OPTIMAL INTERFACE ERROR ESTIMATES FOR THE MEAN CURVATURE FLOW*

R. H. NOCHETTO† M. PAOLINI‡ AND C. VERDI‡

Abstract. We study a double obstacle problem for a singularly perturbed reaction diffusion equation with a small parameter $\varepsilon > 0$. This problem is viewed as an approximation for the classical mean curvature flow. We derive an optimal error estimate of order $\mathcal{O}(\varepsilon^2)$ for the interfaces, which is valid before the onset of singularities. The proof is based on constructing suitable sub and supersolutions, and using a modified distance function that incorporates a shape correction.

AMS(MOS) subject classifications (1985 revision). 49Q05, 35K57, 35B25, 35A35

1. Introduction. It is known that the singularly perturbed reaction-diffusion equation, with a smooth double equal well potential $\Psi : \mathbf{R} \to \mathbf{R}$,

$$\partial_t u_\varepsilon - \Delta u_\varepsilon + \tfrac{1}{2\varepsilon^2}\Psi'(u_\varepsilon) = 0 \qquad \text{in } \mathbf{R}^n \times (0, +\infty),$$

provides an approximation for an interface $\Sigma(t)$ evolving by mean curvature

$$V = \kappa^m,$$

where V is the normal velocity of the interface $\Sigma(t)$ and κ^m the sum of its principal curvatures [1,2,3,4,6,7,8,9,13,14]. Such an equation was introduced by Allen and Cahn [1] in order to describe the motion of antiphase boundaries in crystalline solids, thus showing its relevance in phase transitions. It was independently suggested by De Giorgi [6] as a variational approach to the mean curvature flow. Such a connection has been rigorously established by Evans, Soner, and Souganidis [9], who have proved convergence of the zero level set of u_ε to the generalized motion by mean curvature [10], even beyond the onset of singularities, provided the limit interface does not develop interior; see also [2,4,13]. Asymptotic analyses were carried out prior to those convergence results, but apply only to smooth evolutions [3,7,8,14].

For the quartic double well potential $\Psi(s) = (1-s^2)^2$, the solution u_ε exhibits a transition layer of width $\mathcal{O}(\varepsilon)$ across the interface, satisfies $|u_\varepsilon| < 1$ in the whole $\mathbf{R}^n \times (0, +\infty)$, and tends exponentially fast to the limit values ± 1. This is specially inconvenient for numerical

*This work was partially supported by NSF Grant DMS-9008999, and by MURST (Progetto Nazionale "Equazioni di Evoluzione e Applicazioni Fisico-Matematiche" and "Analisi Numerica e Matematica Computazionale") and CNR (IAN and Contract 91.01312.01) of Italy.

†Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA.

‡Dipartimento di Matematica, Università di Milano, 20133 Milano, and Istituto di Analisi Numerica del CNR, 27100 Pavia, Italy.

purposes because of the need to solve the problem in the entire domain, even though the action takes place in a relatively small region. Computational evidence shows that the location of the approximate interface may be very sensitive to the boundary condition for bounded domains, and thus somehow unstable. It is thus important for computational purposes, and of theoretical interest as well, to examine the effect of substituting the smooth potential by

$$\Psi(s) = \begin{cases} 1 - s^2 & \text{if } s \in [-1,1] \\ +\infty & \text{if } s \notin [-1,1]. \end{cases}$$

In fact, the solution u_ε of the resulting double obstacle PDE attains the values $u_\varepsilon = \pm 1$ outside a narrow transition layer of order $\mathcal{O}(\varepsilon)$, thus reducing the computation of u_ε to a thin n–dimensional strip (see [5,15]).

Suppose the initial interface $\Sigma(0)$ is a smooth, bounded, and closed hypersurface. It is a consequence of the Maximum Principle that $\Sigma(t)$ is contained in the convex hull of $\Sigma(0)$, $\operatorname{conv}(\Sigma(0))$, and that $\Sigma(t)$ eventually disappears for finite time [10, Thm. 7.1b; 3,12]. We can then set the double obstacle problem in a bounded domain $\Omega \subset \mathbf{R}^n$ containing $\operatorname{conv}(\Sigma(0))$, impose a vanishing flux condition on $\partial\Omega$, and study the evolution up to a finite time T.

The goal of this paper is to prove an optimal error estimate, valid before the onset of singularities, for the distance between the mean curvature flow $\Sigma(t)$ and the approximate interface $\Sigma_\varepsilon(t) = \{\mathbf{x} \in \Omega : u_\varepsilon(\mathbf{x}, t) = 0\}$. In fact, assuming that $\Sigma = \{(\mathbf{x}, t) \in \Omega \times [0, T] : \mathbf{x} \in \Sigma(t)\}$ is sufficiently regular (see §5.1) and $u_\varepsilon(\cdot, 0)$ has the correct shape (see §5.2), we prove that

$$\operatorname{dist}\big(\Sigma(t), \Sigma_\varepsilon(t)\big) \le C_T \varepsilon^2 \qquad \forall\, t \in [0, T],$$

where C_T is a constant depending on T but independent of ε and dist is the Hausdorff distance. This estimate is optimal and improves the results obtained by Chen for the regular potential [4] and by Chen and Elliott for the double obstacle potential [5], who show a first order error estimate via comparison arguments.

Both the order and optimality of the interface error estimate are a consequence of the Maximum Principle and the explicit construction of sub and supersolutions, which in turn are inspired by, and indeed rely on, the formal asymptotics developed in [15]. It is not obvious that the resulting asymptotic expansion for u_ε converges, not even for fixed number of terms and $\varepsilon \downarrow 0$, as in [7,8] for the smooth potential. This is an interesting open problem, which would also lead to a rate of convergence for interfaces, via nondegeneracy properties of both the continuous and truncated solutions. In this light, a nontrivial by-product of the rigorous asymptotic analysis of De Mottoni and Schatzman [7,8] would be an $\mathcal{O}(\varepsilon^2)$ interface error estimate for the regular potential, but under higher regularity restrictions on Σ than in the present discussion. Regardless of convergence, the formal asymptotics of §3 still provides valuable information on the shape of u_ε, which, together with a modified distance function and the nondegeneracy property of sub and supersolutions, plays a fundamental role in our subsequent rigorous analysis; see §§5.3, 6, and 7.

The paper is organized as follows. In §2 we introduce a nontrivial 1-D solution associated with the double obstacle potential. Formal asymptotics for a contracting circle is presented in §3. This derivation provides the basis for the construction of several special functions which, properly combined, lead to the barriers of §§5 and 7. A simple but crucial comparison result is proved in §4. The sub and supersolutions are fully examined in §5, and then used in §6 to derive interface error estimates. In §7 we return to the special case of a circle evolving by mean curvature and show rigorously that our estimate is sharp.

2. Double Well Potential. The subdifferential graph $\psi = \frac{1}{2}\Psi'$ of Ψ is given by

$$\psi(s) = \begin{cases} (-\infty, 1] & \text{if } s = -1 \\ -s & \text{if } s \in (-1,1) \\ [-1,+\infty) & \text{if } s = 1. \end{cases}$$

We seek solutions of the double obstacle problem

$$\gamma''(x) - \psi(\gamma(x)) \ni 0 \qquad \text{in } \mathbf{R} \tag{2.1}$$

that correspond to absolute minimizers of the functional

$$\mathcal{F}(\varphi) = \int_{\mathbf{R}} \big(|\varphi'(x)|^2 + \Psi(\varphi(x))\big)\,dx \qquad \text{for } \varphi \in D(\mathcal{F}),$$

where $D(\mathcal{F}) = \{\varphi \in H^1_{\text{loc}}(\mathbf{R}) : |\varphi(x)| \le 1 \text{ in } \mathbf{R},\ \lim_{x\to\pm\infty}\varphi(x) = \pm 1\}$. It is not difficult to see that any absolute minimizer γ must be nondecreasing. Since the inequality $\mathcal{F}(\varphi) \ge 2\int_{\mathbf{R}} \varphi'(x)\sqrt{\Psi(\varphi(x))}dx = \pi$ holds for any nondecreasing function $\varphi \in D(\mathcal{F})$, then the desired minimizers γ satisfy $\gamma'(x) = \sqrt{\Psi(\gamma(x))}$. It turns out that imposing $\gamma(0) = 0$, the unique nondecreasing solution $\gamma \in D(\mathcal{F})$ of (2.1) is given by

$$\gamma(x) = \begin{cases} -1 & \text{if } x < -\frac{\pi}{2} \\ \sin x & \text{if } x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ +1 & \text{if } x > \frac{\pi}{2}. \end{cases}$$

Hence, $\gamma \in C^{1,1}(\mathbf{R})$; see Figure 3.1. This function will be used in §5.2 to define the initial datum u^0_ε for the double obstacle problem.

The following elementary inequality will be useful in the sequel:

$$\gamma(x) - \gamma(x-h) > \tfrac{1}{\pi}h\Big(h + \min\big(\tfrac{\pi}{2} + (x-h), \tfrac{\pi}{2} - x\big)\Big) \qquad \forall\ -\tfrac{\pi}{2} < x-h < x < \tfrac{\pi}{2} \tag{2.2}$$

and $0 < h \ll 1$. It is a trivial consequence of the relation

$$\gamma(x) - \gamma(x-h) = \int_{x-h}^{x} \gamma'(z)dz > \int_{x-h}^{x} \big(1 - \tfrac{2}{\pi}|z|\big)dz \ge \tfrac{1}{\pi}h\big(\pi - (|x| + |x-h|)\big).$$

3. Formal Asymptotics for the Radial Case. We propose here a brief discussion of formal asymptotics for the simplest situation, the radially symmetric case. Even though this presentation is only *formal*, it motivates several crucial aspects of our subsequent rigorous analysis such as the definition of initial datum and sub and supersolutions in §§5 and 7.

Consider in $\mathbf{R}^2$ a circle of radius $\phi(t)$ which evolves by mean curvature, that is

$$\phi'(t) + \tfrac{1}{\phi(t)} = 0, \qquad \phi(0) = 1. \tag{3.1}$$

Since $\phi(t) = \sqrt{1-2t}$, we see that the circle shrinks to a point at time $t^\star = \frac{1}{2}$. If the initial datum $u_\varepsilon(\cdot, 0)$ of the double obstacle problem is radially symmetric, so is the solution $u_\varepsilon(\cdot, t)$. Using polar coordinates, we denote by $r = \phi_\varepsilon(t)$ the zero level set of u_ε, that is

$$\phi_\varepsilon(0) = 1, \qquad u_\varepsilon(\phi_\varepsilon(t), t) = 0. \tag{3.2}$$

In addition, the PDE satisfied by u_ε within the transition layer $\{|u_\varepsilon(r,t)| < 1\}$ reads

$$\partial_t u_\varepsilon - \tfrac{1}{r}\partial_r(r\partial_r u_\varepsilon) - \tfrac{1}{\varepsilon^2}u_\varepsilon = 0. \tag{3.3}$$

Let us introduce the stretched variable $y = \frac{r-\phi_\varepsilon(t)}{\varepsilon}$ and denote

$$U_\varepsilon(y,t) = u_\varepsilon\big(\phi_\varepsilon(t) + \varepsilon y, t\big),$$

whence $u_\varepsilon(r,t) = U_\varepsilon\big(\frac{r-\phi_\varepsilon(t)}{\varepsilon}, t\big)$. With the notation $U_\varepsilon^{(k)} = \partial_y^k U_\varepsilon$, we realize that

$$\partial_t u_\varepsilon = \partial_t U_\varepsilon - \tfrac{1}{\varepsilon}\phi_\varepsilon' U_\varepsilon' \quad \text{and} \quad \tfrac{1}{r}\partial_r(r\partial_r u_\varepsilon) = \tfrac{1}{\varepsilon r}U_\varepsilon' + \tfrac{1}{\varepsilon^2}U_\varepsilon''. \tag{3.4}$$

Suppose that both U_ε and ϕ_ε can be expressed in terms of ε (inner expansion) as follows:

$$U_\varepsilon(y,t) = \textstyle\sum_{i=0}^\infty \varepsilon^i U_i(y,t), \qquad \phi_\varepsilon(t) = \sum_{i=0}^\infty \varepsilon^i \varphi_i(t).$$

This and (3.2) entail $U_i(0,t) = 0$ for all $i \geq 0$, $\varphi_0(0) = 1$, and $\varphi_i(0) = 0$ for all $i \geq 1$.

We now intend to derive explicit expressions of, and compatibility conditions for, the first terms of these formal asymptotic expansions; see [15]. Using (3.4), together with

$$\tfrac{1}{r} = \big(\varepsilon y + \textstyle\sum_{i=0}^\infty \varepsilon^i \varphi_i(t)\big)^{-1} = \frac{1}{\varphi_0} - \varepsilon\frac{y+\varphi_1}{\varphi_0^2} + \varepsilon^2\Big(\frac{(y+\varphi_1)^2}{\varphi_0^3} - \frac{\varphi_2}{\varphi_0^2}\Big) + \mathcal{O}(\varepsilon^3),$$

we can substitute U_ε into (3.3) and collect all resulting terms containing like powers of ε. In the sequel we skip details and simply examine the first four summands in increasing order and equate them to zero, thus starting with the $\frac{1}{\varepsilon^2}$–term:

$$U_0''(y,t) + U_0(y,t) = 0, \qquad U_0(-\tfrac{\pi}{2}, t) = -1, \quad U_0(\tfrac{\pi}{2}, t) = 1.$$

Determining boundary conditions for the inner expansion is a delicate issue that typically involves matching with the outer expansion. The presence of obstacles, however, creates an interesting difference with respect to usual matched asymptotics: the boundary of the transition region should be allowed to move independently of the zero level set, and so adjust for the solution to be globally $C^{1,1}$. Such a formally correct asymptotics, involving a further expansion for the transition region width, is an intriguing open problem that is worth exploring. We simply overcome this difficulty upon imposing continuity of the truncated inner expansion $S_\varepsilon^l = \sum_{i=0}^{l} \varepsilon^i U_i$ with the far field $u_\varepsilon = \pm 1$ at $y = \pm\frac{\pi}{2}$, which in turn yields homogeneous Dirichlet conditions for U_i at $y = \pm\frac{\pi}{2}$ for $i \geq 1$. This arbitrary choice leads to a uniform transition layer $\{|y| \leq \frac{\pi}{2}\}$, and provides the essential information near the interface at the expense of giving up $C^{1,1}$ regularity of S_ε^l. The resulting formal series may not converge, not even in the sense of [7,8], namely for $\varepsilon \downarrow 0$ but fixed l. Adding further corrections partially compensates for the lack of regularity, as explained at the end of this section, and results in the rigorous construction of barriers of §§5,7.

We clearly obtain

$$U_0(y,t) = \gamma(y) \qquad \forall\, y \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

The $\frac{1}{\varepsilon}$–term yields

$$U_1''(y,t) + U_1(y,t) = -U_0'(y)\big(\varphi_0'(t) + \tfrac{1}{\varphi_0(t)}\big), \qquad U_1(-\tfrac{\pi}{2},t) = U_1(\tfrac{\pi}{2},t) = 0.$$

For this problem to be solvable a compatibility condition between the right hand side of the ODE and span$\{U_0'\}$ must be enforced (Fredholm alternative), where span$\{U_0'\}$ is the kernel of $\varphi'' + \varphi$ subject to vanishing Dirichlet boundary conditions. This reads

$$\big(\varphi_0'(t) + \tfrac{1}{\varphi_0(t)}\big)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |U_0'(y)|^2\, dy = 0,$$

and shows that φ_0 must satisfy (3.1); thus

$$\varphi_0(t) = \phi(t) \qquad \text{and} \qquad U_1(y,t) = 0.$$

Since $\partial_t U_0 = 0$, the ε^0–term leads to the boundary value problem

$$U_2''(y,t) + U_2(y,t) = U_0'(y)\big(\tfrac{y+\varphi_1(t)}{\varphi_0^2(t)} - \varphi_1'(t)\big), \qquad U_2(-\tfrac{\pi}{2},t) = U_2(\tfrac{\pi}{2},t) = 0,$$

and corresponding compatibility constraint

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \big(\tfrac{y+\varphi_1(t)}{\varphi_0^2(t)} - \varphi_1'(t)\big)|U_0'(y)|^2\, dy = 0.$$

Then φ_1 must satisfy $\varphi_1'(t) - \frac{\varphi_1(t)}{\varphi_0^2(t)} = 0$ subject to $\varphi_1(0) = 0$, whose solution is $\varphi_1(t) = 0$. Therefore U_2 solves

$$U_2''(y,t) + U_2(y,t) = \frac{yU_0'(y)}{\varphi_0^2(t)}, \qquad U_2(-\tfrac{\pi}{2},t) = U_2(\tfrac{\pi}{2},t) = 0.$$

We introduce the odd function $\xi \in C^{0,1}(\mathbf{R})$ defined by

$$\xi(x) = \tfrac{1}{16}\big(4x\gamma'(x) + (4x^2 - \pi^2)\gamma(x)\big) \quad \text{if } x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \qquad \xi(x) = 0 \quad \text{if } x \notin [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

(see Figure 3.1), which satisfies

$$\xi''(x) + \xi(x) = x\gamma'(x) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}). \tag{3.5}$$

Note that ξ' exhibits a jump discontinuity at $\pm\frac{\pi}{2}$: $[\![\xi']\!](\pm\frac{\pi}{2}) = \mp\frac{\pi}{8}$. We then realize that

$$U_2(y,t) = \frac{\xi(y)}{\varphi_0^2(t)} \qquad \forall\, y \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

We conclude our discussion with the ε–term, which, in view of $U_1 = 0$, reads

$$U_3''(y,t) + U_3(y,t) = U_0'(y)\big(\tfrac{\varphi_2(t)}{\varphi_0^2(t)} - \varphi_2'(t) - \tfrac{y^2}{\varphi_0^3(t)}\big), \qquad U_3(-\tfrac{\pi}{2},t) = U_3(\tfrac{\pi}{2},t) = 0.$$

Imposing the L^2–orthogonality constraint between the above right hand side and U_0', we deduce an ODE for φ_2, namely,

$$\varphi_2'(t) - \frac{\varphi_2(t)}{\varphi_0^2(t)} = -\frac{\int_{-\pi/2}^{\pi/2} y^2|U_0'(y)|^2 dy}{\varphi_0^3(t)\int_{-\pi/2}^{\pi/2}|U_0'(y)|^2 dy} = \frac{6-\pi^2}{12}\frac{1}{\varphi_0^3(t)}, \qquad \varphi_2(0) = 0. \tag{3.6}$$

Consequently

$$\varphi_2(t) = \frac{\pi^2-6}{12}\frac{\log\varphi_0(t)}{\varphi_0(t)} < 0,$$

and U_3 satisfies

$$U_3''(y,t) + U_3(y,t) = \frac{U_0'(y)}{\varphi_0^3(t)}\big(\tfrac{\pi^2-6}{12} - y^2\big), \qquad U_3(-\tfrac{\pi}{2},t) = U_3(\tfrac{\pi}{2},t) = 0.$$

We indicate with $\sigma \in C^{0,1}(\mathbf{R})$ the even function

$$\sigma(x) = \tfrac{1}{24}\big((\pi^2 - 4x^2)x\gamma(x) - 6x^2\gamma'(x)\big) \quad \text{if } x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \qquad \sigma(x) = 0 \quad \text{if } x \notin [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

(see Figure 3.1), which satisfies

$$\sigma''(x) + \sigma(x) = \big(\tfrac{\pi^2-6}{12} - x^2\big)\gamma'(x) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}). \tag{3.7}$$

FIGURE 3.1. Functions γ, ξ, σ, and (dashed lines) ξ_-, ξ_+, and (dotted line) $\overline{\sigma}$.

Note that σ' is discontinuous at $\pm\frac{\pi}{2}$: $[\![\sigma']\!](\pm\frac{\pi}{2}) = \frac{\pi^2}{48}$. Since $\sigma(0) = 0$, we can write

$$U_3(y,t) = \frac{\sigma(y)}{\varphi_0^3(t)} \qquad \forall\ y \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

We stress that the functions

$$\xi_a(x) = \xi(x) + a\gamma'(x), \qquad \sigma_b(x) = \sigma(x) + b\gamma'(x), \qquad \forall\ x \in \mathbf{R},$$

are still solutions of equations (3.5) and (3.7), respectively, for any $a, b \in \mathbf{R}$. The choice of a and b, which can be viewed as extra degrees of freedom, will be crucial in constructing the sub and supersolutions of §§5.3 and 7.

4. Comparison Lemma. In this section we prove an elementary Comparison Lemma similar to that in [5]. Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain. Consider the double obstacle problem

$$\begin{gathered} u_\varepsilon \in L^2(0,T;H^1(\Omega;[-1,1])) \cap H^1(0,T;(H^1(\Omega))') \\ u_\varepsilon(\cdot,0) = u_\varepsilon^0(\cdot) \in L^\infty(\Omega;[-1,1]) \\ \langle \partial_t u_\varepsilon, \varphi - u_\varepsilon \rangle + \langle \nabla u_\varepsilon, \nabla(\varphi - u_\varepsilon) \rangle - \tfrac{1}{\varepsilon^2}\langle u_\varepsilon, \varphi - u_\varepsilon \rangle \geq 0, \end{gathered} \tag{4.1}$$

for all $\varphi \in H^1(\Omega;[-1,1])$ and a.e. $t \in (0,T)$. Hereafter $\langle \cdot, \cdot \rangle$ stands for either the scalar product in $L^2(\Omega)$ or the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$.

Let v satisfy the following auxiliary problem

$$\begin{gathered} v \in L^2(0,T;H^1(\Omega;(-\infty,1])) \cap H^1(0,T;(H^1(\Omega))') \\ v(\cdot,0) \leq u_\varepsilon^0(\cdot) \text{ a.e. in } \Omega \\ \langle \partial_t v, \varphi \rangle + \langle \nabla v, \nabla \varphi \rangle - \tfrac{1}{\varepsilon^2}\langle f, \varphi \rangle \leq 0, \end{gathered} \tag{4.2}$$

for all $\varphi \in H^1(\Omega;[0,+\infty))$ and a.e. $t \in (0,T)$, with

$$f \le v \quad \text{a.e. in } \{v > -1\}.$$

Then v is a subsolution of the double obstacle problem (4.1), i.e.,

$$v \le u_\varepsilon \quad \text{a.e. in } \Omega \times (0,T). \tag{4.3}$$

In order to prove (4.3), set $e = \max(v - u_\varepsilon, 0)$. Then take $\varphi = u_\varepsilon + e \in L^2(0,T;H^1(\Omega;[-1,1]))$ in (4.1) and $\varphi = e \in L^2(0,T;H^1(\Omega;[0,+\infty))$ in (4.2), and subtract the resulting inequalities. After integration on $(0,t)$, we get

$$\begin{aligned} &\tfrac{1}{2}\|e(t)\|^2_{L^2(\Omega)} + \|\nabla e\|^2_{L^2(0,t;L^2(\Omega))} \\ &\quad = \tfrac{1}{2}\|e(0)\|^2_{L^2(\Omega)} + \tfrac{1}{\varepsilon^2}\int_0^t \langle f - v, e\rangle(\tau)d\tau + \tfrac{1}{\varepsilon^2}\int_0^t \|e(\tau)\|^2_{L^2(\Omega)}d\tau \le \tfrac{1}{\varepsilon^2}\int_0^t \|e(\tau)\|^2_{L^2(\Omega)}d\tau. \end{aligned}$$

Assertion (4.3) follows from Gronwall's Lemma, which implies $\|e(t)\|_{L^2(\Omega)} = 0$ and thus $e = 0$ a.e. in $\Omega \times (0,T)$.

The Comparison Lemma for supersolutions can be stated in a similar fashion and is thus omitted.

5. Subsolution and Supersolution. In this section we construct a sub and supersolution for the double obstacle problem (4.1). Such functions will be used in §6 to derive an optimal interface error estimate.

5.1. Interface. Let $\Sigma(t)$ be a mean curvature flow such that $\text{dist}(\text{conv}(\Sigma(0)), \partial\Omega) \ge \delta > 0$, and set $\Sigma = \{(\mathbf{x},t) \in \Omega \times [0,T] : \mathbf{x} \in \Sigma(t)\}$. Assume $\Sigma(t)$ is an oriented closed manifold of codimension 1 and set $O(t) =$ Outside of $\Sigma(t)$, $I(t) =$ Inside of $\Sigma(t)$. The signed distance d is then defined for $t \in [0,T]$ by

$$d(\mathbf{x},t) = \begin{cases} \text{dist}\big(\mathbf{x}, \Sigma(t)\big) & \text{if } \mathbf{x} \in O(t) \\ 0 & \text{if } \mathbf{x} \in \Sigma(t) \\ -\text{dist}\big(\mathbf{x}, \Sigma(t)\big) & \text{if } \mathbf{x} \in I(t). \end{cases}$$

We introduce a tubular neighborhood $\mathcal{T} = \{(\mathbf{x},t) \in \Omega \times [0,T] : \mathbf{x} \in \mathcal{T}(t)\}$ of Σ defined by

$$\mathcal{T}(t) = \{\mathbf{x} \in \Omega : |d(\mathbf{x},t)| \le D\} \qquad \text{for some fixed } 0 < D < \delta,$$

sufficiently small. The interface error estimate in §6 will be proved under the assumption

$$\partial^{i,j}_{\mathbf{x},t} d \in L^\infty(\mathcal{T}) \qquad \text{for } j = 0, i \le 4;\ j = 1, i \le 2. \tag{5.1}$$

We denote by $\mathbf{n}(\mathbf{x},t)$ the inward unit vector normal to $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$ and by $\kappa^s(\mathbf{x},t)$ the sum of the squares of the principal curvatures of $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$.

It follows from (5.1) that, for any $(\mathbf{x},t) \in \mathcal{T}$, there is a unique $\mathbf{s}(\mathbf{x},t) \in \Sigma(t)$ such that

$$\mathrm{dist}\big(\mathbf{s}(\mathbf{x},t),\mathbf{x}\big) = |d(\mathbf{x},t)|.$$

With an abuse of notation, we henceforth indicate with $\mathbf{n}$ and κ^s the composite functions $\mathbf{n}\big(\mathbf{s}(\mathbf{x},t),t\big)$ and $\kappa^s\big(\mathbf{s}(\mathbf{x},t),t\big)$, defined for all $(\mathbf{x},t)\in\mathcal{T}$, and point out that (5.1) yields

$$|\kappa^s|, |\partial_t\kappa^s|, |\nabla\kappa^s|, |\Delta\kappa^s| \le C < +\infty \qquad \text{in } \mathcal{T}.$$

Finally, the following properties of the distance function hold for all $(\mathbf{x},t)\in\mathcal{T}$ [4,7,8,11,15]:

$$\begin{aligned} \nabla d(\mathbf{x},t) &= -\mathbf{n}(\mathbf{x},t) \\ (\partial_t d - \Delta d)(\mathbf{x},t) &= d(\mathbf{x},t)\kappa^s(\mathbf{x},t) + \mathcal{O}(d^2(\mathbf{x},t)). \end{aligned} \tag{5.2}$$

5.2. Initial Datum. In setting the initial datum u_ε^0 of the double obstacle problem we would like to fit the initial interface $\Sigma(0)$, that is $\Sigma_\varepsilon(0)=\Sigma(0)$, and at the same time avoid an initial transient due to inadequate shape of u_ε^0. In case $u_\varepsilon^0 \in W^{1,\infty}(\Omega;[-1,1])$ only satisfies $|u_\varepsilon^0(\mathbf{x})| \ge C\varepsilon|\log\varepsilon|$ for $|d(\mathbf{x},0)| \ge C\varepsilon|\log\varepsilon|$, then it takes a short time, of order $\mathcal{O}(\varepsilon^2|\log\varepsilon|)$, for the solution u_ε to attain the value $|u_\varepsilon(\mathbf{x},t)|=1$ provided $|d(\mathbf{x},t)| > C\varepsilon|\log\varepsilon|$ [5]; here $C>0$ may not be the same at each occurrence but is always independent of ε. Inspired by the asymptotics of §3, we define

$$u_\varepsilon^0(\mathbf{x}) = \gamma\big(\tfrac{d(\mathbf{x},0)}{\varepsilon}\big).$$

Even though this particular choice might seem restrictive, the formal results of §3 strongly suggest that other choices would not perform better in terms of approximating the mean curvature flow. This is known for the PDE involving a smooth double well potential [4,7,8].

5.3. Subsolution. We start out by introducing a *modified distance function*, which combines a shift with a shape correction. For any $(\mathbf{x},t) \in \Omega\times[0,T]$ set

$$d_\varepsilon^-(\mathbf{x},t) = d(\mathbf{x},t) - C_1\varepsilon^2 d^2(\mathbf{x},t) - C_2\varepsilon^2 \mathrm{e}^{2Kt}, \tag{5.3}$$

where $K=\|\kappa^s\|_{L^\infty(\Sigma)}>0$ and $C_1, C_2\ge 1$ will be determined later on independently of ε.

Hereafter, the notation $\zeta = \mathcal{O}(\varepsilon^\alpha)$ will stand for $|\zeta| \le C\varepsilon^\alpha$ with a constant $C>0$ independent of ε, C_1, and C_2, and $\zeta = \mathcal{O}_{C_i}(\varepsilon^\alpha)$ will mean $|\zeta| \le C\varepsilon^\alpha$ with $C>0$ possibly depending on C_1 and C_2 but not on ε; C may change at each occurrence.

For convenience we remove the superscript $^-$, thus denoting $d_\varepsilon = d_\varepsilon^-$, and introduce the layer $\mathcal{T}_\varepsilon = \{(\mathbf{x},t) \in \Omega\times[0,T] : \mathbf{x} \in \mathcal{T}_\varepsilon(t)\}$, where

$$\mathcal{T}_\varepsilon(t) = \{\mathbf{x} \in \Omega : |d_\varepsilon(\mathbf{x},t)| < \tfrac{\pi}{2}\varepsilon\}.$$

Observe that, depending on D, K, C_1, C_2, and T, there exists $0<\mathcal{E}<1$ so that $\mathcal{T}_\varepsilon \subset \mathcal{T}$ for all $0<\varepsilon\leq\mathcal{E}$, that is $|d(\mathbf{x},t)|\leq D$ provided $|d_\varepsilon(\mathbf{x},t)|\leq\frac{\pi}{2}\varepsilon$. Moreover, for $(\mathbf{x},t)\in\mathcal{T}_\varepsilon$ and $0<\varepsilon\leq\mathcal{E}$ we can write $\frac{\pi}{2}\varepsilon>|d_\varepsilon|\geq|d|(1-C_1D\mathcal{E}^2)-C_2\varepsilon^2\mathrm{e}^{2KT}$ and thus deduce

$$d(\mathbf{x},t) = d_\varepsilon(\mathbf{x},t) + C_2\varepsilon^2\mathrm{e}^{2Kt} + \mathcal{O}_{C_i}(\varepsilon^4). \tag{5.4}$$

In view of (5.2) and (5.4), we have

$$\begin{aligned} |\nabla d_\varepsilon(\mathbf{x},t)|^2 &= 1-4C_1\varepsilon^2 d_\varepsilon(\mathbf{x},t)+\mathcal{O}_{C_i}(\varepsilon^4) \\ (\partial_t d_\varepsilon-\Delta d_\varepsilon)(\mathbf{x},t) &= d_\varepsilon(\mathbf{x},t)\kappa^s - C_2\varepsilon^2\mathrm{e}^{2Kt}(2K-\kappa^s)+2C_1\varepsilon^2+\mathcal{O}(\varepsilon^2)+\mathcal{O}_{C_i}(\varepsilon^3) \end{aligned} \tag{5.5}$$

for all $(\mathbf{x},t)\in\mathcal{T}_\varepsilon$. Note the presence of d_ε instead of d on the right hand sides.

Our next task is to introduce the subsolution v_ε^- for the double obstacle problem (4.1). Its definition is motivated by the asymptotics of §3 in that v_ε^- consists of several corrections, in shape and location, of the obvious candidate γ. Set

$$\xi_-(x) = \xi(x) + \tfrac{\pi}{8}\gamma'(x) \qquad \forall\ x\in\mathbf{R},$$

and note that $\xi_-\in C^{0,1}(\mathbf{R})$ is nonnegative and satisfies (3.5), and ξ_-' is discontinuous at $x=-\frac{\pi}{2}$ because $[\![\xi_-']\!](-\frac{\pi}{2})=\frac{\pi}{4}$, but $\xi_-'(\frac{\pi}{2})=\xi_-''(\frac{\pi}{2})=0$; see Figure 3.1.

Then we introduce

$$v_\varepsilon^-(\mathbf{x},t) = \gamma(y(\mathbf{x},t)) + \varepsilon^2\kappa^s(\mathbf{x},t)\xi_-(y(\mathbf{x},t)) \qquad \forall\ (\mathbf{x},t)\in\Omega\times[0,T], \tag{5.6}$$

where $y(\mathbf{x},t)=\frac{d_\varepsilon(\mathbf{x},t)}{\varepsilon}$ is a modified stretched variable, and proceed to prove that v_ε^- is a subsolution for (4.1). Since no confusion is possible, we will use the notation $v_\varepsilon=v_\varepsilon^-$.

First, note that $|v_\varepsilon(\mathbf{x},t)|<1$ in $\mathcal{T}_\varepsilon$, if ε is sufficiently small (depending on K), and $|v_\varepsilon(\mathbf{x},t)|=1$ in $(\Omega\times[0,T])\backslash\mathcal{T}_\varepsilon$. In fact, for any $\alpha>0$ sufficiently small, the function

$$\widehat{\gamma}_-(x) = \gamma(x) + \alpha\xi_-(x) \qquad \forall\ x\in\mathbf{R}$$

satisfies $\widehat{\gamma}_-'(-\frac{\pi}{2}^+) = \alpha\frac{\pi}{4} > 0$, $\widehat{\gamma}_-'(\frac{\pi}{2}) = 0$, and $\widehat{\gamma}_-''(\frac{\pi}{2}^-) = -1$, whence $\widehat{\gamma}_-$ is strictly increasing in $(-\frac{\pi}{2},\frac{\pi}{2})$; thus $|\widehat{\gamma}_-(x)|\leq 1$ for all $x\in\mathbf{R}$. See Figure 5.1.

Then, it is easy to check that

$$v_\varepsilon(\mathbf{x},0)\leq u_\varepsilon^0(\mathbf{x}) \qquad \forall\ \mathbf{x}\in\Omega.$$

In fact, set $z(\mathbf{x})=\frac{d(\mathbf{x},0)}{\varepsilon}$ and note that, for $t=0$ and $\mathbf{x}\in\mathcal{T}_\varepsilon(0)$, $y=z-C_2\varepsilon+\mathcal{O}_{C_i}(\varepsilon^3)<z-\frac{1}{2}C_2\varepsilon$ for ε sufficiently small (depending on C_1 and C_2). Since we only have to consider $-\frac{\pi}{2}<y<z<\frac{\pi}{2}$, (2.2) leads to $\gamma(z)-\gamma(y)>\frac{1}{4\pi}\varepsilon^2C_2^2$. Then

$$v_\varepsilon(\mathbf{x},0) = \gamma(y)+\varepsilon^2\kappa^s\xi_-(y) < \gamma(z)-\tfrac{1}{4\pi}\varepsilon^2C_2^2+\varepsilon^2\kappa^s\xi_-(y) \leq \gamma(z) = u_\varepsilon^0(\mathbf{x}),$$

provided $C_2 \geq 1$ is chosen sufficiently large but independent of ε.

Set now

$$f(\mathbf{x},t) = \begin{cases} v_\varepsilon(\mathbf{x},t) & \text{if } v_\varepsilon(\mathbf{x},t) > -1, \text{ i.e., } d_\varepsilon(\mathbf{x},t) > -\frac{\pi}{2}\varepsilon \\ 0 & \text{if } v_\varepsilon(\mathbf{x},t) = -1, \text{ i.e., } d_\varepsilon(\mathbf{x},t) \leq -\frac{\pi}{2}\varepsilon. \end{cases} \tag{5.7}$$

For all $t \in (0,T)$ and any nonnegative $\varphi \in H^1(\Omega)$, we have

$$\begin{aligned} &\langle \partial_t v_\varepsilon, \varphi \rangle + \langle \nabla v_\varepsilon, \nabla \varphi \rangle - \tfrac{1}{\varepsilon^2}\langle f, \varphi \rangle \\ &\qquad = \int_{\mathcal{T}_\varepsilon(t)} \mathcal{L} v_\varepsilon \varphi + \int_{\partial \mathcal{T}_\varepsilon(t)} \nabla v_\varepsilon \cdot \nu_\varepsilon \varphi - \tfrac{1}{\varepsilon^2} \int_{\Omega \setminus \mathcal{T}_\varepsilon(t)} f\varphi = \mathrm{I} + \mathrm{II} + \mathrm{III}, \end{aligned}$$

where $\mathcal{L} v_\varepsilon = \partial_t v_\varepsilon - \Delta v_\varepsilon - \frac{1}{\varepsilon^2} v_\varepsilon$ and ν_ε is the outward unit vector normal to $\partial \mathcal{T}_\varepsilon(t)$. Since

$$\nabla v_\varepsilon = \tfrac{1}{\varepsilon}\gamma'(y)\nabla d_\varepsilon + \varepsilon \kappa^s \xi_-'(y) \nabla d_\varepsilon + \varepsilon^2 \xi_-(y) \nabla \kappa^s \qquad \text{in } \mathcal{T}_\varepsilon,$$

and

$$\gamma'(y) = \xi_-(y) = 0 \text{ on } \partial\mathcal{T}_\varepsilon(t), \quad \xi_-'(\tfrac{\pi}{2}) = 0, \;\; \xi_-'(-\tfrac{\pi}{2}^+) > 0, \quad \nu_\varepsilon \cdot \nabla d_\varepsilon = -1 \text{ at } y = -\tfrac{\pi}{2},$$

we have $\mathrm{II} \leq 0$. From (5.7) we also have $\mathrm{III} \leq 0$. Hence, it only remains to establish that the first term is nonpositive. A simple calculation yields

$$\begin{aligned} \mathcal{L} v_\varepsilon = &-\tfrac{1}{\varepsilon^2}\big(\gamma''(y)|\nabla d_\varepsilon|^2 + \gamma(y)\big) + \tfrac{1}{\varepsilon}\gamma'(y)\big(\partial_t d_\varepsilon - \Delta d_\varepsilon\big) - \kappa^s\big(\xi_-''(y)|\nabla d_\varepsilon|^2 + \xi_-(y)\big) \\ &+ \varepsilon \xi_-'(y)\big(\kappa^s(\partial_t d_\varepsilon - \Delta d_\varepsilon) - 2\nabla d_\varepsilon \cdot \nabla \kappa^s\big) + \varepsilon^2 \xi_-(y)\big(\partial_t \kappa^s - \Delta \kappa^s\big) \end{aligned}$$

for all $(\mathbf{x},t) \in \mathcal{T}_\varepsilon$. Using (2.1) and (3.5) in $(-\frac{\pi}{2}, \frac{\pi}{2})$, and properties (5.5), we obtain

$$\begin{aligned} -\tfrac{1}{\varepsilon^2}\big(\gamma''(y)|\nabla d_\varepsilon|^2 + \gamma(y)\big) &= -4\varepsilon C_1 y \gamma(y) + \mathcal{O}_{C_i}(\varepsilon^2), \\ \tfrac{1}{\varepsilon}\gamma'(y)\big(\partial_t d_\varepsilon - \Delta d_\varepsilon\big) &\leq \gamma'(y)\big(y\kappa^s - C_2 \varepsilon K \mathrm{e}^{2Kt} + 2C_1\varepsilon + \mathcal{O}(\varepsilon) + \mathcal{O}_{C_i}(\varepsilon^2)\big), \\ -\kappa^s\big(\xi_-''(y)|\nabla d_\varepsilon|^2 + \xi_-(y)\big) &= -y\gamma'(y)\kappa^s + \mathcal{O}_{C_i}(\varepsilon^3), \end{aligned}$$

whence

$$\mathcal{L} v_\varepsilon \leq -\varepsilon\big(4C_1 y\gamma(y) + C_2 K \mathrm{e}^{2Kt}\gamma'(y) - 2C_1\gamma'(y) + \mathcal{O}(1) + \mathcal{O}_{C_i}(\varepsilon)\big) \qquad \text{in } \mathcal{T}_\varepsilon.$$

The first term on the right hand side reveals the correction effect of the middle term in (5.3). In fact, since $x\gamma(x) + \gamma'(x) \geq 1$ for all $x \in \mathbf{R}$, selecting $C_2 \geq \frac{6}{K}C_1$, we have

$$4C_1 y\gamma(y) + C_2 K \mathrm{e}^{2Kt}\gamma'(y) - 2C_1\gamma'(y) \geq 4C_1 \qquad \text{in } \mathcal{T}_\varepsilon.$$

We then realize that a suitable choice of $C_1 \geq 1$, sufficiently large but independent of ε, leads to

$$\mathcal{L} v_\varepsilon \leq -\varepsilon\big(4C_1 + \mathcal{O}(1) + \mathcal{O}_{C_i}(\varepsilon)\big) \leq -\varepsilon\big(C_1 + \mathcal{O}_{C_i}(\varepsilon)\big) \leq 0 \qquad \text{in } \mathcal{T}_\varepsilon,$$

for ε sufficiently small (depending on C_1 and C_2). Hence, applying the Comparison Lemma of §4 we conclude that $v_\varepsilon^- = v_\varepsilon$ verifies

$$v_\varepsilon^-(\mathbf{x}, t) \leq u_\varepsilon(\mathbf{x}, t) \qquad \forall\ (\mathbf{x}, t) \in \Omega \times [0, T].$$

The construction of a supersolution v_ε^+ is entirely similar and is thus omitted, where

$$v_\varepsilon^+(\mathbf{x}, t) = \gamma\big(\tfrac{d_\varepsilon^+(\mathbf{x},t)}{\varepsilon}\big) + \varepsilon^2 \kappa^s(\mathbf{x}, t)\xi_+\big(\tfrac{d_\varepsilon^+(\mathbf{x},t)}{\varepsilon}\big) \qquad \forall\ (\mathbf{x}, t) \in \Omega \times [0, T],$$

$\xi_+(x) = \xi(x) - \frac{\pi}{8}\gamma'(x)$ for all $x \in \mathbf{R}$, and

$$d_\varepsilon^+(\mathbf{x}, t) = d(\mathbf{x}, t) + C_1 \varepsilon^2 d^2(\mathbf{x}, t) + C_2 \varepsilon^2 \mathrm{e}^{2Kt}.$$

FIGURE 5.1. Functions γ, $\widehat{\gamma}_\mp(x \mp c\varepsilon)$ (dashed lines), and $\widetilde{\gamma}_\mp(x \mp c\varepsilon^2)$ (dotted lines).

6. Interface Error Estimate. On using the subsolution v_ε^- and supersolution v_ε^+ defined in §5.3, it is now possible to deduce the following error estimates, valid before the onset of singularities, for interfaces evolving by mean curvature.

THEOREM 6.1. *Let $\Sigma(t)$ be a mean curvature flow which satisfies (5.1). Let $\Sigma_\varepsilon(t) = \{\mathbf{x} \in \Omega : u_\varepsilon(\mathbf{x}, t) = 0\}$ be the zero level set of the solution u_ε of the double obstacle problem (4.1) with $u_\varepsilon^0(\mathbf{x}) = \gamma\big(\frac{d(\mathbf{x},0)}{\varepsilon}\big)$. Then there exist $0 < \mathcal{E} < 1$ and a constant C_T depending on T such that for all $0 < \varepsilon \leq \mathcal{E}$ the following estimate holds:*

$$\operatorname{dist}\big(\Sigma(t), \Sigma_\varepsilon(t)\big) \leq C_T \varepsilon^2 \qquad \forall\ t \in [0, T]. \tag{6.1}$$

Proof. Set $\Sigma_\varepsilon^\mp(t) = \{\mathbf{x} \in \Omega : v_\varepsilon^\mp(\mathbf{x},t) = 0\}$. Invoking the nondegeneracy property of $v_\varepsilon^\mp$ around the interface, namely, $\nabla v_\varepsilon^\mp \cdot \mathbf{n} = -\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)$ provided $d_\varepsilon^\mp(\mathbf{x},t) = 0$, in conjunction with the property $v_\varepsilon^\mp(\mathbf{x},t) = \mathcal{O}(\varepsilon^2)$ for $d_\varepsilon^\mp(\mathbf{x},t) = 0$, we have

$$\mathbf{x} \in \Sigma_\varepsilon^\mp(t) \qquad \text{implies} \qquad d_\varepsilon^\mp(\mathbf{x},t) = \mathcal{O}(\varepsilon^3).$$

Hence, (5.4) shows that $\mathbf{x} \in \Sigma_\varepsilon^\mp(t)$ yields $d(\mathbf{x},t) = \mathcal{O}(\varepsilon^2)$, and so

$$\operatorname{dist}\big(\Sigma_\varepsilon^\mp(t), \Sigma(t)\big) \le C_T \varepsilon^2 \qquad \forall\, t \in [0,T].$$

The inequalities $v_\varepsilon^-(\mathbf{x},t) \le u_\varepsilon(\mathbf{x},t) \le v_\varepsilon^+(\mathbf{x},t)$ for all $(\mathbf{x},t) \in \Omega \times [0,T]$ then lead to

$$\operatorname{dist}\big(\Sigma_\varepsilon(t), \Sigma_\varepsilon^\mp(t)\big) \le \operatorname{dist}\big(\Sigma_\varepsilon^-(t), \Sigma_\varepsilon^+(t)\big) \le C_T \varepsilon^2 \qquad \forall\, t \in [0,T],$$

and the assertion follows. ∎

Without regularity or qualitative assumptions on the initial datum u_ε^0, (6.1) is still valid for any u_ε^0 satisfying

$$v_\varepsilon^-(\mathbf{x},0) \le u_\varepsilon^0(\mathbf{x}) \le v_\varepsilon^+(\mathbf{x},0) \qquad \forall\, \mathbf{x} \in \Omega.$$

Taylor expansion in (5.6) converts these inequalities into the constraint

$$\big|u_\varepsilon^0(\mathbf{x}) - \gamma\big(\tfrac{d(\mathbf{x},0)}{\varepsilon}\big)\big| \le C\varepsilon\gamma'\big(\tfrac{d(\mathbf{x},0)}{\varepsilon}\big) + \mathcal{O}(\varepsilon^2),$$

C independent of ε. We point out that corrections to γ such as $u_\varepsilon^0(\mathbf{x}) = \gamma\big(\frac{d(\mathbf{x},0)}{\varepsilon}\big) + \varepsilon^2\kappa^s(\mathbf{x},0)\xi\big(\frac{d(\mathbf{x},0)}{\varepsilon}\big)$, which could be suggested by the formal asymptotics of §3, verify this constraint but do not improve the approximation quality. Such a claim is a consequence of our next argument.

7. Optimality. The purpose of this section is to show that the interface error estimate stated in Theorem 6.1 is sharp. To this end, we consider a circle evolving by mean curvature, as described in §3, and prove that $\operatorname{dist}\big(\Sigma(t), \Sigma_\varepsilon(t)\big) \ge \frac{1}{2}\varepsilon^2|\varphi_2(t)|$ (see also [7,8,15]). We stick to the notation of §3 and suppose $T < t^\star = \frac{1}{2}$ is fixed.

Let $d(r,t) = r - \phi(t)$ and let $d_\varepsilon^\mp(r,t)$ denote the modified distance function

$$d_\varepsilon^\mp(r,t) = d(r,t) - \varepsilon^2\varphi_2(t) \mp C_1\varepsilon^3 d^2(r,t) \mp C_2\varepsilon^3 \mathrm{e}^{2Kt}, \tag{7.1}$$

where $K = \max_{t\in[0,T]} \frac{1}{\phi^2(t)}$ and $C_1, C_2 \ge 1$ will be determined later on independently of ε. Note that the correction term of order $\mathcal{O}(\varepsilon^2)$ in (7.1) is now written as $-\varepsilon^2\varphi_2(t) + \mathcal{O}_{C_i}(\varepsilon^3)$, which reveals the extra accuracy built into (7.1) with respect to (5.3). Likewise, the sub and supersolutions $v_\varepsilon^\mp$ defined in (5.6) are further corrected to read

$$v_\varepsilon^\mp(r,t) = \gamma\big(\tfrac{d_\varepsilon^\mp(r,t)}{\varepsilon}\big) + \tfrac{\varepsilon^2}{\phi^2(t)}\xi_\mp\big(\tfrac{d_\varepsilon^\mp(r,t)}{\varepsilon}\big) + \tfrac{\varepsilon^3}{\phi^3(t)}\overline{\sigma}\big(\tfrac{d_\varepsilon^\mp(r,t)}{\varepsilon}\big), \tag{7.2}$$

where $\overline{\sigma}$ is the (even nonpositive) function defined by

$$\overline{\sigma}(x) = \sigma(x) - \tfrac{\pi^2}{48}\gamma'(x) \qquad \forall\, x \in \mathbf{R}$$

(see Figure 3.1). Note that $\overline{\sigma}$ still satisfies (3.7) and $\overline{\sigma} \in C^{2,1}(\mathbf{R})$, because $\overline{\sigma}'(\frac{\pi}{2}) = \overline{\sigma}''(\frac{\pi}{2}) = 0$.

We then have the following asymptotic result which, in view of the property $\varphi_2(t) < 0$, shows the optimality of (6.1) for $u_\varepsilon^0(r) = \gamma\big(\frac{r-1}{\varepsilon}\big)$, together with the fact that $\phi_\varepsilon(t)$ moves faster than $\phi(t)$.

THEOREM 7.1. *Let $\phi(t) = \sqrt{1-2t}$ be the radius of the circle shrinking by mean curvature with extinction time $t^\star = \frac{1}{2}$, and let $\varphi_2(t) = \frac{\pi^2-6}{12}\frac{\log\phi(t)}{\phi(t)}$. Let $\phi_\varepsilon(t)$ be the radius of the zero level set of the solution u_ε of the double obstacle problem (4.1) with $u_\varepsilon^0(r) = \gamma(\frac{r-1}{\varepsilon})$. Then, for any fixed $T < t^\star$, there exist $0 < \mathcal{E} < 1$ and a constant $C_T^\star$ depending on T such that for all $0 < \varepsilon \le \mathcal{E}$ the following estimate holds:*

$$\left|\phi_\varepsilon(t) - \big(\phi(t) + \varepsilon^2\varphi_2(t)\big)\right| \le C_T^\star\varepsilon^3 \qquad \forall\ t \in [0,T]. \tag{7.3}$$

Proof. Let $t_\varepsilon^\star$ be the extinction time for $\phi_\varepsilon(t)$, i.e., $\phi_\varepsilon(t_\varepsilon^\star) = 0$. It will turn out, as a by-product of this analysis, that $T < t_\varepsilon^\star < t^\star$ for ε sufficiently small.

We claim that the assertion is a trivial consequence of the estimate

$$v_\varepsilon^-(r,t) \le u_\varepsilon(r,t) \le v_\varepsilon^+(r,t) \qquad \forall\ r > 0,\ t \in [0,T]. \tag{7.4}$$

In fact, denote by $\phi_\varepsilon^\mp(t)$ the radius of the zero level set of $v_\varepsilon^\mp$, i.e., $v_\varepsilon^\mp(\phi_\varepsilon^\mp(t),t) = 0$, and by $r_\varepsilon^\mp(t)$ the radius such that $d_\varepsilon^\mp(r_\varepsilon^\mp(t),t) = 0$. Then, from (7.1), we have

$$0 = d(r_\varepsilon^\mp(t),t) - \varepsilon^2\varphi_2(t) + \mathcal{O}_{C_i}(\varepsilon^3) = r_\varepsilon^\mp(t) - \big(\phi(t) + \varepsilon^2\varphi_2(t)\big) + \mathcal{O}_{C_i}(\varepsilon^3) \quad \forall\ t \in [0,T], \tag{7.5}$$

provided ε is sufficiently small (depending on C_1 and C_2). Invoking now the nondegeneracy property of $v_\varepsilon^\mp$ around the interface, namely $\partial_r v_\varepsilon^\mp(r_\varepsilon^\mp(t),t) = \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)$, in conjunction with $v_\varepsilon^\mp(r_\varepsilon^\mp(t),t) = \mathcal{O}(\varepsilon^2)$, as results from (7.2), we infer that

$$r_\varepsilon^\mp(t) - \phi_\varepsilon^\mp(t) = \mathcal{O}(\varepsilon^3) \qquad \forall\ t \in [0,T],$$

still if ε is sufficiently small. This, together with (7.5), yields

$$\phi_\varepsilon^\mp(t) - \big(\phi(t) + \varepsilon^2\varphi_2(t)\big) = \mathcal{O}_{C_i}(\varepsilon^3) \qquad \forall\ t \in [0,T].$$

But (7.4) implies $\phi_\varepsilon^+(t) \le \phi_\varepsilon(t) \le \phi_\varepsilon^-(t)$, and thus the desired estimate follows.

Consequently, it only remains to establish (7.4). To this end, we just prove

$$v_\varepsilon^-(r,t) \le u_\varepsilon(r,t) \qquad \forall\ r > 0,\ t \in [0,T], \tag{7.6}$$

because the upper bound can be derived similarly. For convenience we drop the superscript $^-$, denote $v_\varepsilon = v_\varepsilon^-$, $d_\varepsilon = d_\varepsilon^-$, and set $y = \frac{d_\varepsilon}{\varepsilon}$. As in Theorem 6.1, we intend to apply the Comparison Lemma. If we set

$$\widetilde{\gamma}_-(x) = \gamma(x) + \alpha\xi_-(x) + \beta\overline{\sigma}(x) = \widehat{\gamma}_-(x) + \beta\overline{\sigma}(x) \qquad \forall\ x \in \mathbf{R},$$

then we see that $\widetilde{\gamma}_-'(-\frac{\pi}{2}^+) = \alpha\frac{\pi}{4} > 0$, $\widetilde{\gamma}_-'(\frac{\pi}{2}) = 0$, and $\widetilde{\gamma}_-''(\frac{\pi}{2}^-) = -1$. Consequently, $\widetilde{\gamma}_-$ is strictly increasing in $(-\frac{\pi}{2},\frac{\pi}{2})$, for any $\alpha,\beta > 0$ sufficiently small; thus $|\widetilde{\gamma}_-(x)| \le 1$ for all

$x \in \mathbf{R}$ (see Figure 5.1). In terms of v_ε, this reads $|v_\varepsilon(r,t)|<1$ in $\mathcal{T}_\varepsilon$, for ε sufficiently small (depending on K). Then, for $t=0$, we have to show that

$$v_\varepsilon(r,0) \le u_\varepsilon^0(r) = \gamma(z) \qquad \forall\, r>0,$$

where $z(r) = \frac{r-1}{\varepsilon}$. On the other hand, since $\varphi_2(0)=0$, we have for $z = \mathcal{O}(1)$

$$y(r,0) = \tfrac{d_\varepsilon(r,0)}{\varepsilon} = z - C_2\varepsilon^2 + \mathcal{O}_{C_i}(\varepsilon^4) < z - \tfrac{1}{2}C_2\varepsilon^2,$$

for ε sufficiently small (depending on C_1 and C_2). We can resort to (2.2) to deduce

$$\gamma(z) - \gamma(y) > \tfrac{1}{4\pi}C_2^2\varepsilon^4 + \tfrac{1}{2\pi}C_2\varepsilon^2 \min\big(y+\tfrac{\pi}{2}, \tfrac{\pi}{2}-z\big)$$

for $-\frac{\pi}{2} < y < z < \frac{\pi}{2}$; hence

$$v_\varepsilon(r,0) < \gamma(z) - \varepsilon^2\Big(\tfrac{1}{4\pi}C_2^2\varepsilon^2 + \tfrac{1}{2\pi}C_2 \min\big(y+\tfrac{\pi}{2}, \tfrac{\pi}{2}-z\big) - \xi_-(y) - \varepsilon\overline{\sigma}(y)\Big) \le \gamma(z),$$

for $C_2 \ge 1$ sufficiently large. It is worth noting that at $z = \frac{\pi}{2}$ the second term between parenthesis vanishes but the first one still provides control because $\xi_-'(\frac{\pi}{2}) = \xi_-''(\frac{\pi}{2}) = \overline{\sigma}'(\frac{\pi}{2}) = \overline{\sigma}''(\frac{\pi}{2}) = 0$, and so $\xi_-(y) + \varepsilon\overline{\sigma}(y) = \mathcal{O}_{C_i}(\varepsilon^6)$ for $y = \frac{\pi}{2} - \mathcal{O}_{C_i}(\varepsilon^2)$.

We now realize that, proceeding as in §5.3, we only have to examine the sign of $\mathcal{L}v_\varepsilon = \partial_t v_\varepsilon - \frac{1}{r}\partial_r(r\partial_r v_\varepsilon) - \frac{1}{\varepsilon^2}v_\varepsilon$ within the layer $\mathcal{T}_\varepsilon$. Making use of the relation

$$\tfrac{1}{r} = \tfrac{1}{\phi+d} = \tfrac{1}{\phi}\tfrac{1}{1+d/\phi} = \tfrac{1}{\phi} - \tfrac{d}{\phi^2} + \tfrac{d^2}{\phi^3} + \mathcal{O}(d^3),$$

and (7.1), and collecting like powers of ε, after tedious calculations we get

$$\begin{aligned}\mathcal{L}v_\varepsilon = &-\tfrac{1}{\varepsilon^2}\big(\gamma''(y)+\gamma(y)\big) - \tfrac{1}{\varepsilon}\gamma'(y)\big(\phi' + \tfrac{1}{\phi}\big) - \tfrac{1}{\phi^2}\big(\xi_-''(y) + \xi_-(y) - y\gamma'(y)\big)\\ &- \varepsilon\Big(\tfrac{1}{\phi^3}\big(\overline{\sigma}''(y) + \overline{\sigma}(y)\big) + \gamma'(y)\big(\varphi_2' - \tfrac{\varphi_2}{\phi^2} + \tfrac{y^2}{\phi^3}\big) + \tfrac{\xi_-'(y)}{\phi^2}\big(\phi' + \tfrac{1}{\phi}\big)\Big)\\ &- \varepsilon^2\big(4C_1 y\gamma(y) + C_2 \mathrm{e}^{2Kt}\big(2K - \tfrac{1}{\phi^2}\big)\gamma'(y) - 2C_1\gamma'(y) + \mathcal{O}(1) + \mathcal{O}_{C_i}(\varepsilon)\big) \quad \text{in } \mathcal{T}_\varepsilon.\end{aligned}$$

In light of the definitions of $\gamma, \xi_-, \overline{\sigma}, \phi$, and φ_2 (see (2.1), (3.5), (3.7), (3.1), and (3.6)), the first four terms on the right hand side vanish. In addition, since $y\gamma(y) + \gamma'(y) \ge 1$, the choice $C_2 \ge \frac{6}{K}C_1$ enables us to control both $-2C_1\gamma'(y)$ and, for C_1 large enough, $\mathcal{O}(1)$. Finally, for ε sufficiently small (depending on C_1 and C_2), we get $\mathcal{L}v_\varepsilon \le 0$ in $\mathcal{T}_\varepsilon$, and as a consequence (7.6). The proof is thus complete. □

The estimate (7.3) is still valid for a perturbed initial datum u_ε^0 satisfying

$$v_\varepsilon^-(r,0) \le u_\varepsilon^0(r) \le v_\varepsilon^+(r,0) \qquad \forall\, r>0,$$

where $v_\varepsilon^\mp$ are defined in (7.2). Since the perturbation $u_\varepsilon^0(r) = \gamma\big(\frac{r-1}{\varepsilon}\big) + \varepsilon^2\xi\big(\frac{r-1}{\varepsilon}\big) + \varepsilon^3\sigma\big(\frac{r-1}{\varepsilon}\big)$ verifies this constraint, we exploit the optimality result to conclude that further adjustment of the shape of γ cannot improve the rate of convergence (6.1).

REFERENCES

[1] S.M. Allen and J.W. Cahn, *A macroscopic theory for antiphase boundary motion and its application to antiphase domain coarsing*, Acta Metall., 27 (1979), pp. 1085–1095.

[2] G. Barles, H.M. Soner, and P.E. Souganidis, *Front propagations and phase field theory*, SIAM J. Control Optim., 31 (1993), pp. 439–469.

[3] L. Bronsard and R.V. Kohn, *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Differential Equations, 90 (1991), pp. 211–237.

[4] X. Chen, *Generation and propagation of interfaces for reaction-diffusion equations*, J. Differential Equations, 96 (1992), pp. 116–141.

[5] X. Chen and C.M. Elliott, *Asymptotics for a parabolic double obstacle problem*, IMA preprint, 922 (1992), pp. 1–25.

[6] E. De Giorgi, *Congetture sui limiti delle soluzioni di alcune equazioni paraboliche quasi lineari*, in Nonlinear Analysis. A Tribute in Honour of G. Prodi, S.N.S. Quaderni, Pisa, 1991, pp. 173–187.

[7] P. De Mottoni and M. Schatzman, *Évolution géométrique d'interfaces*, C.R. Acad. Sci. Paris Sér. I Math., 309 (1989), pp. 453–458.

[8] P. De Mottoni and M. Schatzman, *Geometrical evolution of developed interfaces*, Trans. Amer. Math. Soc. (to appear).

[9] L.C. Evans, H.M. Soner, and P.E. Souganidis, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math., 45 (1992), pp. 1097–1123.

[10] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature I*, J. Differential Geom., 33 (1991), pp. 635–681.

[11] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.

[12] M. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom., 26 (1987), pp. 285–314.

[13] T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, preprint (1991).

[14] J.B. Keller, J. Rubinstein, and P. Sternberg, *Fast reaction, slow diffusion and curve shortening*, SIAM J. Appl. Math., 49 (1989), pp. 116–133.

[15] M. Paolini and C. Verdi, *Asymptotic and numerical analyses of the mean curvature flow with a space-dependent relaxation parameter*, Asymptotic Anal., 5 (1992), pp. 553–574.