SHARP ERROR ANALYSIS FOR CURVATURE DEPENDENT EVOLVING FRONTS

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The evolution of a curvature dependent interface is approximated via a singularly perturbed parabolic double obstacle problem with small parameter $\varepsilon > 0$. The velocity normal to the front is proportional to its mean curvature plus a forcing term. Optimal interface error estimates of order $\mathcal{O}(\varepsilon^2)$ are derived for smooth evolutions, that is before singularities develop. Key ingredients are the construction of sub(super)solutions containing several shape corrections dictated by formal asymptotics, and the use of a modified distance function.

1. Introduction

In this paper we continue our study of the asymptotic behavior of a singularly perturbed parabolic double obstacle problem, which is viewed as a variational approximation of the geometric motion of fronts [9]. Suppose that an interface $\Sigma(t)$ propagates in the normal direction with velocity $V = \kappa^m + g$, where $\kappa^m(\mathbf{x}, t)$ is the sum of the principal curvatures and $g(\mathbf{x}, t)$ is a given forcing term, both evaluated at $(\mathbf{x}, t) \in \Sigma(t)$. For $0 < \varepsilon \ll 1$, let u_ε denote the solution to the following PDE:

$$\varepsilon \partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon + \tfrac{1}{2\varepsilon}\Psi'(u_\varepsilon) = \tfrac{c_0}{2} g \qquad \text{in } \mathbf{R}^n \times (0, +\infty); \tag{1.1}$$

Ψ indicates a double equal well potential and c_0 is just a convenient scaling factor. Reaction diffusion equations like (1.1) arise in a number of applications such as the theory of phase transitions, flame propagation, combustion theory and pattern formation (see [1,6] and the references therein). They also give rise to a formidable number of theoretical issues still to be addressed [4]. It turns out that $u_\varepsilon(\cdot,t)$ converges to ± 1 as $\varepsilon \downarrow 0$, and the transition layer $\mathcal{T}_\varepsilon(t)$ separating the two phases given by $u_\varepsilon(\cdot,t) \approx \pm 1$ converges to $\Sigma(t)$, at least for Ψ smooth, provided $\Sigma(t)$ does not develop interior [1,6]. The function u_ε can thus be regarded as a regularized order parameter. The propagation of $\Sigma(t)$ is interpreted in the viscosity sense via the level set approach, even past singularities [2,7]. The above convergence is a crucial, and rather subtle, result in the theory of nonlinear PDE but it does not give any indication about the quality of such an approximation, not even before singularities occur. This is however the sort of basic question to be asked whenever (1.1) may be used for numerical purposes, but it certainly bears some intrinsic interest as well.

The behavior of u_ε is largely dictated by the specific scaling of (1.1) and is insensitive to the details of Ψ. However, if Ψ is smooth, then u_ε never attains the asymptotic values ± 1. This is numerically inconvenient in that (1.1) must be solved in the entire domain of definition even though the action takes place solely in $\mathcal{T}_\varepsilon(t)$, a relatively thin layer of thickness $\mathcal{O}(\varepsilon)$. This difficulty can be overcome upon choosing Ψ to be

$$\Psi(s) = (1 - s^2) + I_{[-1,1]}(s),$$

where $I_{[-1,1]}$ stands for the indicatrix of the interval [-1,1]. The PDE (1.1) becomes a double obstacle problem, and the corresponding $u_\varepsilon(\cdot,t)$ attains the values ± 1 outside $\mathcal{T}_\varepsilon(t)$. In [9] we study this formulation for $g = 0$, the so-called mean curvature flow, and prove an optimal interface error estimate of order $\mathcal{O}(\varepsilon^2)$. This rate improves upon the linear order derived in [3]. It is the object of this paper to extend our analysis to the case of a forcing term $g \neq 0$. More precisely, if $\Sigma_\varepsilon(t)$ indicates the zero level set of $u_\varepsilon(\cdot,t)$, then we demonstrate the sharp rate of convergence

$$\text{dist}\ (\Sigma(t), \Sigma_\varepsilon(t)) \leq C\varepsilon^2, \tag{1.2}$$

for as long as the evolution remains smooth; dist stands for the Hausdorff distance. As in [9], the key ingredients are a rather technical construction of a subsolution based on formal asymptotics, as well as the use of a modified distance function which combines a shift with a shape correction. The effect of a nonvanishing g is apparent in the asymptotics, and is further reflected in the intricate form of the subsolution and subsequent analysis, as compared with those in [9].

An outline of the paper follows. In §2 we introduce the basic set up and notation, and state the error estimate. Formal asymptotics is discussed in §3 for the radially symmetric case. This study suggests the various corrections embodied in the subsolution, which in turn is defined and rigorously examined in §5. A key tool is the Comparison Lemma of §4, which expresses a distinctive property of this scalar problem, namely the validity of the Maximum Principle. The paper concludes in §6 with the proof of (1.2).

2. Setting and Results

Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with outward unit vector ν normal to $\partial\Omega$. Let $\Sigma(t)$ be a moving front that remains within Ω for all $t \in [0,T]$, and set $\Sigma = \{(\mathbf{x},t) \in \Omega \times [0,T] : \mathbf{x} \in \Sigma(t)\}$. Assume $\Sigma(t)$ is an oriented closed manifold of codimension 1 and set $O(t) =$ Outside of $\Sigma(t)$, $I(t) =$ Inside of $\Sigma(t)$. The signed distance d is then defined by

$$d(\mathbf{x},t) = \begin{cases} \operatorname{dist}\big(\mathbf{x}, \Sigma(t)\big) & \text{if } \mathbf{x} \in O(t) \\ 0 & \text{if } \mathbf{x} \in \Sigma(t) \\ -\operatorname{dist}\big(\mathbf{x}, \Sigma(t)\big) & \text{if } \mathbf{x} \in I(t) \end{cases} \qquad \forall\, t \in [0,T].$$

We introduce a tubular neighborhood $\mathcal{T} = \{(\mathbf{x},t) \in \mathbf{R}^n \times [0,T] : \mathbf{x} \in \mathcal{T}(t)\}$ of Σ defined by

$$\mathcal{T}(t) = \{\mathbf{x} \in \mathbf{R}^n : |d(\mathbf{x},t)| \le D\} \qquad \text{for some fixed } 0 < D \ll 1 \text{ so that } \mathcal{T}(t) \subset \Omega.$$

Throughout the paper we assume the following regularity on d:

$$d, \partial_t d, \partial_{\mathbf{x}}^i d \in C^0(\mathcal{T}) \quad \text{for } i = 1,2,3,4. \tag{2.1}$$

For any $\mathbf{x} \in \Sigma(t)$, we denote by $\mathbf{n}(\mathbf{x},t)$ the inward unit vector normal to $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$. In addition, we indicate with $\kappa^m(\mathbf{x},t)$ the sum of the principal curvatures (positive if $I(t)$ is convex at $\mathbf{x} \in \Sigma(t)$), and with $\kappa^s(\mathbf{x},t)$ the sum of the squares of the principal curvatures of $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$. We suppose that $\Sigma(t)$ moves with normal velocity $V = \kappa^m + g$, that is,

$$\partial_t d(\mathbf{x},t) = \kappa^m(\mathbf{x},t) + g(\mathbf{x},t) \qquad \forall\, \mathbf{x} \in \Sigma(t).$$

The forcing term g in turn satisfies

$$g \in W^{3,\infty}(\Omega \times (0,T)).$$

It follows from (2.1) that, for any $(\mathbf{x},t) \in \mathcal{T}$, there is a unique $\mathbf{s}(\mathbf{x},t) \in \Sigma(t)$ such that

$$\operatorname{dist}\big(\mathbf{s}(\mathbf{x},t), \mathbf{x}\big) = |d(\mathbf{x},t)|.$$

We indicate with $\bar{\mathbf{n}}, \bar{\kappa}^s$, and $\bar{g}$ the composite functions $\mathbf{n}\big(\mathbf{s}(\mathbf{x},t),t\big), \kappa^s\big(\mathbf{s}(\mathbf{x},t),t\big)$, and $g\big(\mathbf{s}(\mathbf{x},t),t\big)$, defined for all $(\mathbf{x},t) \in \mathcal{T}$, and point out that (2.1) yields

$$|\bar{\kappa}^s|, |\partial_t \bar{\kappa}^s|, |\nabla \bar{\kappa}^s|, |\Delta \bar{\kappa}^s| \le C < +\infty \qquad \text{in } \mathcal{T}.$$

The following properties of the distance function are known to hold for all $(\mathbf{x},t) \in \mathcal{T}$ [5,8,10]:

$$\begin{aligned} \nabla d(\mathbf{x},t) &= -\bar{\mathbf{n}}(\mathbf{x},t) \\ (\partial_t d - \Delta d)(\mathbf{x},t) &= \bar{g}(\mathbf{x},t) + d(\mathbf{x},t)\bar{\kappa}^s(\mathbf{x},t) + \mathcal{O}(d^2(\mathbf{x},t)). \end{aligned} \tag{2.2}$$

We now introduce a variational approximation to $\Sigma(t)$ via a singularly perturbed double obstacle problem. We define the potential Ψ and the subdifferential graph $\psi = \frac{1}{2}\Psi'$ by

$$\Psi(s) = \begin{cases} 1-s^2 & \text{if } s \in [-1,1] \\ +\infty & \text{if } s \notin [-1,1] \end{cases}, \qquad \psi(s) = \begin{cases} (-\infty,1] & \text{if } s=-1 \\ -s & \text{if } s \in (-1,1) \\ [-1,+\infty) & \text{if } s=1 \end{cases}.$$

We seek absolute minimizers γ of the functional $\mathcal{F}(\varphi) = \int_{\mathbf{R}} \big(|\varphi'(x)|^2 + \Psi(\varphi(x))\big)\,dx$ for $\varphi \in D(\mathcal{F}) = \{\varphi \in H^1_{\mathrm{loc}}(\mathbf{R}) : |\varphi(x)| \le 1 \text{ in } \mathbf{R},\ \lim_{x\to\pm\infty}\varphi(x) = \pm 1\}$. It is not difficult to see [9] that, imposing $\gamma(0)=0$, the unique absolute minimizer $\gamma \in D(\mathcal{F})$ of $\mathcal{F}$ is given by

$$\gamma(x) = \begin{cases} -1 & \text{if } x < -\frac{\pi}{2} \\ \sin x & \text{if } x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ +1 & \text{if } x > \frac{\pi}{2} \end{cases}$$

(see Fig. 1). Hence, $\gamma \in C^{1,1}(\mathbf{R})$ and solves the (elliptic) double obstacle problem

$$\gamma''(x) - \psi(\gamma(x)) \ni 0 \qquad \text{in } \mathbf{R}. \tag{2.3}$$

Set

$$c_0 = \int_{\mathbf{R}} (\gamma'(x))^2 dx = \int_{-1}^{1} \sqrt{\Psi(s)}\,ds = \frac{\pi}{2},$$

and let u_ε be the solution to the parabolic double obstacle problem

$$\begin{aligned} &\varepsilon\partial_t u_\varepsilon - \varepsilon\Delta u_\varepsilon + \tfrac{1}{\varepsilon}\psi(u_\varepsilon) \ni \tfrac{c_0}{2} g \qquad \text{in } \Omega \times (0,T) \\ &u_\varepsilon(\mathbf{x},0) = u_\varepsilon^0(\mathbf{x}) = \gamma\big(\tfrac{d(\mathbf{x},0)}{\varepsilon}\big) \qquad \forall\, \mathbf{x} \in \Omega \\ &\tfrac{\partial u_\varepsilon}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{2.4}$$

Let $\Sigma_\varepsilon(t) = \{\mathbf{x} \in \Omega : u_\varepsilon(\mathbf{x},t) = 0\}$ be the zero level set of u_ε. The purpose of this paper is to prove the following sharp interface error estimate.

Theorem 2.1. There exist $0 < \mathcal{E} \ll 1$ and a constant C_T depending on T and $\mathcal{E}$ such that for all $0 < \varepsilon \le \mathcal{E}$ the following optimal estimate holds:

$$\operatorname{dist}\big(\Sigma(t), \Sigma_\varepsilon(t)\big) \le C_T \varepsilon^2 \qquad \forall\, t \in [0,T].$$

3. Asymptotics

We propose here a brief discussion of formal asymptotics for the simplest situation, the radially symmetric case. Even though this presentation is only formal, it motivates several crucial aspects of the sub(super)solution construction of §5.

Consider in $\mathbf{R}^2$ a circle of radius $\phi(t)$ which evolves with velocity

$$V = -\phi'(t) = \frac{1}{\phi(t)} + g(\phi(t),t) = \kappa^m + g, \qquad \phi(0) = 1. \tag{3.1}$$

If the initial datum $u_\varepsilon(\cdot,0)$ of the double obstacle problem is radially symmetric, so is the solution $u_\varepsilon(\cdot,t)$. Using polar coordinates, the PDE in (2.4) reads

$$\varepsilon\partial_t u_\varepsilon - \tfrac{\varepsilon}{r}\partial_r(r\partial_r u_\varepsilon) - \tfrac{1}{\varepsilon}u_\varepsilon = \tfrac{c_0}{2}g \qquad \text{in } \{|u_\varepsilon(r,t)|<1\}. \tag{3.2}$$

We denote by $r=\phi_\varepsilon(t)$ the zero level set of u_ε, that is $u_\varepsilon(\phi_\varepsilon(t),t)=0$, and assume $\phi_\varepsilon(0)=1$. Let us introduce the stretched variable $y=\frac{r-\phi_\varepsilon(t)}{\varepsilon}$ and denote

$$U_\varepsilon(y,t)=u_\varepsilon\big(\phi_\varepsilon(t)+\varepsilon y,t\big),$$

whence $u_\varepsilon(r,t)=U_\varepsilon\big(\frac{r-\phi_\varepsilon(t)}{\varepsilon},t\big)$. With the notation $U_\varepsilon^{(k)}=\partial_y^k U_\varepsilon$, we realize that

$$\varepsilon\partial_t u_\varepsilon = \varepsilon\partial_t U_\varepsilon - \phi_\varepsilon' U_\varepsilon' \quad \text{and} \quad \tfrac{\varepsilon}{r}\partial_r(r\partial_r u_\varepsilon)=\tfrac{1}{r}U_\varepsilon' + \tfrac{1}{\varepsilon}U_\varepsilon''. \tag{3.3}$$

Suppose that both U_ε and ϕ_ε can be expressed in terms of ε (inner expansion) as

$$U_\varepsilon(y,t)=\sum_{i=0}^{\infty}\varepsilon^i U_i(y,t), \qquad \phi_\varepsilon(t)=\sum_{i=0}^{\infty}\varepsilon^i\varphi_i(t).$$

Boundary conditions for the U_i's are provided by the outer expansion and read [10]

$$U_0(\mp\tfrac{\pi}{2},t)=\mp 1, \qquad U_i(\mp\tfrac{\pi}{2},t)=0 \quad \text{for all } i\geq 1. \tag{3.4}$$

It readily follows that

$$\tfrac{1}{r}=\big(\varepsilon y+\textstyle\sum_{i=0}^{\infty}\varepsilon^i\varphi_i(t)\big)^{-1}=\tfrac{1}{\varphi_0}-\varepsilon\tfrac{y+\varphi_1}{\varphi_0^2}+\mathcal{O}(\varepsilon^2), \tag{3.5}$$

and, with the notation $g_0^{(k)}(t)=\big(\partial_r^{(k)}g\big)(\varphi_0(t),t)$, that

$$g(r,t)=g\big(\varepsilon y+\textstyle\sum_{i=0}^{\infty}\varepsilon^i\varphi_i(t),t\big)=g_0+\varepsilon(y+\varphi_1)g_0'+\mathcal{O}(\varepsilon^2). \tag{3.6}$$

We now intend to derive explicit expressions of, and compatibility conditions for, the first three terms of these formal asymptotic expansions; see [9,10]. Using (3.3), together with (3.5) and (3.6), we can substitute U_ε into (3.2) and collect all resulting terms containing like powers of ε. In the sequel we skip details and simply examine the resulting summands in increasing order and equate them to zero, thus starting with the $\frac{1}{\varepsilon}$–term:

$$U_0''(y,t)+U_0(y,t)=0.$$

In view of (3.4) and imposing $U_0(0,t)=0$, we clearly obtain

$$U_0(y,t)=\gamma(y) \qquad \forall\, y\in[-\tfrac{\pi}{2},\tfrac{\pi}{2}].$$

The ε^0–term yields

$$U_1''(y,t)+U_1(y,t)=-\Big(\varphi_0'(t)+\tfrac{1}{\varphi_0(t)}\Big)U_0'(y)-\tfrac{c_0}{2}g_0(t)=q_1(y,t).$$

For this problem to be solvable a compatibility condition between the right hand side q_1 and U_0' must be enforced (Fredholm alternative), because span$\{U_0'\}$ is the kernel of $\mathcal{L}\varphi = \varphi'' + \varphi$ subject to vanishing Dirichlet boundary conditions. Such constraint reads

$$0 = \int_{-\pi/2}^{\pi/2} q_1(y,t)U_0'(y)dy = -c_0\Big(\varphi_0'(t) + \tfrac{1}{\varphi_0(t)} + g_0(t)\Big),$$

and, imposing $\varphi_0(0) = 1$, shows that φ_0 must satisfy (3.1). Hence $\varphi_0(t) = \phi(t)$ and

$$\mathcal{L}U_1(y,t) = g_0(t)\big(U_0'(y) - \tfrac{c_0}{2}\big).$$

Let $\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ denote the characteristic function of the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$. We introduce the even function $\eta \in C^{1,1}(\mathbf{R})$ defined by

$$\eta(x) = \tfrac{1}{2}\big(x\gamma(x) + \gamma'(x) - c_0\big)\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x) \qquad \forall\, x \in \mathbf{R}$$

(see Fig. 1), which satisfies $\eta(\mp\frac{\pi}{2}) = \eta'(\mp\frac{\pi}{2}) = 0$ and

$$\mathcal{L}\eta(x) = \gamma'(x) - \tfrac{c_0}{2} \qquad \text{in } (-\tfrac{\pi}{2},\tfrac{\pi}{2}). \tag{3.7}$$

In view of (3.4), we then realize that

$$U_1(y,t) = g_0(t)\eta(y) \qquad \forall\, y \in [-\tfrac{\pi}{2},\tfrac{\pi}{2}].$$

Since $\partial_t U_0 = 0$ and $\varphi_0' + \frac{1}{\varphi_0} = -g_0$, the ε–term leads to

$$\mathcal{L}U_2(y,t) = -\Big(\varphi_1'(t) - \tfrac{y+\varphi_1(t)}{\varphi_0^2(t)}\Big)U_0'(y) + g_0(t)U_1'(y,t) - \tfrac{c_0}{2}g_0'(t)\big(y+\varphi_1\big) = q_2(y,t),$$

and corresponding compatibility constraint

$$0 = \int_{-\pi/2}^{\pi/2} q_2(y,t)U_0'(y)dy = -c_0\Big(\varphi_1'(t) - \tfrac{\varphi_1(t)}{\varphi_0^2(t)} + \varphi_1(t)g_0'(t)\Big).$$

Imposing $\varphi_1(0) = 0$, we readily get $\varphi_1(t) = 0$ and, since $U_1'(y,t) = \frac{1}{2}g_0(t)yU_0'(y)$,

$$\mathcal{L}U_2(y,t) = \Big(\tfrac{1}{\varphi_0^2(t)} + \tfrac{g_0^2(t)}{2}\Big)yU_0'(y) - \tfrac{c_0}{2}g_0'(t)y.$$

We have now the first clear indication that the rate of convergence for the interfaces should be $\mathcal{O}(\varepsilon^2)$, because $\phi_\varepsilon(t) = \phi(t) + \mathcal{O}(\varepsilon^2)$. In order to separate y and t dependences, let us introduce the odd functions $\xi, \mu \in C^{0,1}(\mathbf{R})$ defined by

$$\xi(x) = \tfrac{1}{16}\big(4x\gamma'(x) + (4x^2-\pi^2)\gamma(x)\big)\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x), \qquad \mu(x) = \big(x - \tfrac{\pi}{2}\gamma(x)\big)\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x),$$

for all $x \in \mathbf{R}$. They satisfy $\xi(\mp\frac{\pi}{2}) = 0$, $\mu(\mp\frac{\pi}{2}) = 0$, and

$$\mathcal{L}\xi(x) = x\gamma'(x), \qquad \mathcal{L}\mu(x) = x, \qquad \forall\, x \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}). \tag{3.8}$$

Moreover ξ' and μ' exhibit a jump discontinuity at $\mp\frac{\pi}{2}$, precisely, $[\![\xi']\!](\mp\frac{\pi}{2}) = \pm\frac{\pi}{8}$, $[\![\mu']\!](\mp\frac{\pi}{2}) = \pm 1$. Using (3.4), we then realize that

$$U_2(y,t) = \Big(\frac{1}{\varphi_0^2(t)} + \frac{g_0^2(t)}{2}\Big)\xi(y) - \frac{c_0}{2} g_0'(t)\mu(y) \qquad \forall\; y \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

It is not obvious that the above asymptotics converges, not even in the sense of [5], that is for $\varepsilon \downarrow 0$ but the number of terms fixed. This topic deserves further investigation.

4. Comparison Lemma

We recall an elementary Comparison Lemma similar to that in [3,9]. The double obstacle problem (2.4) can be written equivalently as follows:

$$\begin{gathered} u_\varepsilon \in L^2(0,T; H^1(\Omega;[-1,1])) \cap H^1(0,T;(H^1(\Omega))') \\ u_\varepsilon(\cdot,0) = u_\varepsilon^0(\cdot) \in L^\infty(\Omega;[-1,1]) \\ \varepsilon\langle \partial_t u_\varepsilon, \varphi - u_\varepsilon\rangle + \varepsilon\langle \nabla u_\varepsilon, \nabla(\varphi - u_\varepsilon)\rangle - \tfrac{1}{\varepsilon}\langle u_\varepsilon, \varphi - u_\varepsilon\rangle - \tfrac{c_0}{2}\langle g, \varphi - u_\varepsilon\rangle \ge 0, \end{gathered} \tag{4.1}$$

for all $\varphi \in H^1(\Omega;[-1,1])$ and a.e. $t \in (0,T)$. Hereafter, $\langle\cdot,\cdot\rangle$ stands for either the scalar product in $L^2(\Omega)$ or the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. Let $v \in L^2(0,T;H^1(\Omega;(-\infty,1])) \cap H^1(0,T;(H^1(\Omega))')$ satisfy the auxiliary problem

$$\begin{gathered} v(\cdot,0) \le u_\varepsilon^0(\cdot) \text{ a.e. in } \Omega \\ \varepsilon\langle \partial_t v, \varphi\rangle + \varepsilon\langle \nabla v, \nabla\varphi\rangle - \langle f, \varphi\rangle \le 0, \end{gathered} \tag{4.2}$$

for all $\varphi \in H^1(\Omega;[0,+\infty))$ and a.e. $t \in (0,T)$, with $f \le \frac{1}{\varepsilon}v + \frac{c_0}{2}g$ a.e. in $\{v > -1\}$. Then v is a subsolution of the double obstacle problem (4.1):

$$v \le u_\varepsilon \qquad \text{a.e. in } \Omega \times (0,T).$$

To see this, simply take $\varphi = \max(v, u_\varepsilon)$ in (4.1) and $\varphi = \max(v - u_\varepsilon, 0)$ in (4.2), subtract the resulting expressions and integrate them on $(0,t)$, and finally use Gronwall's Lemma.

5. Sub(Super)solution

In this section we construct a sub(super)solution for the double obstacle problem (4.1). The first relevant issue is to mimic the shape of u_ε as well as possible, which in turn hinges upon the asymptotic analysis of §3. The second crucial ingredient is a modified distance function, which combines a shift with a shape correction.

Fig. 1. Functions γ, η (dashed line), and ξ_-, μ_- (dotted lines).

Fig. 2. Function γ and sub(super)solutions $\gamma_{\mp}(x \mp C\varepsilon)$ (dashed lines).

We start out by introducing the modified distance function. Set

$$K = \|\kappa^s\|_{L^\infty(\Sigma)} + \tfrac{c_0}{2}\|\nabla g\|_{L^\infty(\Omega\times(0,T))} > 0.$$

For any $(\mathbf{x},t) \in \Omega \times [0,T]$ define

$$d_\varepsilon^-(\mathbf{x},t) = d(\mathbf{x},t) - \varepsilon^2 \mathrm{e}^{2Kt}\big(C_1 d^2(\mathbf{x},t) + C_2\big),$$

where $C_1, C_2 \geq 1$ are constants independent of ε to be determined later on.

We need two auxiliary functions $\xi_-, \mu_- \in C^{0,1}(\mathbf{R})$, which can be interpreted as suitable shifts of the original ξ and μ of §3. They are defined by

$$\xi_-(x) = \xi(x) + \tfrac{\pi}{8}\gamma'(x), \qquad \mu_-(x) = \mu(x) + \gamma'(x), \qquad \forall\ x \in \mathbf{R},$$

and still solve (3.8). Furthermore, ξ_-' and μ_-' are discontinuous at $-\frac{\pi}{2}$ because $\llbracket \xi_-' \rrbracket(-\frac{\pi}{2}) = \frac{\pi}{4}$, $\llbracket \mu_-' \rrbracket(-\frac{\pi}{2}) = 2$, but $\xi_-'(\frac{\pi}{2}) = \mu_-'(\frac{\pi}{2}) = 0$ (see Fig. 1). We then introduce the function $\gamma_-(\,\cdot\,;\varepsilon,\alpha_1,\alpha_2,\alpha_3) : \mathbf{R} \to \mathbf{R}$ defined by

$$\gamma_-(x;\varepsilon,\alpha_1,\alpha_2,\alpha_3) = \gamma(x) + \varepsilon\alpha_1\eta(x) + \varepsilon^2\big(\alpha_2 + \tfrac{1}{2}\alpha_1^2\big)\xi_-(x) - \varepsilon^2\tfrac{c_0}{2}\alpha_3\mu_-(x),$$

for any $\varepsilon > 0$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ (see Fig. 2). Note that $\gamma_-(x) = 1$ (-1) for all $x \geq \frac{\pi}{2}$ $(x \leq -\frac{\pi}{2})$, $\gamma_-'(\frac{\pi}{2}) = 0$, and, for $\varepsilon \leq \mathcal{E} \ll 1$ (depending on the α_i's),

$\gamma_-''(\frac{\pi}{2}^-) \le -\frac{1}{2}$, $\gamma_-''(-\frac{\pi}{2}^+) \ge \frac{1}{2}$. Hence, γ_- is strictly convex in a right neighborhood of $-\frac{\pi}{2}$ uniformly in $\varepsilon \le \mathcal{E} \ll 1$, which in turn shows the existence of at most one point $a_\varepsilon \in (-\frac{\pi}{2}, 0)$ such that $\gamma_-(a_\varepsilon) = -1$. If such point does exist, then $\gamma_-'(a_\varepsilon) > 0$. Otherwise we set $a_\varepsilon = -\frac{\pi}{2}$ and point out that in any event $\gamma_-(a_\varepsilon) = -1$, $\gamma_-'(a_\varepsilon^+) \ge 0$, and $\gamma_-'(x) > 0$, $|\gamma_-(x)| < 1$ for all $x \in (a_\varepsilon, \frac{\pi}{2})$. Since no confusion is possible, we henceforth denote by γ_- the function $\max(-1, \gamma_-)$.

Let $y(\mathbf{x}, t) = \frac{d_\varepsilon^-(\mathbf{x},t)}{\varepsilon}$ be the modified stretched variable. Recall the notation $\bar\kappa^s(\mathbf{x}, t) = \kappa^s(\mathbf{s}(\mathbf{x}, t), t)$, $\bar g(\mathbf{x}, t) = g(s(\mathbf{x}, t), t)$, and set

$$\bar g'(\mathbf{x}, t) = \big(\nabla g\big)\big(\mathbf{s}(\mathbf{x}, t), t\big) \cdot \nabla d(\mathbf{x}, t).$$

The subsolution v_ε^- for the double obstacle problem (4.1) is given by

$$v_\varepsilon^-(\mathbf{x}, t) = \gamma_-(y(\mathbf{x}, t); \varepsilon, \bar g(\mathbf{x}, t), \bar\kappa^s(\mathbf{x}, t), \bar g'(\mathbf{x}, t)) \qquad \forall\ (\mathbf{x}, t) \in \Omega \times [0, T].$$

Note that $|v_\varepsilon^-(\mathbf{x}, t)| \le 1$ in $\Omega \times [0, T]$, provided $\varepsilon \le \mathcal{E} \ll 1$ (depending on K and $\|g\|_{L^\infty(\Omega\times(0,T))}$). Then, we define the layer $\mathcal{T}_\varepsilon = \{(\mathbf{x}, t) \in \mathbf{R}^n \times [0, T] : \mathbf{x} \in \mathcal{T}_\varepsilon(t)\}$, where

$$\mathcal{T}_\varepsilon(t) = \{\mathbf{x} \in \mathbf{R}^n : |v_\varepsilon^-(\mathbf{x}, t)| < 1\},$$

and note that $|d_\varepsilon^-(\mathbf{x}, t)| < \frac{\pi}{2}\varepsilon$ in $\mathcal{T}_\varepsilon$. For $\varepsilon \le \mathcal{E} \ll 1$ (depending on D, K, C_1, C_2, and T), we have $\mathcal{T}_\varepsilon \subset \mathcal{T}$, that is $|d(\mathbf{x}, t)| \le D$ provided $|d_\varepsilon^-(\mathbf{x}, t)| \le \frac{\pi}{2}\varepsilon$, and

$$d(\mathbf{x}, t) = d_\varepsilon^-(\mathbf{x}, t) + C_2\varepsilon^2 \mathrm{e}^{2Kt} + \mathcal{O}(\varepsilon^3) = \mathcal{O}(\varepsilon) \qquad \forall\ (\mathbf{x}, t) \in \mathcal{T}_\varepsilon. \tag{5.1}$$

Hereafter, the notation $\zeta = \mathcal{O}(\varepsilon^\alpha)$ will stand for $|\zeta| \le C_*\varepsilon^\alpha + C_{C_1,C_2}\varepsilon^{\alpha+1}$ with constants $C_*, C_{C_1,C_2} \ge 0$ that may vary at the various occurrences, and C_{C_1,C_2} is independent of ε whereas C_* is independent of ε, C_1, and C_2. In view of (2.2) and (5.1), we have, for all $(\mathbf{x}, t) \in \mathcal{T}_\varepsilon$,

$$\begin{aligned} |\nabla d_\varepsilon^-|^2 &= 1 - 4C_1\varepsilon^2 d_\varepsilon^- \mathrm{e}^{2Kt} + \mathcal{O}(\varepsilon^3) \\ \partial_t d_\varepsilon^- - \Delta d_\varepsilon^- &= \bar g + d_\varepsilon^- \bar\kappa^s - C_2\varepsilon^2 \mathrm{e}^{2Kt}(2K - \bar\kappa^s) + 2C_1\varepsilon^2 \mathrm{e}^{2Kt} + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{5.2}$$

if $\varepsilon \le \mathcal{E} \ll 1$. Note the occurrence of d_ε^- instead of d on the right hand sides.

We now intend to prove that v_ε^- is a subsolution for the double obstacle problem (4.1). First, we check that

$$v_\varepsilon^-(\mathbf{x}, 0) \le u_\varepsilon^0(\mathbf{x}) \qquad \forall\ \mathbf{x} \in \Omega.$$

Set $z(\mathbf{x}) = \frac{d(\mathbf{x},0)}{\varepsilon}$ and note that, for $t = 0$ and $\mathbf{x} \in \mathcal{T}_\varepsilon(0)$, $y = z - C_2\varepsilon + \mathcal{O}(\varepsilon^2) < z - \frac{1}{2}C_2\varepsilon$ for $\varepsilon \le \mathcal{E} \ll 1$ (depending on C_1 and C_2). Since we only have to consider $-\frac{\pi}{2} \le a_\varepsilon(\mathbf{x}, 0) < y < z < \frac{\pi}{2}$, we use the elementary inequality [9]

$$\gamma(z) - \gamma(y) \ge \tfrac{1}{2\pi} C_2\varepsilon\big(\tfrac{1}{2}C_2\varepsilon + \min\big(\tfrac{\pi}{2} + y, \tfrac{\pi}{2} - z\big)\big)$$

to conclude that, for $C_2 \geq 1$ sufficiently large but independent of $\varepsilon \leq \mathcal{E} \ll 1$,

$$v_\varepsilon^-(\mathbf{x},0) = \gamma_-(y) < \gamma(z) - \varepsilon\Big(\tfrac{1}{8\pi}C_2^2\varepsilon + \tfrac{1}{2\pi}C_2 \min\big(\tfrac{\pi}{2}+y, \tfrac{\pi}{2}-z\big) - \bar{g}\eta(y)\Big)$$
$$- \varepsilon^2\Big(\tfrac{1}{8\pi}C_2^2 - (\bar{\kappa}^s + \tfrac{1}{2}\bar{g}^2)\xi_-(y) + \tfrac{c_0}{2}\bar{g}'\mu_-(y)\Big) \leq \gamma(z) = u_\varepsilon^0(\mathbf{x}).$$

We point out that the last inequality results from the property $\eta(y) = \mathcal{O}(\varepsilon)$ for $z = \frac{\pi}{2}$. To demonstrate the second inequality in (4.2) we would like to integrate the middle term by parts. This would involve regularity of $\partial\mathcal{T}_\varepsilon(t)$, which is not a priori guaranteed. To circumvent such a difficulty, we introduce a set $\mathcal{T}_\varepsilon^\rho(t)$ slightly smaller than $\mathcal{T}_\varepsilon(t)$, namely,

$$\mathcal{T}_\varepsilon^\rho(t) = \{\mathbf{x} \in \mathcal{T}_\varepsilon(t) : -1+\rho < v_\varepsilon^-(\mathbf{x},t) < 1\},$$

along with its boundaries

$$\partial^-\mathcal{T}_\varepsilon^\rho(t) = \{x \in \partial\mathcal{T}_\varepsilon(t) : v_\varepsilon^-(\mathbf{x},t) = -1+\rho\}, \qquad \partial^+\mathcal{T}_\varepsilon^\rho(t) = \partial\mathcal{T}_\varepsilon(t)\backslash\partial^-\mathcal{T}_\varepsilon^\rho(t),$$

where $0 < \rho \ll \varepsilon$ will tend to 0. Since $\eta(\frac{\pi}{2}) = \xi_-(\frac{\pi}{2}) = \mu_-(\frac{\pi}{2}) = 0$, it turns out that $\partial^+\mathcal{T}_\varepsilon^\rho(t)$ is also the $\frac{\pi}{2}$–level set of $y(\cdot,t)$ which, in view of (2.1) and (5.2), is a smooth function with nonvanishing gradient. The Implicit Function Theorem yields the existence of an outward unit vector ν_ε normal to $\partial^+\mathcal{T}_\varepsilon^\rho(t)$. Moreover, the expression

$$\nabla v_\varepsilon^- = \tfrac{1}{\varepsilon}\gamma_-'(y)\nabla d_\varepsilon^- + \varepsilon\eta(y)\nabla\bar{g} + \varepsilon^2\Big(\xi_-(y)\nabla\big(\bar{\kappa}^s + \tfrac{1}{2}\bar{g}^2\big) - \tfrac{c_0}{2}\mu_-(y)\nabla\bar{g}'\Big) \tag{5.3}$$

in $\mathcal{T}_\varepsilon$, shows that $\nabla v_\varepsilon^- = 0$ on $\partial^+\mathcal{T}_\varepsilon^\rho(t)$, because $\gamma_-'(\frac{\pi}{2}) = 0$. On the other hand, if $x_\rho > a_\varepsilon$ is defined by $\gamma_-(x_\rho) = -1+\rho$, then we infer that $\gamma_-'(x_\rho) \geq C(x_\rho - a_\varepsilon)$ with $C > 0$ independent of ε. This is certainly a consequence of the strict convexity of γ_- in a right neighborhood of a_ε. With the aid of (5.3), we deduce

$$\big|\nabla v_\varepsilon^-(\mathbf{x},t)\big| \geq \tfrac{1}{\varepsilon}C(x_\rho(\mathbf{x},t) - a_\varepsilon(\mathbf{x},t))\big(1 - \mathcal{O}(\varepsilon^2)\big) \neq 0 \qquad \forall\, \mathbf{x} \in \partial^-\mathcal{T}_\varepsilon^\rho(t),$$

and $\varepsilon \leq \mathcal{E} \ll 1$. The Implicit Function Theorem applies again and shows the existence of a unit outward vector ν_ε orthogonal to $\partial^-\mathcal{T}_\varepsilon^\rho(t)$. We then realize that $\nabla v_\varepsilon^- \cdot \nu_\varepsilon = -|\nabla v_\varepsilon^-| \leq 0$ on $\partial^-\mathcal{T}_\varepsilon^\rho(t)$, because $\partial^-\mathcal{T}_\varepsilon^\rho(t)$ is the $(-1+\rho)$–level set of v_ε^- and $v_\varepsilon^- > -1+\rho$ in $\mathcal{T}_\varepsilon^\rho(t)$. Set

$$f(\mathbf{x},t) = \begin{cases} \frac{1}{\varepsilon}v_\varepsilon^-(\mathbf{x},t) + \frac{c_0}{2}g(\mathbf{x},t) & \text{if } v_\varepsilon^-(\mathbf{x},t) > -1 \\ 0 & \text{if } v_\varepsilon^-(\mathbf{x},t) = -1 \end{cases} \tag{5.4}$$

and observe that $f(\mathbf{x},t) > 0$ in $\{v_\varepsilon^-(\mathbf{x},t) = 1\}$, for $\varepsilon \leq \mathcal{E} \ll 1$. Integration by parts in $\mathcal{T}_\varepsilon^\rho(t)$ for all $t \in (0,T)$ leads to

$$\varepsilon\langle\partial_t v_\varepsilon^-, \varphi\rangle + \varepsilon\langle\nabla v_\varepsilon^-, \nabla\varphi\rangle - \langle f, \varphi\rangle = \int_{\mathcal{T}_\varepsilon(t)} \mathcal{J}v_\varepsilon^-\varphi + \varepsilon\int_{\partial\mathcal{T}_\varepsilon^\rho(t)} \nabla v_\varepsilon^- \cdot \nu_\varepsilon\varphi$$
$$- \int_{\Omega\backslash\mathcal{T}_\varepsilon(t)} f\varphi + \varepsilon\int_{\mathcal{T}_\varepsilon(t)\backslash\mathcal{T}_\varepsilon^\rho(t)} (\nabla v_\varepsilon^- \cdot \nabla\varphi + \Delta v_\varepsilon^-\varphi) = \mathrm{I} + \mathrm{II}^\rho + \mathrm{III} + \mathrm{IV}^\rho \tag{5.5}$$

for all $0 \le \varphi \in H^1(\Omega)$, where $\mathcal{J} v_\varepsilon^- = \varepsilon \partial_t v_\varepsilon^- - \varepsilon \Delta v_\varepsilon^- - \frac{1}{\varepsilon} v_\varepsilon^- - \frac{c_0}{2} g$. In the sequel we examine each term separately. Since $\nabla v_\varepsilon^- \cdot \nu_\varepsilon \le 0$ on $\partial \mathcal{T}_\varepsilon^\rho(t)$, we get $\mathrm{II}^\rho \le 0$. From (5.4) we also have $\mathrm{III} \le 0$ for $\varepsilon \le \mathcal{E} \ll 1$. Hence, it only remains to establish that the first term is nonpositive, because $|\mathrm{IV}^\rho| \le \frac{1}{\varepsilon} |\mathcal{T}_\varepsilon(t) \backslash \mathcal{T}_\varepsilon^\rho(t)|^{1/2} \|\varphi\|_{H^1(\Omega)} \downarrow 0$ as $\rho \downarrow 0$. First, we can write

$$g(\mathbf{x},t) = \bar g(\mathbf{x},t) + \varepsilon y(\mathbf{x},t) \bar g'(\mathbf{x},t) + C_2 \varepsilon^2 \mathrm{e}^{2Kt} \bar g'(\mathbf{x},t) + \mathcal{O}(\varepsilon^2),$$

as results from (5.1), and then use the definition of v_ε^- to arrive at

$$\begin{aligned} \varepsilon \partial_t v_\varepsilon^- &= \gamma_-'(y) \partial_t d_\varepsilon^- + \mathcal{O}(\varepsilon^2) \\ \varepsilon \Delta v_\varepsilon^- &= \tfrac{1}{\varepsilon} \gamma_-''(y) |\nabla d_\varepsilon^-|^2 + \gamma_-'(y) \Delta d_\varepsilon^- + 2\varepsilon \eta'(y) \nabla d_\varepsilon^- \cdot \nabla \bar g + \mathcal{O}(\varepsilon^2) \end{aligned}$$

for all $(\mathbf{x},t) \in \mathcal{T}_\varepsilon$. Since $\nabla \bar g(\mathbf{x},t)$ is tangent to the front $\Sigma(t)$ at $\mathbf{s}(\mathbf{x},t)$ whereas $\nabla d_\varepsilon^-(\mathbf{x},t)$ is orthogonal, we obtain the crucial relation $\nabla d_\varepsilon^- \cdot \nabla \bar g = 0$. Consequently

$$\begin{aligned} \mathcal{J} v_\varepsilon^- = &- \tfrac{1}{\varepsilon} \big(\gamma_-''(y) |\nabla d_\varepsilon^-|^2 + \gamma_-(y) \big) + \gamma_-'(y) \big(\partial_t d_\varepsilon^- - \Delta d_\varepsilon^- \big) \\ &- \tfrac{c_0}{2} \bar g - \tfrac{c_0}{2} \varepsilon y \bar g' - C_2 \varepsilon^2 \tfrac{c_0}{2} \mathrm{e}^{2Kt} \bar g' + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Hence, using properties (5.2) and $\eta'(y) = \frac{1}{2} y \gamma'(y)$, we get

$$\begin{aligned} \mathcal{J} v_\varepsilon^- &= -\tfrac{1}{\varepsilon} \big(\gamma''(y) + \gamma(y) \big) - \bar g \big(\eta''(y) + \eta(y) - \gamma'(y) + \tfrac{c_0}{2} \big) \\ -\varepsilon \big(\bar\kappa^s &+ \tfrac{1}{2} \bar g^2 \big) \big(\xi_-''(y) + \xi_-(y) - y\gamma'(y) \big) + \varepsilon \tfrac{c_0}{2} \bar g' \big(\mu_-''(y) + \mu_-(y) - y \big) \\ -\varepsilon^2 \mathrm{e}^{2Kt} \big(&- 4 C_1 y \gamma''(y) + C_2 \gamma'(y) (2K - \bar\kappa^s) + C_2 \tfrac{c_0}{2} \bar g' - 2 C_1 \gamma'(y) \big) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

In light of the definitions of γ, η, ξ_-, and μ_- (see (2.3), (3.7), and (3.8)), the first four terms on the right hand side vanish. In addition, since $y\gamma(y) + \gamma'(y) \ge 1$ and $K \ge \bar\kappa^s + \frac{c_0}{2} |\bar g'|$, we get

$$\begin{aligned} \mathcal{J} v_\varepsilon^- &\le -\varepsilon^2 \mathrm{e}^{2Kt} \Big(y\gamma(y) \big(4C_1 - C_2 \tfrac{c_0}{2} |\bar g'| \big) + \gamma'(y) \big(C_2 (2K - \bar\kappa^s - \tfrac{c_0}{2} |\bar g'|) - 2C_1 \big) \Big) \\ &+ \mathcal{O}(\varepsilon^2) \le -\varepsilon^2 \mathrm{e}^{2Kt} \Big(y\gamma(y) \big(4C_1 - C_2 K \big) + \gamma'(y) \big(C_2 K - 2C_1 \big) \Big) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Recall that $\mathcal{O}(\varepsilon^2)$ stands for $C_* \varepsilon^2 + C_{C_1,C_2} \varepsilon^3$. The choice $C_2 K = 3C_1$ then leads to

$$\mathcal{J} v_\varepsilon^- \le -\varepsilon^2 \big(C_1 - C_* - \varepsilon C_{C_1,C_2} \big) \le 0,$$

for C_1 sufficiently large but independent of ε, say $C_1 = 2C_*$, and $\varepsilon \le \mathcal{E} \ll 1$ depending on C_1, C_2, say $\mathcal{E} \le C_* / C_{C_1,C_2}$; thus $\mathrm{I} \le 0$.

Upon taking $\rho \downarrow 0$ in (5.5) we conclude that v_ε^- satisfies (4.2). Finally, applying the Comparison Lemma of §4 we end up with

$$v_\varepsilon^-(\mathbf{x},t) \le u_\varepsilon(\mathbf{x},t) \qquad \forall\ (\mathbf{x},t) \in \Omega \times [0,T].$$

The construction of a supersolution v_ε^+ can be carried out as follows. Let $\tilde{d}_\varepsilon^-$ and $\tilde{v}_\varepsilon^-$ indicate the modified distance function and the subsolution of the auxiliary problem corresponding to $\tilde{g} = -g$ and $\tilde{d} = -d$. We clearly obtain that $v_\varepsilon^+ = -\tilde{v}_\varepsilon^-$ is a supersolution and $d_\varepsilon^+ = -\tilde{d}_\varepsilon^-$ is the modified distance function for the original problem.

6. Proof of Theorem 2.1

Set $\Sigma_\varepsilon^\mp(t) = \{\mathbf{x} \in \Omega : v_\varepsilon^\mp(\mathbf{x},t) = 0\}$. Invoking the nondegeneracy property of $v_\varepsilon^\mp$ around the interface, namely, $\nabla v_\varepsilon^\mp \cdot \bar{\mathbf{n}} = -\frac{1}{\varepsilon} + \mathcal{O}(1)$ provided $d_\varepsilon^\mp(\mathbf{x},t) = 0$, in conjunction with the property $v_\varepsilon^\mp(\mathbf{x},t) = \mathcal{O}(\varepsilon)$ for $d_\varepsilon^\mp(\mathbf{x},t) = 0$, we have

$$\mathbf{x} \in \Sigma_\varepsilon^\mp(t) \qquad \text{implies} \qquad d_\varepsilon^\mp(\mathbf{x},t) = \mathcal{O}(\varepsilon^2).$$

Hence, (5.1) shows that $\mathbf{x} \in \Sigma_\varepsilon^\mp(t)$ yields $d(\mathbf{x},t) = \mathcal{O}(\varepsilon^2)$, and so

$$\operatorname{dist}\big(\Sigma_\varepsilon^\mp(t), \Sigma(t)\big) \le C_T \varepsilon^2 \qquad \forall\, t \in [0,T].$$

The inequalities $v_\varepsilon^-(\mathbf{x},t) \le u_\varepsilon(\mathbf{x},t) \le v_\varepsilon^+(\mathbf{x},t)$ for all $(\mathbf{x},t) \in \Omega \times [0,T]$ then lead to

$$\operatorname{dist}\big(\Sigma_\varepsilon(t), \Sigma_\varepsilon^\mp(t)\big) \le \operatorname{dist}\big(\Sigma_\varepsilon^-(t), \Sigma_\varepsilon^+(t)\big) \le C_T \varepsilon^2 \qquad \forall\, t \in [0,T],$$

and the error estimate in Theorem 2.1 follows. Optimality has been already proved in [9] for the radially symmetric case with a vanishing forcing term $g = 0$. Theorem 2.1 is thus complete.

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