1,1), f = 0, \psi(x) = \max(0, 1 - \alpha|x|)$ with $\alpha > 0$ large; thus $u(x) = 1 - |x|$ and $\mathcal{C} = \{0\}$. Let $\mathcal{P}_h$ be a uniform partition not including $x = 0$, and $\alpha > \frac{2}{h}$. Since $I_h\psi(x) = 0$ we obtain $u_h(x) = 0$, and realize that $\mathcal{E}_3$ cannot be eliminated from (4.1). It is reasonable to expect that for an adaptive strategy to succeed, the initial mesh should resolve the elementwise oscillations of ψ in the entire Ω. In particular, if ψ exhibits straight line singularities, the first mesh as well as subsequent meshes should match these singularities with interelement boundaries. $\Box$

Remark 4.6. The estimator $\mathcal{E}_4$ measures global oscillations of $I_h\psi$ in $\mathcal{C}_h$. The restriction to $\mathcal{C}_h$ is important in practice to avoid overrefinement in a region where $I_h\psi$ is inactive. $\Box$

For any $T \in \mathcal{P}_h$, let T^* be the enlarged set

$$T^* := \cup\{S \in \mathcal{P}_h : \ S \text{ and } T \text{ has a common side } \}$$

and denote by $\mathcal{E}(T)$ the following local error estimator

$$\mathcal{E}(T)^2 := \sum_{S \subset T^*} h_S^2 \| f \|_{L^2(S)}^2 + \sum_{e \subset \partial T} h_e \| [\![\nabla u_h]\!] \|_{L^2(e)}^2. \tag{4.3}$$

Our next objective is to give a partial lower bound. In fact, the next estimate is not fully a posteriori in that it involves the measure μ.

Theorem 4.2 (Lower Bound) *Under the conditions of Theorem 4.1, there exist constants $C_2, C_3 > 0$ depending on the minimum angle of $\mathcal{P}_h$ such that, for any $T \in \mathcal{P}_h$,*

$$\| \nabla(u - u_h) \|^2_{L^2(T^*)} \geq C_2 \mathcal{E}(T)^2 - C_3 \Big(\| h(f - \bar{f}) \|^2_{L^2(T^*)} + h_T^{2-d} \mu(T^*)^2 \Big), \tag{4.4}$$

where $\bar{f}|_S := \frac{1}{|S|} \int_S f dx$ is the average of $f \in L^2(\Omega)$ over $S \in \mathcal{P}_h$, and μ is the nonnegative Radon measure defined in (2.1).

Remark 4.7. From (2.2) we know that $\text{supp}(\mu) \subset \mathcal{C}$. If $\mathcal{C}$ happens to be empty, i.e. u is the solution to a linear elliptic problem, then Theorem 4.2 yields

$$\| \nabla(u - u_h) \|^2_{L^2(\Omega)} \geq C_2 \Big(\mathcal{E}_1^2 + \mathcal{E}_2^2 \Big) - C_3 \| h(f - \bar{f}) \|^2_{L^2(\Omega)},$$

and this estimate is sharp [23]. In general, Theorem 4.2 implies that

$$\| \nabla(u - u_h) \|^2_{L^2(T^*)} \geq C_2 \mathcal{E}(T)^2 - C_3 \| h(f - \bar{f}) \|^2_{L^2(T^*)}$$

for any $T \in \mathcal{P}_h$ such that $T^* \subset \mathcal{N}$. Consequently, the local estimator $\mathcal{E}(T)$ is sharp for elements T in the interior of $\mathcal{N}$ and thus overrefinement will not occur in $\mathcal{N}$. $\Box$

Remark 4.8. Let $\Omega = [-1, 1], f = 0$ and $\psi(x) = 1 - 2|x|$; thus $u(x) = 1 - |x|$ and $\mu = 2\delta$, where δ is the Dirac mass at $x = 0$. If $\mathcal{P}_h$ is a uniform partition of $[-1, 1]$ containing $x = 0$, then $u = u_h$. It is clear that $\mathcal{E}(T)^2 = 4h_T \neq 0$ for any of the elements T containing $x = 0$ but $\| \nabla(u - u_h) \|_{L^2(T^*)} = 0$. We conclude that the term involving the measure μ in (4.4) cannot be eliminated in the contact set $\mathcal{C}$, which in this case is $\mathcal{C} = \text{supp}(\mu) = \{0\}$. $\Box$

Remark 4.9. We explore now whether the jumps of $\nabla I_h \psi$ in $\mathcal{P}_h^C$ could be included in the definition of (4.3). Let $\Omega = (-1, 1), f = 0$, and $\psi(x) = \max(1 - 2|x|, \frac{1}{4} - |x - \frac{3}{4}|)$; thus $u(x) = 1 - |x|, \mu = 2\delta$, and $\mathcal{C} = \{0\} \cup [\frac{3}{4}, 1)$. Let $\mathcal{P}_h$ be a uniform partition of $[0, 1]$ including $x = 0, 1/2, 3/4$; thus $I_h \psi = \psi$ and $u = u_h$. If T is an element containing $x = 3/4$, then all terms in (4.4) vanish but $[\![\psi']\!](3/4) = -2$. We conclude that it is not possible to obtain an estimate such as (4.4) with $\nabla I_h \psi$ in $\mathcal{E}(T)$. $\Box$

5 Proof of Theorem 4.1

1. We first derive an expression for the error $e_u := u - u_h$. To this end, we would like to take $v = u_h$ in (1.1), but such a v_h may not be admissible. We take instead $v = u_h^* \in \mathcal{K}$, where

$$u_h^* := \max(u_h, \psi),$$

to get

$$(\nabla u, \nabla e_u) - (f, e_u) \le (\nabla u, \nabla(u_h^* - u_h)) - (f, u_h^* - u_h). \tag{5.1}$$

From (2.4) it is easy to derive the inequality

$$(\nabla u_h, \nabla(u_h - v)) - (f, u_h - v) \le (\nabla u_h, \nabla(v_h - v)) - (f, v_h - v) \quad \forall v \in \mathcal{K}, v_h \in \mathcal{K}_h. \tag{5.2}$$

We would like to take $v_h = \Pi_h u$ in (5.2), but such a v_h may not be admissible. We thus take $v_h = u^* \in \mathcal{K}_h$, where

$$u^* := I_h(\max(\Pi_h u, I_h \psi)),$$

together with $v = u$ to arrive at

$$\begin{aligned} -(\nabla u_h, \nabla e_u) + (f, e_u) \ &\le\ \Big((\nabla u_h, \nabla(u^* - \Pi_h u)) - (f, u^* - \Pi_h u)\Big) \\ &+\ \Big((\nabla u_h, \nabla(\Pi_h u - u)) - (f, \Pi_h u - u)\Big). \end{aligned} \tag{5.3}$$

Adding (5.1) to (5.3) and noticing that

$$\Pi_h u - u = (\Pi_h e_u - e_u) + (\Pi_h u_h - u_h),$$

we obtain the following variant of (1.6) which accounts for a nontrivial obstacle ψ

$$\begin{aligned} \| \nabla e_u \|_{L^2(\Omega)}^2 \ &\le\ \Big((\nabla u, \nabla(u_h^* - u_h)) - (f, u_h^* - u_h)\Big) \\ &+\ \Big((\nabla u_h, \nabla(u^* - \Pi_h u)) - (f, u^* - \Pi_h u)\Big) \\ &+\ \Big((\nabla u_h, \nabla(\Pi_h e_u - e_u)) - (f, \Pi_h e_u - e_u)\Big) \\ &+\ \Big((\nabla u_h, \nabla(\Pi_h u_h - u_h)) - (f, \Pi_h u_h - u_h)\Big) =: \mathrm{I} + \cdots + \mathrm{IV}. \end{aligned}$$

The rest of the proof consists of estimating in order the four terms on the right-hand side. We stress that making the (nonpractical) choice of $\Pi_h \psi$ as a discrete obstacle, as opposed to the actual $I_h \psi$, leads to $u^* = \Pi_h u$ and thus to $\mathrm{II} = 0$. This suggests that $\Pi_h \psi - I_h \psi$ should dictate the size of II, as becomes apparent in step 4 below. We also point out that IV $= 0$ provided Π_h is a projector, which is not the case here.

2. We introduce a distribution $\Lambda_h \in H^{-1}(\Omega)$ related to λ_h and defined by

$$\langle \Lambda_h, v\rangle := (\nabla u_h, \nabla v) - (f, v) \quad \forall\ v \in H_0^1(\Omega). \tag{5.4}$$

Combining (2.10) with the (scaled) trace inequality (3.4) with $p = 2$ we easily obtain

$$\begin{aligned} |\langle \Lambda_h, v\rangle| &\le \sum_{e\in\mathcal{B}_h} \int_e |[\![\nabla u_h]\!] v| ds + |(f, v)| \\ &\le C\Big(|\!|\!|\, h^{1/2}[\![\nabla u_h]\!] \,|\!|\!|_{L^2(\Omega)} + \|\, hf \,\|_{L^2(\Omega)}\Big)\Big(\|\, \nabla v \,\|_{L^2(\Omega)} + \|\, h^{-1} v \,\|_{L^2(\Omega)}\Big), \end{aligned} \tag{5.5}$$

where $C > 0$ depends only on the minimum angle of $\mathcal{P}_h$. Since

$$\lambda_h(x_j)\frac{|S_j|}{d+1} = (\lambda_h, \phi_j)_h = \langle \Lambda_h, \phi_j\rangle,$$

(5.6) immediately gives rise to the following pointwise estimate for λ_h

$$\lambda_h(x_j) \le C h_j^{-\frac{d}{2}-1}\Big(|\!|\!|\, h^{1/2}[\![\nabla u_h]\!] \,|\!|\!|_{L^2(S_j)} + \|\, hf \,\|_{L^2(S_j)}\Big). \tag{5.6}$$

3. In order to estimate I we observe that $u_h^* \in H_0^1(\Omega)$ and

$$0 \le u_h^* - u_h = \max(u_h, \psi) - u_h \le (\psi - I_h\psi)_+.$$

In light of (2.1) and the property $\mu \ge 0$, we have

$$\mathrm{I} = (\nabla u, \nabla(u_h^* - u_h)) - (f, u_h^* - u_h) = \int_\Omega (u_h^* - u_h) d\mu \le \int_\Omega (\psi - I_h\psi)_+ d\mu.$$

We now resort to (2.5), for the first and only time, to infer that $(\psi - I_h\psi)_+ \in H_0^1(\Omega)$. This enables us to use (2.1) again and thereby get

$$\begin{aligned} \mathrm{I} &\le (\nabla u, \nabla(\psi - I_h\psi)_+) - (f, (\psi - I_h\psi)_+) \\ &= \Big(\nabla(u - u_h), \nabla(\psi - I_h\psi)_+\Big) \\ &+ \Big((\nabla u_h, \nabla(\psi - I_h\psi)_+) - (f, (\psi - I_h\psi)_+)\Big) =: \mathrm{V} + \mathrm{VI}. \end{aligned}$$

We estimate V and VI separately. It is clear that for all $\eta > 0$

$$\mathrm{V} \le \eta \|\, \nabla(u - u_h) \,\|_{L^2(\Omega)}^2 + \frac{1}{4\eta} \|\, \nabla(\psi - I_h\psi) \,\|_{L^2(\Omega)}^2.$$

On the other hand, in view of (5.5) with $v = (\psi - I_h\psi)_+$ we deduce

$$\mathrm{VI} \le C\Big(|\!|\!|\, h^{1/2}[\![\nabla u_h]\!] \,|\!|\!|_{L^2(\Omega)}^2 + \|\, hf \,\|_{L^2(\Omega)}^2 + \|\, \nabla(\psi - I_h\psi) \,\|_{L^2(\Omega)}^2 + \|\, h^{-1}(\psi - I_h\psi) \,\|_{L^2(\Omega)}^2\Big).$$

Collecting the estimates for V and VI and choosing $\eta = 1/4$ we arrive at

$$\mathrm{I} \le C\Big(\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_3^2\Big) + \frac{1}{4}\|\, \nabla(u - u_h) \,\|_{L^2(\Omega)}^2. \tag{5.7}$$

4. We point out that II is problematic because it involves u, which is not computable and must be removed. However, since Π_h is positivity preserving, we realize that

$$u \ge \psi \quad \Rightarrow \quad \Pi_h u \ge \Pi_h \psi.$$

For each interior node x_j we thus have

$$0 \le u^*(x_j) - (\Pi_h u)(x_j) \le ((I_h\psi)(x_j) - (\Pi_h\psi)(x_j))_+,$$

which eliminates the explicit appearance of u. Using (2.7) together with $\lambda_h \ge 0$, we obtain

$$\mathrm{II} = \sum_{j=1}^{J} \lambda_h(x_j)(v_h - \Pi_h u)(x_j) \int_{S_j} \phi_j dx \le C \sum_{j=1}^{J} h_j^d \lambda_h(x_j)(I_h\psi(x_j) - \Pi_h\psi(x_j))_+.$$

To proceed further, we note that (3.7) in conjunction with the definition (3.1) of Π_h yields

$$\begin{aligned}(I_h\psi(x_j) - \Pi_h\psi(x_j))_+ &\le (I_h\psi(x_j) - \Pi_h(I_h\psi)(x_j))_+ + (\Pi_h(I_h\psi - \psi)(x_j))_+ \\ &\le C h_j^{\frac{3-d}{2}} |\!|\!|\, [\![\nabla I_i\psi]\!] \,|\!|\!|_{L^2(S_j)} + C h_j^{-\frac{d}{2}} \|\, \psi - I_h\psi \,\|_{L^2(S_j)}.\end{aligned}$$

Since $\lambda_h = 0$ on $\mathcal{N}_h$, combining this estimate with (5.6), we get

$$\mathrm{II} \le C\Big(\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_3^2 + \mathcal{E}_4^2\Big).$$

5. Term III is rather standard in a posteriori error analysis. In fact, in light of (5.5) with $v = \Pi_h e_u - e_u$, and (3.5) and (3.6) with $\varphi = e_u$, we have

$$\mathrm{III} \le C\Big(\mathcal{E}_1^2 + \mathcal{E}_2^2\Big) + \frac{1}{4}\|\, \nabla(u - u_h) \,\|_{L^2(\Omega)}^2.$$

We recall that IV does not vanish because Π_h is not a projection. This important difference with respect to standard error analysis can be handled via Corollary 3.2, which applied with $v_h = u_h$ leads to

$$\mathrm{VI} \le C\Big(|\!|\!|\, h^{1/2}[\![\nabla u_h]\!] \,|\!|\!|_{L^2(\Omega)}^2 + \|\, hf \,\|_{L^2(\Omega)}^2\Big) = C\Big(\mathcal{E}_1^2 + \mathcal{E}_2^2\Big).$$

6. Substituting the estimates for I to IV into (5.4), we derive the asserted estimate (4.1).

6 Proof of Theorem 4.2

The proof is based on a constructive local argument due to Verfürth [22]. We first introduce the *local* estimator

$$\eta_T^2 := \sum_{S\subset T^*} h_S^2 \| f_S \|_{L^2(S)}^2 + \sum_{e\subset\partial T} h_e \| [\![\nabla u_h]\!] \|_{L^2(e)}^2, \tag{6.1}$$

where $f_S := \bar{f}|_S$. We seek a function $\varphi \in H_0^1(T^*) \subset H_0^1(\Omega)$ of the form

$$\varphi := \sum_{S\subset T^*} \alpha_S \varphi_S + \sum_{e\subset\partial T} \beta_e \varphi_e,$$

where $\varphi_S \in P_{d+1}(S)$ is the canonical bubble function of S, i.e., the product of the barycentric coordinates of S, and φ_e is the canonical basis function of $P_d(S)$ which vanishes at all other nodes except at the barycenter x_e of $e \subset \partial S$, at which $\varphi_e(x_e) = 1$. The numbers α_S and β_e are chosen so that

$$\langle [\![\nabla u_h]\!], \varphi\rangle_e = h_e \| [\![\nabla u_h]\!] \|_{L^2(e)}^2, \tag{6.2}$$

$$(f_S, \varphi)_S = h_S^2 \| f_S \|_{L^2(S)}^2. \tag{6.3}$$

Since $[\![\nabla u_h]\!]_e$ is constant on e and $\varphi|_e = \beta_e \varphi_e$, we deduce from (6.2) that

$$\beta_e [\![\nabla u_h]\!]_e \langle 1, \varphi_e\rangle_e = h_e \| [\![\nabla u_h]\!] \|_{L^2(e)}^2,$$

which defines β_e uniquely; $\beta_e := 0$ provided $[\![\nabla u_h]\!] = 0$. A simple scaling argument implies

$$|\beta_e| \le C h_e^{\frac{3-d}{2}} \| [\![\nabla u_h]\!] \|_{L^2(e)}. \tag{6.4}$$

To satisfy (6.3) we select $\alpha_S \in \mathbf{R}$ such that

$$\alpha_S f_S (1, \varphi_S)_S = \alpha_S (f_S, \varphi_S)_S = h_S^2 \| f_S \|_{L^2(S)}^2 - \sum_{e\subset\partial T} \beta_e (f_S, \varphi_e)_S,$$

which in turn defines α_S uniquely; $\alpha_S := 0$ if $f_S = 0$. Again, a simple scaling argument combined with (6.4) yields

$$\begin{aligned} |\alpha_S| &\le C h_S^{\frac{4-d}{2}} \| f_S \|_{L^2(S)} + C \sum_{e\subset\partial T} |\beta_e| \\ &\le C h_S^{\frac{4-d}{2}} \| f_S \|_{L^2(S)} + C \sum_{e\subset\partial T} h_e^{\frac{3-d}{2}} \| [\![\nabla u_h]\!] \|_{L^2(e)}. \end{aligned} \tag{6.5}$$

Since $\mathcal{P}_h$ is shape regular, then φ_S and φ_e satisfy the obvious bounds

$$\|\nabla\varphi_S\|_{L^2(T^*)} + \|\nabla\varphi_e\|_{L^2(T^*)} \leq C h_T^{-1}|T^*|^{1/2} \leq C h_T^{\frac{d-2}{2}},$$

whence (6.4) and (6.5) clearly yield

$$\begin{aligned}
\|\nabla\varphi\|_{L^2(T^*)} &\leq \sum_{S\subset T^*}|\alpha_S|\|\nabla\varphi_S\|_{L^2(T^*)} + \sum_{e\subset\partial T}|\beta_e|\|\nabla\varphi_e\|_{L^2(T^*)} \\
&\leq C\Big(\sum_{S\subset T^*} h_S^2\|f_S\|^2_{L^2(S)} + \sum_{e\subset\partial T} h_e\|[\![\nabla u_h]\!]\|^2_{L^2(e)}\Big)^{1/2} \\
&\leq C\eta_T.
\end{aligned}\tag{6.6}$$

Similarly, we obtain

$$\|\varphi\|_{L^2(T^*)} \leq C h_T\eta_T, \quad \|\varphi\|_{L^\infty(T^*)} \leq C h_T^{\frac{2-d}{2}}\eta_T.\tag{6.7}$$

Combining (6.1)-(6.3) with (2.10) we have

$$\begin{aligned}
\eta_T^2 &= \sum_{S\subset T^*}(f_S,\varphi)_S + \sum_{e\subset\partial T}\langle[\![\nabla u_h]\!],\varphi\rangle_e \\
&= \sum_{S\subset T^*}(f_S - f,\varphi)_S + (f,\varphi)_{T^*} - (\nabla u_h,\nabla\varphi)_{T^*} \\
&= \sum_{S\subset T^*}(f_S - f,\varphi)_S + (\nabla(u-u_h),\nabla\varphi) - \int_\Omega \varphi\, d\mu,
\end{aligned}$$

where we have used (2.1). Finally the asserted estimate (4.4) follows from (6.6) and (6.7) together with the fact that $\mathrm{supp}(\varphi)\subset T^*$.

7 A Priori Error Analysis

We recall the following convergence result under minimal regularity [12]

$$\lim_{H\to 0}\|u-u_h\|_{H^1(\Omega)} = 0,\tag{7.1}$$

where $H := \max_{S\in\mathcal{P}_h} h_S$. We now wonder about a similar result for the a posteriori estimator

$$\mathcal{E}(u_h, f, \psi, h) := \sum_{i=1}^{4}\mathcal{E}_i.\tag{7.2}$$

Since $\mathcal{E}$ depends on u_h, which changes with $\mathcal{P}_h$, the asymptotics of $\mathcal{E}$ is unclear. We start with a weak result that mimics (7.1).

Lemma 7.1 *Let $f \in L^2(\Omega)$ and $\psi \in W^{1,p}(\Omega)$ for some $p > d$. Then we have*

$$\lim_{H \to 0} \mathcal{E}(u_h, f, \psi, h) = 0.$$

Proof. Since $f \in L^2(\Omega)$ and $\psi \in W^{1,p}(\Omega)$ with $p > d$, interpolation theory implies $\mathcal{E}_2 + \mathcal{E}_3 \to 0$ as $H \to 0$. If u were smooth, we would have $[\![u_h]\!] = [\![u_h - u]\!]$ and, on adding and subtracting $I_h u$ and removing the jumps, we could write

$$\mathcal{E}_1^2 \leq C \| \nabla(u_h - I_h u) \|_{L^2(\Omega)}^2 + C \| \nabla(u - I_h u) \|_{L^2(\Omega)}^2 + C \| h D^2 u \|_{L^2(\Omega)}^2;$$

here we have used the inverse estimate $\| h^{1/2} [\![\nabla v_h]\!] \|_{L^2(\Omega)} \leq C \| \nabla v_h \|_{L^2(\Omega)}$ and the trace inequality (3.4). This estimate, coupled with a regularization argument, leads to $\mathcal{E}_1 \to 0$ as $H \to 0$ if u is rough. Similar reasoning applies to $\mathcal{E}_4$ and concludes the proof. $\square$

We now derive an optimal *a priori* error estimate for $\mathcal{E}(u_h, f, \psi, h)$ under additional regularity assumptions on the data and the solution. Let $\psi \in H^2(\Omega)$ with $\psi \leq 0$ on $\partial\Omega$, and $f \in L^2(\Omega)$. The Radon measure $d\mu = \lambda dx$ is absolutely continuous with respect to the Lebesgue measure and $\lambda = -\Delta u - f \in L^2(\Omega)$ [11], [17], [20]. Under these conditions, the following a priori error estimate is valid [5], [7]

$$\begin{aligned} \| u - u_h \|_{H^1(\Omega)} \;&\leq\; C\Big(\| h\lambda \|_{L^2(\Omega)} + \| hD^2\psi \|_{L^2(\Omega)}\Big) \\ &+\; C \inf_{v_h \in \mathcal{K}_h} \Big(\| \nabla(u - v_h) \|_{L^2(\Omega)} + \| h^{-1}(u - v_h) \|_{L^2(\Omega)}\Big). \end{aligned} \tag{7.3}$$

We conclude this section with an *optimal a priori* upper bound for $\mathcal{E}(u_h, f, \psi, h)$, that is of the same order as (7.3).

Lemma 7.2 *Let $f \in L^2(\Omega)$ and $\psi \in H^2(\Omega)$ satisfy (2.5) and $\psi \leq 0$ on $\partial\Omega$. Then there exists a constant $C > 0$ independent of h such that*

$$\begin{aligned} \mathcal{E}(u_h, f, \psi, h) \;&\leq\; C\Big(\| hf \|_{L^2(\Omega)} + \| hD^2\psi \|_{L^2(\Omega)}\Big) \\ &+\; C \inf_{v_h \in K_h} \Big(\| \nabla(u - v_h) \|_{L^2(\Omega)} + \| h^{-1}(u - v_h) \|_{L^2(\Omega)}\Big). \end{aligned}$$

Proof. Since $\psi \in H^2(\Omega)$, interpolation theory directly yields $\mathcal{E}_3^2 + \mathcal{E}_4^2 \leq C \| hD^2\psi \|_{L^2(\Omega)}^2$. The estimate of $\mathcal{E}_1$ and $\mathcal{E}_2$ resorts to Theorem 4.2, and is thus indirect. We see that

$$\mathcal{E}_1^2 + \mathcal{E}_2^2 \leq C \| u - u_h \|_{H^1(\Omega)}^2 + C \| h(f - \bar{f}) \|_{L^2(\Omega)}^2 + C \sum_{S \in \mathcal{P}_h} h_S^{2-d} \mu(S)^2.$$

Since $\lambda = -\Delta\psi - f$ in $\mathcal{C}$ and $\lambda = 0$ otherwise, we infer that

$$\mu(S) = \int_S d\mu = \int_S \lambda dx \le \| \lambda \|_{L^2(S)} |S|^{1/2} \le C h_S^{d/2} \| f \|_{L^2(S)} + C h_S^{d/2} \| D^2\psi \|_{L^2(S)},$$

whence

$$\mathcal{E}_1^2 + \mathcal{E}_2^2 \le C \| u - u_h \|_{H^1(\Omega)}^2 + C \| hf \|_{L^2(\Omega)}^2 + C \| hD^2\psi \|_{L^2(\Omega)}^2.$$

The desired estimate follows from (7.3). $\square$

8 Double Obstacle Problems

In this section we extend the results in previous sections to double obstacle problems. Let $\psi_1, \psi_2 \in H^1(\Omega) \cap C(\bar{\Omega})$ such that $\psi_1 \le \psi_2$ on Ω and $\psi_1 \le 0, \psi_2 \ge 0$ on $\partial\Omega$. Let

$$\mathcal{K} := \{ v \in H_0^1(\Omega) : \psi_1 \le v \le \psi_2 \;\text{ a.e. in } \Omega \}$$

be the convex set of constraints, and consider the following problem: solve

$$u \in \mathcal{K} : \quad (\nabla u, \nabla(v-u)) \ge (f, v-u) \quad \forall v \in \mathcal{K}. \tag{8.1}$$

If $f \in H^{-1}(\Omega)$, then (8.1) admits a unique solution $u \in \mathcal{K}$ and there exists a Radon measure μ such that [17]

$$\int_\Omega \zeta d\mu = (\nabla u, \nabla \zeta) - (f, \zeta) \quad \forall \zeta \in H_0^1(\Omega). \tag{8.2}$$

It is easy to see that $\text{supp}(\mu) \subset \mathcal{C}_1 \cup \mathcal{C}_2$, where the coincidence sets $\mathcal{C}_1, \mathcal{C}_2$ are given by

$$\mathcal{C}_1 := \{ x \in \Omega : u(x) = \psi_1(x) \}, \quad \mathcal{C}_2 := \{ x \in \Omega : u(x) = \psi_2(x) \}.$$

From (8.1) we deduce that $\mu \ge 0$ in $\mathcal{C}_1$ whereas $\mu \le 0$ in $\mathcal{C}_2$. Let

$$\mathcal{K}_h := \{ v_h \in \mathcal{V}_h : I_h\psi_1 \le v_h \le I_h\psi_2 \;\text{ in } \Omega \}$$

be the discrete convex set of constraints. The discrete problem reads [5]:

$$u_h \in \mathcal{K}_h : \quad (\nabla u_h, \nabla(v_h - u_h)) \ge (f, v_h - u_h) \quad \forall v_h \in \mathcal{K}_h. \tag{8.3}$$

We denote by $\mathcal{C}_h^1$ and $\mathcal{C}_h^2$ the discrete coincidence or contact sets

$$\mathcal{C}_h^1 := \cup \{ S_j : u_h(x_j) = \psi_1(x_j) \}, \quad \mathcal{C}_h^2 := \cup \{ S_j : u_h(x_j) = \psi_2(x_j) \}.$$

Theorem 8.1 *Let $f \in L^2(\Omega)$ and $\psi_1, \psi_2 \in H^1(\Omega) \cap C(\bar{\Omega})$ with $\psi_1 \le \psi_2$ on Ω and $\psi_1 \le I_h\psi_1 \le 0, \psi_2 \ge I_h\psi_2 \ge 0$ on $\partial\Omega$. Then there exists a constant $C_4 > 0$ depending on the minimum angle of $\mathcal{P}_h$ such that the following a posteriori error estimate holds*

$$\| u - u_h \|_{H^1(\Omega)} \le C_4 \sum_{i=1}^{6} \mathcal{E}_i,$$

where the error indicators $\mathcal{E}_i$ are given by

$\mathcal{E}_1 :=$	$\vert\vert\vert\, h^{1/2} [\![\nabla u_h]\!] \,\vert\vert\vert_{L^2(\Omega)}$	*jump residual,*
$\mathcal{E}_2 :=$	$\| hf \|_{L^2(\Omega)}$	*interior residual,*
$\mathcal{E}_3 :=$	$\| \nabla(\psi_1 - I_h\psi_1) \|_{L^2(\Omega)} + \| h^{-1}(\psi_1 - I_h\psi_1) \|_{L^2(\Omega)}$	*obstacle 1 oscillation,*
$\mathcal{E}_4 :=$	$\| \nabla(\psi_2 - I_h\psi_2) \|_{L^2(\Omega)} + \| h^{-1}(\psi_2 - I_h\psi_2) \|_{L^2(\Omega)}$	*obstacle 2 oscillation,*
$\mathcal{E}_5 :=$	$\vert\vert\vert\, h^{1/2} [\![\nabla I_h\psi_1]\!] \,\vert\vert\vert_{L^2(\mathcal{C}_h^1)}$	*obstacle 1 jump residual,*
$\mathcal{E}_6 :=$	$\vert\vert\vert\, h^{1/2} [\![\nabla I_h\psi_2]\!] \,\vert\vert\vert_{L^2(\mathcal{C}_h^2)}$	*obstacle 2 jump residual.*

Proof. We argue as in §5 but with the auxiliary functions ($\varphi_+ := \max(0, \varphi)$, $\varphi_- := (-\varphi)_+$)

$$u_h^* := u_h + (\psi_1 - u_h)_+ - (\psi_2 - u_h)_- \in \mathcal{K}$$

and

$$u^* := \Pi_h u + I_h((I_h\psi_1 - \Pi_h u)_+) - I_h((I_h\psi_2 - \Pi_h u)_-) \in \mathcal{K}_h. \quad \square$$

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