QUADRATIC RATE OF CONVERGENCE FOR CURVATURE DEPENDENT SMOOTH INTERFACES: A SIMPLE PROOF[1]

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Abstract. We consider the approximation of curvature dependent geometric front evolutions by singularly perturbed parabolic double obstacle problems with small parameter ε. We give a simplified proof of optimal interface error estimates of order $\mathcal{O}(\varepsilon^2)$, valid in the smooth regime, which is based on constructing precise barriers, perturbing the forcing term and exploiting the maximum principle.

1. INTRODUCTION

Let $\Sigma_0 \subset \mathbb{R}^n$ be a smooth closed oriented manifold of codimension 1, and let d_0 denote the signed distance function to Σ_0 that is negative inside Σ_0. Let $\Sigma(t)$ be the surface that emanating from Σ_0 moves in its normal direction (pointing toward the set $I(t)$ inside $\Sigma(t)$) with velocity $V = \kappa + g$. Hereafter κ indicates the sum of the principal curvatures, positive if $I(t)$ is convex, and g is a given forcing term defined in $\mathbb{R}^n \times [0,T]$. This motion is regular for some time, but may eventually develop singularities in which case has to be interpreted in a viscosity sense. We just refer to [1,5] and references therein because our main concern here is the smooth regime. We suppose that for all $t \in [0,T]$ the propagating front $\Sigma(t)$ is contained in a smooth domain Ω, and set $Q := \Omega \times (0,T)$.

Based on the Landau-Ginzburg theory of phase transitions, $\Sigma(t)$ can be viewed as the asymptotic limit as $\varepsilon \downarrow 0$ of the zero level set $\Sigma_\varepsilon(t) := \{\mathbf{x} \in \Omega : \chi_\varepsilon(\mathbf{x},t) = 0\}$ of the solution χ_ε to the singularly perturbed reaction-diffusion equation:

$$\varepsilon \partial_t \chi_\varepsilon - \varepsilon \Delta \chi_\varepsilon + \tfrac{1}{2\varepsilon} \Psi'(\chi_\varepsilon) = \tfrac{c_0}{2} g \quad \text{in } Q. \tag{1}$$

Here Ψ is a double well potential with wells of equal depth at ± 1 and $c_0 := \int_{-1}^{1} \sqrt{\Psi(s)}ds$ is a proper scaling factor. A typical example is $\Psi(s) := (s^2-1)^2$ and $c_0 = \frac{4}{3}$. Convergence of $\Sigma_\varepsilon(t)$ to $\Sigma(t)$ as $\varepsilon \downarrow 0$ has been proved in [2,3] for smooth evolutions, and in [1,5] even past singularities, in case of no fattening; [4] is a related work about rigorous asymptotics. The solution $\chi_\varepsilon(\cdot,t)$ exhibits a transition layer across $\Sigma_\varepsilon(t)$, perhaps after a short initial transient, but never attains the equilibrium values $+1$ or -1. Consequently (1) has to be solved in the whole Ω, which increases by one the dimension of the original problem and is thus numerically inconvenient. The specific form of Ψ is however immaterial to achieve the geometric law $V = \kappa + g$ in the limit, provided the two wells possess equal depth. This observation suggests that other choices of Ψ, possibly singular, may improve upon the above quartic. In this vein, we consider the alternative potential

$$\Psi(s) := \begin{cases} 1 - s^2 & \text{if } s \in [-1,1] \\ +\infty & \text{if } s \notin [-1,1], \end{cases}$$

for which $c_0 = \frac{\pi}{2}$. Such a Ψ converts equation (1) into the following double obstacle problem:

$$\varepsilon \partial_t \chi_\varepsilon - \varepsilon \Delta \chi_\varepsilon - \tfrac{1}{\varepsilon} \chi_\varepsilon + \operatorname{sgn}^{-1}(\chi_\varepsilon) \ni \tfrac{c_0}{2} g \quad \text{in } Q, \tag{2}$$

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with boundary condition $\chi_\varepsilon = 1$ on $\partial\Omega \times (0,T)$ and initial condition $\chi_\varepsilon(\cdot,0) = \chi_\varepsilon^0(\cdot)$ in Ω. In contrast to the usual reaction-diffusion approach with a quartic-like nonlinearity, the solution χ_ε to (2) attains the values $+1$ or -1, irrespective of g, outside an $\mathcal{O}(\varepsilon)$-wide transition region where $|\chi_\varepsilon| < 1$. Since the resulting problem has to be solved within such a thin noncoincidence set, where all the action takes place, we realize that this approach retains the geometric (or local) structure of the original problem while taking advantage of the variational structure of (2) and being insensitive to singularity formation. This is numerically very attractive and of some intrinsic interest as well [7].

If the evolution is smooth, then an optimal rate of convergence $\mathcal{O}(\varepsilon^2)$ for interfaces was proved in [8,9], provided $\chi_\varepsilon^0 := \gamma\big(\frac{d_0}{\varepsilon}\big)$ with γ the standing wave of §2. The interface error estimates in [8,9] are a consequence of the maximum principle and the explicit construction of tight barriers which, in turn, are dictated by the formal asymptotics developed in [10]. An essential but very technical ingredient is the use of a modified distance function which incorporates a shape correction. The purpose of this note is to rederive the optimal rate via a simple argument without the modified distance function. Its correction effect is achieved here with a straightforward perturbation of the forcing term.

2. BARRIERS: DEFINITION

Formal asymptotics suggests a number of shape corrections to the basic standing wave profile γ that give rise to the barriers γ_+ and γ_-. We now introduce those functions in close form without any motivation, and refer to [8,9,10] for details. Since $\gamma \in C^{1,1}(\mathbb{R})$ solves the (elliptic) double obstacle problem $\gamma''(x) - \frac{1}{2}\Psi'(\gamma(x)) \ni 0$ in $\mathbb{R}$, γ is given by

$$\gamma(x) := \sin x \text{ if } x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \quad \gamma(x) := -1 \text{ if } x < -\tfrac{\pi}{2}, \quad \gamma(x) := +1 \text{ if } x > \tfrac{\pi}{2};$$

thus $\mathcal{L}(\gamma) := \gamma'' + \gamma = 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. The remaining functions are given by

$$\eta(x) := \tfrac{1}{2}\big(x\gamma(x) - c_0 + \gamma'(x)\big), \quad \xi(x) := \tfrac{1}{16}\big(4x\gamma'(x) + (4x^2 - \pi^2)\gamma(x)\big), \quad \mu(x) := \big(x - \tfrac{\pi}{2}\gamma(x)\big),$$
$$\xi_\pm(x) := \xi(x) \mp \tfrac{\pi}{8}\gamma'(x), \quad \mu_\pm(x) := \mu(x) \mp \gamma'(x),$$

for $|x| \le \frac{\pi}{2}$, and extended by zero for $|x| > \frac{\pi}{2}$. It is worth noting, and easy to check, that $\eta \in C^{1,1}(\mathbb{R})$, $\xi, \mu \in C^{0,1}(\mathbb{R})$ with discontinuous first derivatives at $x = \pm\frac{\pi}{2}$, ξ'_+, μ'_+ (resp. ξ'_-, μ'_-) are continuous at $x = -\frac{\pi}{2}$ (resp. $x = \frac{\pi}{2}$), and for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\mathcal{L}(\eta)(x) = \gamma'(x) - \tfrac{c_0}{2}, \quad \mathcal{L}(\xi_\pm)(x) = x\gamma'(x), \quad \mathcal{L}(\mu_\pm)(x) = x. \tag{3}$$

Let $d(\cdot,t)$ denote the signed distance function to the front $\Sigma(t)$ that is negative in $I(t)$; thus $d(\mathbf{x},0) = d_0(\mathbf{x})$. Note that d is smooth in a neighborhood $\mathcal{T} := \{(\mathbf{x},t) \in Q : |d(\mathbf{x},t)| \le D\}$ of $\Sigma := \{(\mathbf{x},t) \in Q : \mathbf{x} \in \Sigma(t)\}$ and

$$\mathbf{s}(\mathbf{x},t) := \mathbf{x} - d(\mathbf{x},t)\nabla d(\mathbf{x},t) \in \Sigma(t) \quad \forall\ (\mathbf{x},t) \in \mathcal{T}$$

is the unique point of $\Sigma(t)$ at distance $|d(\mathbf{x},t)|$. Consequently $\nabla d(\mathbf{x},t)$ is the outward unit vector normal to $\Sigma(t)$ at $\mathbf{s}(\mathbf{x},t)$. Let $\kappa^s(\mathbf{x},t)$ denote the sum of the squares of the principal curvatures of $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$, and

$$\bar\kappa^s(\mathbf{x},t) := \kappa^s\big(\mathbf{s}(\mathbf{x},t),t\big), \quad \bar g(\mathbf{x},t) := g\big(\mathbf{s}(\mathbf{x},t),t\big), \quad \bar g'(\mathbf{x},t) := (\nabla g)\big(\mathbf{s}(\mathbf{x},t),t\big)\cdot\nabla d(\mathbf{x},t).$$

In view of $\partial_t d = V = \kappa + g$ and the explicit form of Δd [6], d satisfies the property:

$$\partial_t d(\mathbf{x},t) - \Delta d(\mathbf{x},t) - \bar g(\mathbf{x},t) = d(\mathbf{x},t)\bar\kappa^s(\mathbf{x},t) + \mathcal{O}(d^2(\mathbf{x},t)) \quad \forall\ (\mathbf{x},t) \in \mathcal{T}. \tag{4}$$

The candidate for supersolution reads

$$\gamma_+(x;\mathbf{x},t) := \min\Big(1, \gamma(x) + \varepsilon(\bar g + \varepsilon^2 a)\eta(x) + \varepsilon^2\big(\bar\kappa^s + \tfrac{1}{2}\bar g^2\big)\xi_+(x) - \tfrac{c_0}{2}\varepsilon^2\bar g'\mu_+(x)\Big), \tag{5}$$

for all $x \in \mathbb{R}$, $(\mathbf{x},t) \in Q$, where $a = a(t) > 0$ is to be selected depending on t. Note that $\gamma_+(\cdot;\mathbf{x},t) \in C^{0,1}(\mathbb{R})$, because $[\![\gamma_+']\!](\frac{\pi}{2}) \neq 0$, γ_+' is continuous at $-\frac{\pi}{2}$, and γ_+ is strictly increasing and satisfies $|\gamma_+| < 1$ in $(-\frac{\pi}{2}, x_\varepsilon)$ for a proper $0 < x_\varepsilon \leq \frac{\pi}{2}$, provided ε is sufficiently small. Similarly the subsolution reads

$$\gamma_-(x;\mathbf{x},t) := \max\Big(-1, \gamma(x) + \varepsilon(\bar{g} - \varepsilon^2 a)\eta(x) + \varepsilon^2\big(\bar{\kappa}^s + \tfrac{1}{2}\bar{g}^2\big)\xi_-(x) - \tfrac{c_0}{2}\varepsilon^2\bar{g}'\mu_-(x)\Big).$$

We point out that the perturbation $\varepsilon^2 a$ to $\bar{g}$ is reminiscent of that in [1] but different from it. In fact, the idea in [1] is to use the geometric motion corresponding to the perturbed forcing in constructing barriers, which may not be smooth for as long as $\Sigma(t)$ is.

3. BARRIERS: COMPARISON

We intend to construct an explicit supersolution to the variational inequality (2) with convex set

$$K := \{\varphi \in H^1(\Omega) : |\varphi| \leq 1 \text{ in } \Omega, \varphi = 1 \text{ on } \partial\Omega\}.$$

The variational formulation reads: *Seek $\chi_\varepsilon \in L^2(0,T;K) \cap H^1(0,T;H^{-1}(\Omega))$ such that $\chi_\varepsilon(\cdot,0) = \chi_\varepsilon^0(\cdot)$ a.e. in Ω and, for all $\varphi \in K$ and a.e. $t \in (0,T)$,*

$$\varepsilon\langle\partial_t\chi_\varepsilon, \varphi - \chi_\varepsilon\rangle + \varepsilon\langle\nabla\chi_\varepsilon, \nabla(\varphi - \chi_\varepsilon)\rangle - \tfrac{1}{\varepsilon}\langle\chi_\varepsilon, \varphi - \chi_\varepsilon\rangle - \tfrac{c_0}{2}\langle g, \varphi - \chi_\varepsilon\rangle \geq 0. \tag{6}$$

Hereafter $\langle\cdot,\cdot\rangle$ stands for the L^2 inner product or the usual duality pairing between H^{-1} and H_0^1. We say that a function $\chi \in H^1(Q)$ *is a supersolution of (6) if $\chi \geq -1$ and*

$$\varepsilon\langle\partial_t\chi, \varphi\rangle + \varepsilon\langle\nabla\chi, \nabla\varphi\rangle - \langle f, \varphi\rangle \geq 0 \quad \forall\ 0 \leq \varphi \in H_0^1(\Omega), \text{ a.e. } t \in (0,T). \tag{7}$$

with $f \geq \frac{1}{\varepsilon}\chi + \frac{c_0}{2}g$ a.e. in $\{\chi < +1\}$. The following comparison lemma holds [8].

Lemma 1. *Let χ be a supersolution of (6). If $\chi(\cdot,0) \geq \chi_\varepsilon^0(\cdot)$ a.e. in Ω and $\chi \geq 1$ on $\partial\Omega \times (0,T)$, then $\chi \geq \chi_\varepsilon$ for all $(\mathbf{x},t) \in Q$.*

Proof: Take $\varphi = \min(\chi, \chi_\varepsilon)$ in (6) and $\varphi = \min(\chi - \chi_\varepsilon, 0)$ in (7), subtract the resulting expressions and integrate them on $(0,t)$, and finally use Gronwall's lemma. ∎

Lemma 2. *Let γ_+ be as in (5) for $a(t) := 2b\Lambda e^{3\Lambda t}$, where $\Lambda := \|\kappa^s\|_{L^\infty(\Sigma)} + \|\nabla g\|_{L^\infty(Q)} > 0$. For $b > 0$ sufficiently large and shift $\sigma(t) := be^{3\Lambda t}$, the following function is a supersolution of (6):*

$$\chi(\mathbf{x},t) := \gamma_+\big(\tfrac{d(\mathbf{x},t)}{\varepsilon} + \varepsilon\sigma(t); \mathbf{x}, t\big) \quad \forall\ (\mathbf{x},t) \in Q. \tag{8}$$

Proof: Since $|\chi| \leq 1$, we only have to demonstrate (7). Let the stretched variable be

$$y(\mathbf{x},t) := \tfrac{d(\mathbf{x},t)}{\varepsilon} + \varepsilon\sigma(t) \quad \forall\ (\mathbf{x},t) \in Q, \tag{9}$$

and define the layer $\mathcal{T}_\varepsilon := \{(\mathbf{x},t) \in Q : \mathbf{x} \in \mathcal{T}_\varepsilon(t)\}$, where $\mathcal{T}_\varepsilon(t) := \{\mathbf{x} \in \Omega : |\chi(\mathbf{x},t)| < 1\}$. Then $|y| < \frac{\pi}{2}$ in $\mathcal{T}_\varepsilon$, whence $\mathcal{T}_\varepsilon \subset \mathcal{T}$ for ε sufficiently small. In view of (4) and (9), for all $(\mathbf{x},t) \in \mathcal{T}_\varepsilon$, we have $|\nabla y| = \frac{1}{\varepsilon}$ and

$$\varepsilon(\partial_t y - \Delta y) = \bar{g} + \varepsilon y\bar{\kappa}^s + \varepsilon^2\big(\sigma'(t) - \sigma(t)\bar{\kappa}^s\big) + \mathcal{O}(d^2), \tag{10}$$

$$g = \bar{g} + \varepsilon y\bar{g}' - \varepsilon^2\sigma(t)\bar{g}' + \mathcal{O}(d^2). \tag{11}$$

Note the occurrence of y instead of d on the right hand sides. Now set $\mathcal{J}(\chi) := \varepsilon\partial_t\chi - \varepsilon\Delta\chi - \frac{1}{\varepsilon}\chi - \frac{c_0}{2}g$ and

$$f(\mathbf{x},t) := \begin{cases} \frac{1}{\varepsilon}\chi(\mathbf{x},t) + \frac{c_0}{2}g(\mathbf{x},t) & \text{if } \chi(\mathbf{x},t) < 1 \\ 0 & \text{if } \chi(\mathbf{x},t) = 1. \end{cases}$$

Integration by parts in $\mathcal{T}_\varepsilon(t)$ for all $t\in(0,T)$ leads to

$$\varepsilon\langle\partial_t\chi,\varphi\rangle+\varepsilon\langle\nabla\chi,\nabla\varphi\rangle-\langle f,\varphi\rangle=\int_{\mathcal{T}_\varepsilon(t)}\mathcal{J}(\chi)\varphi+\varepsilon\int_{\partial\mathcal{T}_\varepsilon(t)}\nabla\chi\cdot\nu\varphi-\int_{\Omega\backslash\mathcal{T}_\varepsilon(t)}f\varphi,$$

for all $0\le\varphi\in H_0^1(\Omega)$. This would involve regularity of $\partial\mathcal{T}_\varepsilon(t)$, which is not a priori guaranteed. To circumvent such a difficulty, we can introduce a set $\mathcal{T}_\varepsilon^\rho(t)$ slightly smaller than $\mathcal{T}_\varepsilon(t)$, namely, $\mathcal{T}_\varepsilon^\rho(t):=\{\mathbf{x}\in\mathcal{T}_\varepsilon(t):-1<\chi(\mathbf{x},t)<1-\rho\}$, where $0<\rho\ll\varepsilon$ will tend to 0 [9]. Since $\nabla\chi\cdot\nu\ge0$ on $\partial\mathcal{T}_\varepsilon(t)$ and $f\le0$ in $\Omega\backslash\mathcal{T}_\varepsilon(t)$ for ε sufficiently small, it only remains to establish that $\mathcal{J}(\chi)\ge0$ in $\mathcal{T}_\varepsilon(t)$. Using (8) we can write

$$\varepsilon\partial_t\chi=\varepsilon\gamma_+'\partial_t y+\varepsilon\partial_t\gamma_+,$$
$$\varepsilon\Delta\chi=\tfrac{1}{\varepsilon}\gamma_+''+\varepsilon\gamma_+'\Delta y+2\nabla\gamma_+'\cdot\nabla d+\varepsilon\Delta\gamma_+.$$

Hence, in view of (5), we have

$$\mathcal{J}(\chi)=-\tfrac{1}{\varepsilon}(\gamma_+''+\gamma_+)+\varepsilon\gamma_+'(\partial_t y-\Delta y)-\tfrac{c_0}{2}g-2\varepsilon\eta'\nabla\bar g\cdot\nabla d+\mathcal{O}(\varepsilon^2).$$

Since $\nabla\bar g(\mathbf{x},t)$ is tangent to $\Sigma(t)$ at $\mathbf{s}(\mathbf{x},t)$ whereas $\nabla d(\mathbf{x},t)$ is orthogonal, we obtain $\nabla\bar g\cdot\nabla d=0$. Using (3) and (5), the first term in the expression above reads

$$-\tfrac{1}{\varepsilon}(\gamma_+''+\gamma_+)=-\big(\bar g+\varepsilon^2a(t)\big)\big(\gamma'-\tfrac{c_0}{2}\big)-\varepsilon\big(\bar\kappa^s+\tfrac12\bar g^2\big)y\gamma'+\varepsilon\tfrac{c_0}{2}\bar g'y.$$

On the other hand, we make use of (10) and the fact that $\sigma'(t)=3\Lambda\sigma(t)$ to get

$$\varepsilon\gamma_+'(\partial_t y-\Delta y)=\bar g\gamma'+\varepsilon\bar g^2\eta'+\varepsilon\bar\kappa^s y\gamma'+\varepsilon^2\sigma(t)(3\Lambda-\bar\kappa^s)\gamma'+\mathcal{O}(\varepsilon^2)+b\mathcal{O}(\varepsilon^3),$$

where we use that $d^2=\mathcal{O}(\varepsilon^2)+b\mathcal{O}(\varepsilon^3)$ and $a(t)=b\mathcal{O}(1)$. With the aid of property $\eta'(y)=\frac12 y\gamma'(y)$ and (11), after reordering and cancellation we arrive at

$$\tfrac{1}{\varepsilon^2}\mathcal{J}(\chi)=a(t)\big(\tfrac{c_0}{2}-\gamma'\big)+\sigma(t)(3\Lambda-\bar\kappa^s)\gamma'+\sigma(t)\tfrac{c_0}{2}\bar g'+\mathcal{O}(1)+b\mathcal{O}(\varepsilon).$$

Since $a(t)=2\Lambda\sigma(t)\ge2b\Lambda$ and $\Lambda\ge\bar\kappa^s+|\bar g'|$, we easily obtain

$$\tfrac{1}{\varepsilon^2}\mathcal{J}\chi\ge a(t)\big(\tfrac{c_0}{2}-\gamma'\big)+a(t)\gamma'-\tfrac{c_0}{2}\Lambda\sigma(t)+\mathcal{O}(1)+b\mathcal{O}(\varepsilon)\ge\tfrac{c_0}{2}b\Lambda+\mathcal{O}(1)+b\mathcal{O}(\varepsilon)\ge0,$$

for b sufficiently large, but independent of ε, and ε sufficiently small. ∎

4. INTERFACE ERROR ESTIMATE

We finally prove optimal error estimates for interfaces before the onset of singularities. Let χ_ε be the solution to (2) with initial datum $\chi_\varepsilon^0(\mathbf{x}):=\gamma\big(\frac{d_0(\mathbf{x})}{\varepsilon}\big)$, and $\chi_\pm$ be the barriers in (8) for $\gamma_\pm$. It is easy to check that, for b sufficiently large [9],

$$\chi_+(\mathbf{x},0)=\gamma_+\big(\tfrac{d_0(\mathbf{x})}{\varepsilon}+b\varepsilon\big)\ge\gamma\big(\tfrac{d_0(\mathbf{x})}{\varepsilon}\big)=\chi_\varepsilon^0(\mathbf{x})\ge\chi_-(\mathbf{x},0)\quad\text{a.e. }\mathbf{x}\in\Omega.$$

Since $\chi_-=\chi_+=\chi_\varepsilon=1$ on $\partial\Omega\times(0,T)$, Lemma 1 yields

$$\chi_-(\mathbf{x},t)\le\chi_\varepsilon(\mathbf{x},t)\le\chi_+(\mathbf{x},t)\quad\forall\ (\mathbf{x},t)\in Q.\tag{12}$$

THEOREM 1. *There exist $0<\varepsilon_0$ such that for all $\varepsilon\le\varepsilon_0$ and $0<t\le T$ the following optimal interface error estimate holds:*

$$\operatorname{dist}\big(\Sigma(t),\Sigma_\varepsilon(t)\big)\le C\varepsilon^2.$$

PROOF: First note that

$$\big(\nabla\chi_\pm\cdot\nabla d\big)(\mathbf{z},t)>0\quad\forall\ \mathbf{z}\in\mathcal{T}_\varepsilon(t),\qquad\big(\nabla\chi_\pm\cdot\nabla d\big)(\mathbf{x},t)\ge\tfrac{1}{2\varepsilon}\quad\forall\ \mathbf{x}\in\Sigma(t).\tag{13}$$

For any $\mathbf{x}\in\Sigma(t)$, let $N(\mathbf{x},t)$ denote a straight segment of size $2D$ perpendicular to $\Sigma(t)$ at $\mathbf{x}$. For a given $\mathbf{y}\in\Sigma_\varepsilon(t)$, we have that $(\mathbf{y},t)\in\mathcal{T}=\{|d(\mathbf{z},t)|\le D\}$, whence $\mathbf{x}=\mathbf{s}(\mathbf{y},t)\in\Sigma(t)$ is uniquely defined and $\mathbf{y}\in N(\mathbf{x},t)$. Since $\chi_\pm(\mathbf{x},t)=\sigma\mathcal{O}(\varepsilon)$ for $\mathbf{x}\in\Sigma(t)$, (13) yields the existence of unique $\mathbf{y}_\pm\in N(\mathbf{x},t)$ such that $\chi_\pm(\mathbf{y}_\pm,t)=0$ and

$$|\mathbf{x}-\mathbf{y}_\pm|=d(\mathbf{y}_\pm,t)\le C\sigma(t)\varepsilon^2.\tag{14}$$

With the aid of (12), we conclude that all the zeros $\mathbf{y}$ of $\chi_\varepsilon(\cdot,t)$ on $N(\mathbf{x},t)$ must lie between $\mathbf{y}^-$ and $\mathbf{y}^+$, which, in turn, applies to the original $\mathbf{y}\in\Sigma_\varepsilon(t)$, because $\mathbf{y}\in N(\mathbf{x},t)$. Therefore, from (14) we readily get

$$\operatorname{dist}\big(\mathbf{x},\Sigma_\varepsilon(t)\big),\operatorname{dist}\big(\mathbf{y},\Sigma(t)\big)\le|\mathbf{x}-\mathbf{y}|\le\max_{\mp}|\mathbf{x}-\mathbf{y}^\mp|\le C\sigma(t)\varepsilon^2.$$

Optimality has been shown in [8]. The proof of Theorem 1 is thus complete. ∎

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