DOUBLE OBSTACLE FORMULATION WITH VARIABLE RELAXATION PARAMETER FOR SMOOTH GEOMETRIC FRONT EVOLUTIONS: LOCAL INTERFACE ERROR ESTIMATES*

R. H. NOCHETTO† M. PAOLINI‡ AND C. VERDI‡

Abstract. A singularly perturbed double obstacle problem is examined as a variational tool for the approximation of the geometric motion of fronts. The relaxation parameter is space-time dependent, thereby allowing the control of transition layer thickness and related interface pointwise accuracy. Optimal order interface error estimates of local character, namely depending on the relaxation parameter local magnitude, are derived for smooth evolutions. The proof is based on constructing suitable sub and supersolutions, which incorporate a number of shape corrections to the basic standing wave profile, and using a modified distance function to the front. Numerical simulations illustrate how such a local property can be exploited in dealing with large curvatures and ultimately in resolving singularities.

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1. Introduction. We study the geometric motion of a front $\Sigma(t) \subset \mathbb{R}^n$ that propagates in its normal direction with velocity $V = \kappa^m + g$; hereafter κ^m is the sum of the principal curvatures and $g = g(\mathbf{x}, t)$, $(\mathbf{x}, t) \in \mathbb{R}^n \times [0, T]$, is a given forcing term. In contrast to its simple classical formulation, such an evolution exhibits a complex structure because topological changes, such as breaking and merging, and extinction may occur. Geometric front motions can be defined in a generalized (viscosity) sense, even past singularities, via the so-called level set approach [7,10]. In addition to its intrinsic geometric interest, this problem has relevant applications in the theory of phase transitions in material sciences, flame propagation, crystal growth, etc.; we refer to [4] for references and applications.

Based on the Landau-Ginzburg theory of phase transitions, $\Sigma(t)$ can be recovered as a limit as $\varepsilon \downarrow 0$ of the zero level set $\Sigma_\varepsilon(t)$ of the solution u_ε to the singularly perturbed reaction-diffusion PDE

$$\varepsilon \partial_t (a u_\varepsilon) - \varepsilon \nabla \cdot (a \nabla u_\varepsilon) + \frac{1}{2\varepsilon a} \Psi'(u_\varepsilon) = \frac{c_0}{2} g \qquad \text{in } \mathbb{R}^n \times (0, +\infty). \tag{1.1}$$

Here $\Psi(s)$ is a double well potential, with wells of equal depth located at $s = \pm 1$, and $c_0 = \int_{-1}^{1} \sqrt{\Psi(s)} ds$ is a convenient scaling factor. The density function $a = a(\mathbf{x}, t)$ is space-time dependent and satisfies $0 < a_0 \le a(\mathbf{x}, t) < +\infty$. For $a = 1$, convergence of $\Sigma_\varepsilon(t)$ to $\Sigma(t)$ as $\varepsilon \downarrow 0$ has been proved in [5] for smooth evolutions, and in [2,9] even past

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†Department of Mathematics, University of Maryland, and Institute for Physical Science and Technology, College Park, MD 20742, USA.

‡Dipartimento di Matematica, Università di Milano, 20133 Milano, and Istituto di Analisi Numerica del CNR, 27100 Pavia, Italy.

singularities. Since formal asymptotics reveals that the limit of $\Sigma_\varepsilon(t)$ as $\varepsilon \downarrow 0$ is insensitive to the details of Ψ, the precise form of Ψ is just a matter of convenience [17]. Motivated by numerical considerations, we propose the following double obstacle potential:

$$\Psi(s) = \begin{cases} 1 - s^2 & \text{if } s \in [-1,1] \\ +\infty & \text{if } s \notin [-1,1]. \end{cases} \tag{1.2}$$

The virtue of this choice lies on the simple but crucial facts that $|u_\varepsilon(\mathbf{x},t)| \le 1$ irrespective of g and that $u_\varepsilon(\mathbf{x},t)$ attains the values ± 1 outside a narrow transition layer $\mathcal{T}_\varepsilon(t)$. Both formal asymptotics and numerics indicate that the local thickness of $\mathcal{T}_\varepsilon(t)$ is of order $\mathcal{O}(\varepsilon a(\mathbf{x},t))$, thereby justifying the use of a variable relaxation parameter $\varepsilon a(\mathbf{x},t)$ as a means of controling layer thickness and related interface pointwise accuracy. Since u_ε has to be computed only in $\mathcal{T}_\varepsilon(t)$, which is a relatively small region, the double obstacle formulation retains the local character of the geometric problem at hand while taking advantage of the variational structure of (1.1). These properties are exploited numerically by the *dynamic mesh algorithm* of [14], which is a finite element method that solely triangulates $\mathcal{T}_\varepsilon(t)$ according to a mesh density function proportional to $a(\mathbf{x},t)$, and then updates the resulting graded mesh to follow the layer motion. Singularity formation does not require any special care!

The function $a(\mathbf{x},t)$ should be designed so as to become progressively small in a space-time vicinity of singularities, thereby enhancing resolution. A number of numerical experiments illustrate the gains in computing time and local resolution obtained upon using a suitably designed variable relaxation parameter $\varepsilon a(\mathbf{x})$ for the motion by mean curvature ($g = 0$) [14,17]. Because of the parabolic nature of (1.1), $a(\mathbf{x},t)$ can in principle be adjusted as time evolves to meet the above requirement. We then expect that by the time a singularity develops, the value of $a(\mathbf{x},t)$ will already be the minimum a_0 and, therefore, the full strength of the method will be used to enhance the resolution of the singulatity and investigate its structure. These issues make the dynamic mesh algorithm quite an attractive and effective numerical tool, with a lot of potential to be explored in connection with phase transitions. In the present discussion, however, $a(\mathbf{x},t)$ is assumed to be given in advance and smooth. Its adaptive design and implementation deserves further analysis and experimentation.

In spite of numerical evidence [14,17], the local effect of a variable relaxation parameter still lacks of solid mathematical basis. The main purpose of this paper is to provide a rigorous justification to the crucial fact that, for smooth evolutions, the local magnitude of $\varepsilon a(\mathbf{x},t)$ dictates the local distance between $\Sigma(t)$ and $\Sigma_\varepsilon(t)$. We in fact prove the following *optimal local interface error estimate*:

$$\operatorname{dist}\big(\mathbf{x}, \Sigma_\varepsilon(t)\big) \;\; (\text{resp. } \Sigma(t)) \le C_* \varepsilon^2 a^2(\mathbf{x},t) \qquad \forall\, \mathbf{x} \in \Sigma(t) \;\; (\text{resp. } \Sigma_\varepsilon(t)), \tag{1.3}$$

where $C_* > 0$ is independent of ε, $\mathbf{x}$, t, and a_0. This estimate is shown under the numerically reasonable assumption that a number of derivatives of a can be bounded in terms of

a. Since singularity formation should be preceded by the condition $a(\mathbf{x},t) = a_0$, we realize that only small oscillations of a could be expected in the vicinity of singularities. The fact that the interface error in (1.3) is proportional to $\varepsilon^2 a^2(\mathbf{x},t)$ shows that several extra digits of accuracy can be gained when most needed. This result extends our previous investigations valid for $a(\mathbf{x},t) = 1$ [16,15], and leads to an optimal *global* interface error estimate without structural assumptions on a as well. The proof is based on explicitly constructing sub and supersolutions of (1.1) which incorporate a number of shape corrections to the basic standing wave $\gamma(x) = \sin x$. They, in turn, account for $g \neq 0$ and variations of a. An important tool, as in [16,15], is the use of a modified signed distance function to the front consisting of a shift and shape correction.

An outline of the paper follows. In §2 we set some further notation and state the local error estimate precisely (see Theorem 2.1). In §3 we prove a comparison lemma which extends those in [6,16] to variable a. In §4 we introduce several shape corrections to γ along with the candidate for subsolution γ_-. We omit any reference to formal asymptotics because it will not clarify the already technical discussion. The effect of shape corrections is apparent in §5, where a number of parameters of γ_- are chosen according to g, a, and their derivatives in order to compensate several terms and eventually give rise to subsolutions. We prove Theorem 2.1 in §6, and finally conclude in §7 with a numerical simulation that illustrates the practical significance of (1.3).

2. Setting and Results.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\Sigma(t) \subset \Omega$ for $t \in [0,T]$ be an oriented closed manifold of codimension 1, and set

$$\Sigma = \{(\mathbf{x},t) \in \Omega \times [0,T] : \mathbf{x} \in \Sigma(t)\}, \qquad O(t) = \text{Outside of } \Sigma(t), \quad I(t) = \text{Inside of } \Sigma(t).$$

For any $\mathbf{x} \in \Sigma(t)$, we denote by $\mathbf{n}(\mathbf{x},t)$ the inward unit vector normal to $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$. In addition, we indicate with $\kappa^m(\mathbf{x},t)$ the sum of the principal curvatures (positive if $I(t)$ is convex at $\mathbf{x} \in \Sigma(t)$) and with $\kappa^s(\mathbf{x},t)$ the sum of the squares of the principal curvatures of $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$. The signed distance $d(\cdot,t)$ is defined, for all $t \in [0,T]$, by

$$d(\mathbf{x},t) = \begin{cases} \operatorname{dist}\big(\mathbf{x},\Sigma(t)\big) & \text{if } \mathbf{x} \in O(t) \\ 0 & \text{if } \mathbf{x} \in \Sigma(t) \\ -\operatorname{dist}\big(\mathbf{x},\Sigma(t)\big) & \text{if } \mathbf{x} \in I(t). \end{cases}$$

We introduce a neighborhood $\mathcal{T}$ of Σ defined by $\mathcal{T} = \{(\mathbf{x},t) \in \mathbb{R}^n \times [0,T] : |d(\mathbf{x},t)| \leq D\}$ for some fixed $0 < D \ll 1$ so that $\mathcal{T} \subset \Omega \times [0,T]$. Throughout the paper we assume that

$$\partial^{i,j}_{\mathbf{x},t} d \in L^\infty(\mathcal{T}), \qquad j = 0, i \leq 4;\ j = 1, i \leq 2. \tag{2.1}$$

It follows that, for any $(\mathbf{x},t) \in \mathcal{T}$, there is a unique $\mathbf{s}(\mathbf{x},t) \in \Sigma(t)$ such that $\operatorname{dist}\big(\mathbf{s}(\mathbf{x},t),\mathbf{x}\big) =$

$|d(\mathbf{x},t)|$. For any function φ sufficiently regular, we introduce the notation

$$
\begin{aligned}
\overline{\varphi}(\mathbf{x},t) &= \varphi\big(\mathbf{s}(\mathbf{x},t),t\big) \\
\overline{\varphi}'(\mathbf{x},t) &= (\nabla\varphi)\big(\mathbf{s}(\mathbf{x},t),t\big)\cdot\nabla d(\mathbf{x},t) = \overline{(\nabla\varphi)}(\mathbf{x},t)\cdot\nabla d(\mathbf{x},t) \\
\overline{(\nabla\varphi)}'(\mathbf{x},t) &= (H\varphi)\big(\mathbf{s}(\mathbf{x},t),t\big)\nabla d(\mathbf{x},t) \\
\overline{\varphi}''(\mathbf{x},t) &= \big((H\varphi)\big(\mathbf{s}(\mathbf{x},t),t\big)\nabla d(\mathbf{x},t)\big)\cdot\nabla d(\mathbf{x},t) = \overline{(\nabla\varphi)}'(\mathbf{x},t)\cdot\nabla d(\mathbf{x},t)
\end{aligned}
\tag{2.2}
$$

for all $(\mathbf{x},t)\in\mathcal{T}$, where $H\varphi$ stands for the Hessian matrix of φ. Since $\nabla\overline{\varphi}(\mathbf{x},t)$ is tangent to the front $\Sigma(t)$ whereas $\nabla d(\mathbf{x},t) = -\overline{\mathbf{n}}(\mathbf{x},t)$ is normal [13], both at $\mathbf{s}(\mathbf{x},t)$, we obtain the crucial orthogonality property

$$
\nabla d(\mathbf{x},t)\cdot\nabla\overline{\varphi}(\mathbf{x},t) = 0 \qquad \forall\ (\mathbf{x},t)\in\mathcal{T}. \tag{2.3}
$$

In addition, since $\mathbf{x}-\mathbf{s}(\mathbf{x},t) = d(\mathbf{x},t)\nabla d(\mathbf{x},t)$ for all $(\mathbf{x},t)\in\mathcal{T}$, we have

$$
\varphi(\mathbf{x},t) = \overline{\varphi}(\mathbf{x},t) + d(\mathbf{x},t)\overline{\varphi}'(\mathbf{x},t) + \tfrac{1}{2}d^2(\mathbf{x},t)\overline{\varphi}''(\mathbf{x},t) + \mathcal{O}_{\|\partial^3_{\mathbf{x}}\varphi\|_{L^\infty(\mathcal{T})}}\big(d^3(\mathbf{x},t)\big). \tag{2.4}
$$

Hereafter, the notation $\zeta = \mathcal{O}_\theta(\xi)$ stands for $|\zeta|\le C\xi$, where $C>0$ is a constant independent of $0<\xi\le X\ll 1$ but dependent on θ, that may change at the various occurrences.

Let the forcing term g satisfy

$$
\|\partial^{i,j}_{\mathbf{x},t}g\|_{L^\infty(\Omega\times(0,T))}\le G<\infty, \qquad j=0, i\le 3;\ j=1, i\le 1. \tag{2.5}
$$

Assuming that $\Sigma(t)$ moves with normal velocity $V=\kappa^m+g$, it follows that

$$
\begin{aligned}
\partial_t d(\mathbf{x},t) &= \overline{\kappa^m}(\mathbf{x},t) + \overline{g}(\mathbf{x},t) \\
\Delta d(\mathbf{x},t) &= \overline{\kappa^m}(\mathbf{x},t) - d(\mathbf{x},t)\overline{\kappa^s}(\mathbf{x},t) + \mathcal{O}_{\|\kappa^s\|_{L^\infty(\Sigma)}}\big(d^2(\mathbf{x},t)\big)
\end{aligned}
\tag{2.6}
$$

for all $(\mathbf{x},t)\in\mathcal{T}$; see [8,17] and [12,§14.6;13], respectively. We also stress that (2.1) yields

$$
\|\overline{\kappa^s}\|_{W^{2,1}_\infty(\mathcal{T})} = K<\infty. \tag{2.7}
$$

We now deal with the variational approximation to $\Sigma(t)$ via a singularly perturbed double obstacle problem. In view of (1.2), the unique absolute minimizer of the functional $\mathcal{F}(\varphi) = \int_{\mathbb{R}}\big(|\varphi'(x)|^2 + \Psi(\varphi(x))\big)dx$, such that $\gamma(0)=0$, $|\gamma(x)|\le 1$ in $\mathbb{R}$, and $\lim_{x\to\pm\infty}\gamma(x)=\pm 1$, is given by [6,14,16,15,17]

$$
\gamma(x) = \begin{cases} -1 & \text{if } x < -\frac{\pi}{2} \\ \sin x & \text{if } x\in[-\frac{\pi}{2},\frac{\pi}{2}] \\ +1 & \text{if } x > \frac{\pi}{2}. \end{cases}
$$

Hence, $\gamma \in C^{1,1}(\mathbb{R})$ and solves the (elliptic) double obstacle problem $\gamma''(x)+\frac{1}{2}\Psi'(\gamma(x)) \ni 0$ in $\mathbb{R}$. In particular,

$$\gamma''(x)+\gamma(x)=0 \qquad \forall\, x\in(-\tfrac{\pi}{2},\tfrac{\pi}{2}). \tag{2.8}$$

Let the density function $a \in W^{4,2}_{\infty}(\Omega\times(0,T))$ satisfy $0<a_0\le a(\mathbf{x},t)\le A$ for all $(\mathbf{x},t)\in\Omega\times(0,T)$ together with

$$\Big|\frac{\partial^{i,j}_{\mathbf{x},t}a(\mathbf{x},t)}{a(\mathbf{x},t)}\Big|\le A \qquad j=0, i=1,2,3,4;\ j=1, i=0,1,2;\ j=2, i=0, \tag{2.9}$$

where A is a constant independent of a_0. This relative boundedness of derivatives of a guarantees that only small variations of a are allowed whenever a becomes small, and is quite acceptable for numerical purposes, as observed in §1. Let

$$c_0=\int_{\mathbb{R}}(\gamma'(x))^2dx=\int_{-1}^{1}\sqrt{\Psi(s)}ds=\tfrac{\pi}{2},$$

be a scaling factor and $u_\varepsilon\in W^{2,1}_2(\Omega\times(0,T))$ be the solution to the singularly perturbed $(0<\varepsilon\ll 1)$ parabolic double obstacle problem [11]:

$$\begin{aligned} \varepsilon\partial_t(au_\varepsilon)-\varepsilon\nabla\cdot(a\nabla u_\varepsilon)+\tfrac{1}{2\varepsilon a}\Psi'(u_\varepsilon) &\ni \tfrac{c_0}{2}g \qquad \text{in } \Omega\times(0,T)\\ u_\varepsilon(\mathbf{x},0)=u^0_\varepsilon(\mathbf{x}) &= \gamma\big(\tfrac{d(\mathbf{x},0)}{\varepsilon\overline{a}(\mathbf{x},0)}\big) \qquad \forall\,\mathbf{x}\in\Omega\\ u_\varepsilon &= 1 \qquad \text{on } \partial\Omega\times(0,T). \end{aligned} \tag{2.10}$$

Let $\Sigma_\varepsilon(t)=\{\mathbf{x}\in\Omega: u_\varepsilon(\mathbf{x},t)=0\}$ be the zero level set of u_ε. The purpose of this paper is to prove the following *local* sharp interface error estimate between $\Sigma(t)$ and $\Sigma_\varepsilon(t)$.

THEOREM 2.1. *Let d, g and a satisfy* (2.1), (2.5), *and* (2.9) *respectively. Then there exist $0<\mathcal{E}\ll 1$ and a constant C_*, both depending on T, G, K, and A but not on a_0, $\mathbf{x}$, and t, such that for all $0<\varepsilon\le\mathcal{E}$ the following optimal estimate holds:*

$$\operatorname{dist}\big(x,\Sigma_\varepsilon(t)\big)\ \ (\textit{resp. } \Sigma(t))\le C_*\varepsilon^2a^2(\mathbf{x},t) \quad \forall\, t\in[0,T],\ \mathbf{x}\in\Sigma(t)\ \ (\textit{resp. } \Sigma_\varepsilon(t)). \tag{2.11}$$

We stress the local character of (2.11), expressed through the dependence of C_* only on A and not on the minimum a_0 of the density function a: the estimate becomes more accurate where $a(\mathbf{x},t)$ is small.

3. Comparison Lemma. We now extend the Comparison Lemma of [6,16] to the case of variable a.

LEMMA 3.1. *Let $v \in W_2^{1,1}(\Omega \times (0,T))$, $v \le 1$, and $f \in L^2(\Omega \times (0,T))$ satisfy the auxiliary problem*

$$
\begin{aligned}
&\varepsilon\partial_t(av) - \varepsilon\nabla\cdot(a\nabla v) - f \le 0 \qquad \text{in } \mathcal{D}'(\Omega\times(0,T))\\
&v(\cdot,0) \le u_\varepsilon^0(\cdot) \qquad \text{in } \Omega \\
&v = 1 \qquad \text{on } \partial\Omega\times(0,T),
\end{aligned}
\tag{3.1}
$$

with $f \le \frac{1}{\varepsilon a}v + \frac{c_0}{2}g$ a.e. in $\{v > -1\}$. Then v is a subsolution of the double obstacle problem (2.10), that is,

$$
v \le u_\varepsilon \qquad \text{a.e. in } \Omega\times(0,T). \tag{3.2}
$$

Proof. For any $s \in \mathbb{R}$, set $[s]^+ = \max(s,0)$. Let S denote the sign graph, namely,

$$
S(s) = -1 \quad \text{if } s<0, \qquad S(0) = [-1,1], \qquad S(s) = 1 \quad \text{if } s > 0,
$$

and $H(s) = \frac{1}{2}(S(s)+1)$ indicate the Heaviside graph; note that $\psi = \frac{1}{2}\Psi' = S^{-1} - I$. We now introduce a convenient regularization of ψ that, for any $\rho > 0$, is defined by

$$
\psi_\rho = S_\rho^{-1} - I \qquad \text{where} \qquad S_\rho(s) = 1 + \rho[s]^+ - 2\tanh\left(\tfrac{[-s]^+}{\rho}\right).
$$

Let $u_\varepsilon^\rho \in W_2^{2,1}(\Omega\times(0,T))$ be a smooth approximation of u_ε which satisfies the following (classical) initial boundary value problem [11]:

$$
\begin{aligned}
&\varepsilon\partial_t(au_\varepsilon^\rho) - \varepsilon\nabla\cdot(a\nabla u_\varepsilon^\rho) + \tfrac{1}{\varepsilon a}\psi_\rho(u_\varepsilon^\rho) = \tfrac{c_0}{2}g \qquad \text{in } \Omega\times(0,T)\\
&u_\varepsilon^\rho(\cdot,0) = u_\varepsilon^0(\cdot) + \rho[-u_\varepsilon^0(\cdot)]^+ \qquad \text{in } \Omega\\
&u_\varepsilon^\rho = 1 \qquad \text{on } \partial\Omega\times(0,T);
\end{aligned}
\tag{3.3}
$$

thus $u_\varepsilon^\rho > -1$ in $\Omega\times(0,T)$. Note that the modified initial datum is compatible with the nonlinearity ψ_ρ and converges to u_ε^0 as $\rho \downarrow 0$. Set $e_\rho = v - u_\varepsilon^\rho$ and take the difference between the PDEs in (3.1) and (3.3) to obtain

$$
\varepsilon\partial_t(ae_\rho) - \varepsilon\nabla\cdot(a\nabla e_\rho) \le f + \tfrac{1}{\varepsilon a}\psi_\rho(u_\varepsilon^\rho) - \tfrac{c_0}{2}g \qquad \text{in } \mathcal{D}'(\Omega\times(0,T)). \tag{3.4}
$$

For any $\sigma > 0$, let $H_\sigma(s) = \tanh\left(\frac{[s]^+}{\sigma}\right)$ be a regularization of H. Multiply (3.4) by $0 \le H_\sigma(e_\rho) \in H^1(\Omega)$ and integrate on $\Omega\times(0,t)$. We get

$$
\begin{aligned}
\varepsilon\int_\Omega\int_0^t \partial_t(ae_\rho)H_\sigma(e_\rho) + \varepsilon\int_0^t\int_\Omega a|\nabla e_\rho|^2 H'_\sigma(e_\rho) \le \int_0^t\int_\Omega \tfrac{1}{\varepsilon a}e_\rho H_\sigma(e_\rho)\\
+ \int_0^t\int_\Omega \left(f - \tfrac{1}{\varepsilon a}v - \tfrac{c_0}{2}g\right)H_\sigma(e_\rho) + \int_0^t\int_\Omega \tfrac{1}{\varepsilon a}S_\rho^{-1}(u_\varepsilon^\rho)H_\sigma(e_\rho).
\end{aligned}
$$

We see that $H_\sigma(e_\rho) > 0$ yields $-1 < u_\varepsilon^\rho < v \le 1$, whence $f - \frac{1}{\varepsilon a} v - \frac{c_0}{2} g \le 0$ by assumption and $S_\rho^{-1}(u_\varepsilon^\rho) \le 0$ by construction. Therefore

$$\int_\Omega \int_0^t \partial_t(ae_\rho) H_\sigma(e_\rho) \le \tfrac{1}{\varepsilon^2 a_0} \int_0^t \int_\Omega [e_\rho]^+ \le \tfrac{1}{\varepsilon^2 a_0^2} \int_0^t \int_\Omega [ae_\rho]^+.$$

Since $\lim_{\sigma \downarrow 0} H_\sigma(e_\rho) = H(e_\rho) = H(ae_\rho)$, we can take $\sigma \downarrow 0$ and use the Dominated Convergence Theorem to get

$$\int_\Omega [ae_\rho]^+(\cdot, t) = \int_\Omega \int_0^t \partial_t [ae_\rho]^+(\cdot, \tau) d\tau \le \tfrac{1}{\varepsilon^2 a_0^2} \int_0^t \int_\Omega [ae_\rho]^+(\cdot, \tau) d\tau,$$

because $[ae_\rho]^+(\cdot, 0) = 0$. Finally we use Gronwall's Lemma to conclude $\int_\Omega [ae_\rho]^+(\cdot, t) = 0$ for all $t \in [0, T]$. The assertion (3.2) then follows easily upon letting $\rho \downarrow 0$, and using the fact that $u_\varepsilon^\rho \to u_\varepsilon$ a.e. in $\Omega \times (0, T)$. □

4. Shape Corrections. Let us consider the linear operator $\mathcal{L}\varphi(x) = \varphi''(x) + \varphi(x)$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, subject to vanishing Dirichlet boundary conditions at $x = \mp\frac{\pi}{2}$. For the problem $\mathcal{L}\varphi = q$ to be solvable, the compatibility condition $\int_{-\pi/2}^{\pi/2} q(x)\gamma'(x)dx = 0$ must be enforced (Fredholm alternative), because $\operatorname{span}\{\gamma'\}$ is the kernel of $\mathcal{L}$. So we set

$$\mathcal{P} = \left\{ \varphi \in C^{0,1}(\mathbb{R}) : \textstyle\int_{-\pi/2}^{\pi/2} \varphi(x)\gamma'(x)dx = 0 \right\},$$

and let $\chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$ be the characteristic function of the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, we introduce the following auxiliary functions $\eta_1, \eta_2, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \in C^{0,1}(\mathbb{R})$, defined by

$$\begin{aligned}
\eta_1(x) &= \tfrac{1}{2}\big(x\gamma(x) - c_0(1 - \gamma'(x))\big)\chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(x) \\
\eta_2(x) &= \tfrac{1}{2} x^2 \gamma'(x) \\
\xi_1(x) &= -\tfrac{1}{2} x\gamma'(x) \\
\xi_2(x) &= \tfrac{1}{16}\big(4x\gamma'(x) + (4x^2 - \pi^2)\gamma(x)\big)\chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(x) \\
\xi_3(x) &= \tfrac{1}{48}\big(3(4x^2 - \pi^2)\gamma(x) + 4x(3 - 2x^2)\gamma'(x)\big)\chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(x) \\
\xi_4(x) &= \tfrac{1}{32}\Big(\big(\pi^2(3 - \tfrac{\pi^2}{4}) + 4x^2(x^2 - 3)\big)\gamma(x) + 4x(2x^2 - 3)\gamma'(x)\Big)\chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(x) \\
\xi_5(x) &= \big(x - \tfrac{\pi}{2}\gamma(x)\big)\chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(x),
\end{aligned}$$

which are solution of the following problems for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\begin{aligned}
\mathcal{L}\eta_1(x) &= q_1(x) = \gamma'(x) - \tfrac{c_0}{2} \in \mathcal{P} \\
\mathcal{L}\eta_2(x) &= q_2(x) = \gamma'(x) - 2x\gamma(x) \in \mathcal{P} \\
\mathcal{L}\xi_1(x) &= q_3(x) = \gamma(x) \in \mathcal{P} \\
\mathcal{L}\xi_2(x) &= q_4(x) = x\gamma'(x) \in \mathcal{P} \\
\mathcal{L}\xi_3(x) &= q_5(x) = x^2\gamma(x) \in \mathcal{P} \\
\mathcal{L}\xi_4(x) &= q_6(x) = x^3\gamma'(x) \in \mathcal{P} \\
\mathcal{L}\xi_5(x) &= q_7(x) = x \in \mathcal{P}.
\end{aligned} \tag{4.1}$$

Despite these functions are suggested by formal asymptotics, we do not insist in such a connection. Their effect though will become clear in §5. It is important to realize that they are not better than Lipschitz continuous:

$$[\![\eta_1']\!](\mp\tfrac{\pi}{2}) = \tfrac{1}{2}(c_0-1), \quad [\![\eta_2']\!](\mp\tfrac{\pi}{2}) = \tfrac{\pi^2}{8}, \quad [\![\xi_1']\!](\mp\tfrac{\pi}{2}) = \pm\tfrac{\pi}{4}, \quad [\![\xi_2']\!](\mp\tfrac{\pi}{2}) = \pm\tfrac{\pi}{8},$$
$$[\![\xi_3']\!](\mp\tfrac{\pi}{2}) = \pm\tfrac{\pi}{48}(6+\pi^2), \quad [\![\xi_4']\!](\mp\tfrac{\pi}{2}) = \pm\tfrac{\pi}{32}(\pi^2-6), \quad [\![\xi_5']\!](\mp\tfrac{\pi}{2}) = \pm 1.$$

We also need the auxiliary functions $\widehat{\eta}_i, \xi_j^- \in C^{0,1}(\mathbb{R})$, which can be interpreted as suitable shifts of η_i, ξ_j. They are defined, for all $x \in \mathbb{R}$, by

$$\widehat{\eta}_i(x) = \eta_i(x) + \eta_i'(\tfrac{\pi}{2}^-)\gamma'(x), \qquad \xi_j^-(x) = \xi_j(x) + \xi_j'(\tfrac{\pi}{2}^-)\gamma'(x),$$

and still solve (4.1). These further corrections amount to eliminating discontinuities of η_i' at $x = \mp\frac{\pi}{2}$ and ξ_j' at $x = \frac{\pi}{2}$, because $\widehat{\eta}_i'(\mp\frac{\pi}{2}) = 0$ and $(\xi_j^-)'(\frac{\pi}{2}) = 0$ but $[\![(\xi_j^-)']\!](-\frac{\pi}{2}) = 2[\![\xi_j']\!](-\frac{\pi}{2}) \neq 0$. Finally, we will also need the relations

$$\begin{aligned} \widehat{\eta}_1'(x) = \tfrac{1}{2}q_4(x), \qquad \widehat{\eta}_2'(x) &= \tfrac{\pi^2}{8}q_3(x) + q_4(x) - \tfrac{1}{2}q_5(x), \\ x\big(\widehat{\eta}_1''(x) - \widehat{\eta}_1(x)\big) &= -q_5(x) + \tfrac{c_0}{2}q_7(x), \\ x\big(\widehat{\eta}_2''(x) - \widehat{\eta}_2(x)\big) &= (1+\tfrac{\pi^2}{4})q_4(x) - 2q_5(x) - q_6(x). \end{aligned} \tag{4.2}$$

We then introduce the candidate for subsolution γ_- that incorporates several shape corrections to γ. Let $\gamma_-(\,\cdot\,;\varepsilon,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5) : \mathbb{R} \to \mathbb{R}$ be defined, for any $\varepsilon > 0$ and $\alpha_i, \beta_j \in \mathbb{R}$, by

$$\gamma_-(x;\varepsilon,\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5) = \gamma(x) + \varepsilon\textstyle\sum_{i=1}^{2}\alpha_i\widehat{\eta}_i(x) + \varepsilon^2\sum_{j=1}^{5}\beta_j\xi_j^-(x), \tag{4.3}$$

and note that $\gamma_-(x) = 1$ (-1) for all $x \geq \frac{\pi}{2}$ $(x \leq -\frac{\pi}{2})$, $\gamma_-'(\frac{\pi}{2}) = 0$, and, for $\varepsilon \leq \mathcal{E} \ll 1$ (depending on the α_i's and β_j's), $\gamma_-''(\frac{\pi}{2}^-) \leq -\frac{1}{2}$, $\gamma_-''(-\frac{\pi}{2}^+) \geq \frac{1}{2}$. Hence, γ_- is strictly convex in a right neighborhood of $-\frac{\pi}{2}$ uniformly in $\varepsilon \leq \mathcal{E}$, which in turn shows the existence of at most one point $x_\varepsilon \in (-\frac{\pi}{2}, 0)$ such that $\gamma_-(x_\varepsilon) = -1$. If such point does exist, then $\gamma_-'(x_\varepsilon) > 0$. Otherwise we set $x_\varepsilon = -\frac{\pi}{2}$ and point out that in any event $\gamma_-(x_\varepsilon) = -1$, $\gamma_-'(x_\varepsilon^+) \geq 0$, and $\gamma_-'(x) > 0$, $|\gamma_-(x)| < 1$ for all $x \in (x_\varepsilon, \frac{\pi}{2})$. Since no confusion is possible, we henceforth denote by γ_- the function $\max(-1, \gamma_-)$.

Finally, note that $\gamma(x)$, $\gamma'(x)$, $\gamma''(x)$, $\widehat{\eta}_i(x)$, $\widehat{\eta}_i'(x)$, $\widehat{\eta}_i''(x)$, $\xi_j^-(x)$, $(\xi_j^-)'(x)$, $(\xi_j^-)''(x) = \mathcal{O}(1)$ for all $x \in \mathbb{R}$, whence $\gamma_-(x), \gamma_-'(x), \gamma_-''(x) = \mathcal{O}(1)$ for $\varepsilon \leq \mathcal{E}$.

5. Subsolution. In this section we construct a subsolution for the double obstacle problem (2.10), and just indicate how to get a supersolution. In doing so there are two crucial issues involved. The first one is to mimic the shape of u_ε as well as possible. The second ingredient is the use of a modified distance function, which combines a shift with a shape correction. This idea was already exploited in [16,15] for $a = 1$.

Hereafter, we will assume that $0 < \varepsilon \le \mathcal{E} \ll 1$, where we will use the symbol $\mathcal{E}$ to indicate an upper bound of ε that will be chosen sufficiently small depending on T, G, K, and A but not on a_0. The constants G, K, and A are defined in (2.5), (2.7), and (2.9), respectively. Upon restricting $\varepsilon \le \mathcal{E}$ beforehand, we will sistematically resort to the replacement

$$\zeta = \mathcal{O}(\varepsilon^\alpha) + \mathcal{O}_{G,K,A,T}(\varepsilon^{\alpha+1}) \qquad \to \qquad \zeta = \mathcal{O}(\varepsilon^\alpha), \tag{5.1}$$

thereby eliminating the explicit dependence on G, K, A, and T.

We start out by introducing the *modified distance function*. For all $(\mathbf{x}, t) \in \mathcal{T}$, set

$$\overline{\varepsilon}(\mathbf{x}, t) = \varepsilon \overline{a}(\mathbf{x}, t) = \varepsilon a\big(\mathbf{s}(\mathbf{x}, t), t\big) \tag{5.2}$$

and define

$$d_\varepsilon^-(\mathbf{x}, t) = d(\mathbf{x}, t) - \big(\sigma_1(t) d^2(\mathbf{x}, t) + \sigma_2(t)\big)\overline{\varepsilon}^2(\mathbf{x}, t), \tag{5.3}$$

where $\sigma_i(t) = C_i \mathrm{e}^{2Ht}$ for $i = 1, 2$; $H > 0$ and $C_1, C_2 \ge 1$ are suitable constants, to be determined later, depending on G, K, and A, but independent of a_0 and ε. We stress that the effect of the shape correction $d^2(\mathbf{x}, t)$ cannot be replaced by a simpler vertical shift in the definition of subsolution, which is in striking contrast with the case of regular potential [3,9]. Let

$$y_\varepsilon^-(\mathbf{x}, t) = \tfrac{d_\varepsilon^-(\mathbf{x}, t)}{\overline{\varepsilon}(\mathbf{x}, t)} \tag{5.4}$$

be the *stretched variable*. Since no confusion is possible, we will use the notation $d_\varepsilon = d_\varepsilon^-$ and $y = y_\varepsilon^-$. Note that $|y(\mathbf{x}, t)| \le \frac{\pi}{2}$ implies $|d(\mathbf{x}, t)| \le D$, i.e., $(\mathbf{x}, t) \in \mathcal{T}$, and

$$d = \overline{\varepsilon} y + \sigma_2 \overline{\varepsilon}^2 + \mathcal{O}_{A, \sigma_i(T)}(\overline{\varepsilon}^4) = \mathcal{O}(\overline{\varepsilon}). \tag{5.5}$$

Our first task is to write a number of asymptotic expansions for quantities of interest, starting with d_ε. In view of (2.3), (2.6), (5.4), and (5.5), we have

$$\begin{gathered}
\nabla d_\varepsilon = (1 - 2\sigma_1 \overline{\varepsilon}^3 y)\nabla d - 2\sigma_2 \overline{\varepsilon}^2 \tfrac{\nabla \overline{a}}{\overline{a}} + \mathcal{O}_{A, \sigma_i(T)}(\overline{\varepsilon}^4) \\
|\nabla d_\varepsilon|^2 = 1 - 4\sigma_1 \overline{\varepsilon}^3 y + \mathcal{O}_{A, \sigma_i(T)}(\overline{\varepsilon}^4) \\
\partial_t d_\varepsilon = \overline{\kappa^m} + \overline{g} - 2H\sigma_2 \overline{\varepsilon}^2 - 2\sigma_2 \overline{\varepsilon}^2 \tfrac{\partial_t \overline{a}}{\overline{a}} + \mathcal{O}_{A, \sigma_i(T)}(\overline{\varepsilon}^3) \\
\Delta d_\varepsilon = \overline{\kappa^m} - \overline{\varepsilon} y \overline{\kappa^s} - 2\sigma_1 \overline{\varepsilon}^2 - \sigma_2 \overline{\kappa^s} \overline{\varepsilon}^2 - 2\sigma_2 \overline{\varepsilon}^2 \big(\tfrac{\Delta \overline{a}}{\overline{a}} + \tfrac{|\nabla \overline{a}|^2}{\overline{a}^2}\big) + \mathcal{O}_{\|\kappa^s\|_{L^\infty(\Sigma)}}(\overline{\varepsilon}^2),
\end{gathered} \tag{5.6}$$

which lead to the following modifications of (2.3) and (2.4) in terms of d_ε and y:

$$\nabla d_\varepsilon \cdot \nabla \overline{\varphi} = -2\sigma_2 \overline{\varepsilon}^2 \tfrac{\nabla \overline{a}}{\overline{a}} \cdot \nabla \overline{\varphi} + |\nabla \overline{\varphi}| \mathcal{O}_{A, \sigma_i(T)}(\overline{\varepsilon}^4) \tag{5.7}$$

$$\varphi = \overline{\varphi} + \overline{\varepsilon} y \overline{\varphi}' + \tfrac{1}{2}\overline{\varepsilon}^2 y^2 \overline{\varphi}'' + \sigma_2 \overline{\varepsilon}^2 (\overline{\varphi}' + \overline{\varepsilon} y \overline{\varphi}'') + \mathcal{O}_{\|\partial_{\mathbf{x}}^3 \varphi\|_{L^\infty(\mathcal{T})}}(\overline{\varepsilon}^3), \tag{5.8}$$

together with

$$\frac{1}{\varphi} = \frac{1}{\overline{\varphi}} - \overline{\varepsilon} y \frac{\overline{\varphi}'}{\overline{\varphi}^2} + \overline{\varepsilon}^2 y^2 \big(\frac{(\overline{\varphi}')^2}{\overline{\varphi}^3} - \frac{1}{2} \frac{\overline{\varphi}''}{\overline{\varphi}^2} \big) \tag{5.9}$$
$$- \sigma_2 \overline{\varepsilon}^2 \Big(\frac{\overline{\varphi}'}{\overline{\varphi}^2} - 2\overline{\varepsilon} y \big(\frac{(\overline{\varphi}')^2}{\overline{\varphi}^3} - \frac{1}{2} \frac{\overline{\varphi}''}{\overline{\varphi}^2} \big) \Big) + \mathcal{O}_{\left\| \partial^3_{\mathbf{x}} \frac{1}{\varphi} \right\|_{L^\infty(\mathcal{T})}} (\overline{\varepsilon}^3).$$

We also have the following expansions, which result from (5.4), (5.6), and (5.7),

$$\begin{aligned}
\nabla y &= \tfrac{1}{\varepsilon} \nabla d_\varepsilon - y \tfrac{\nabla \overline{a}}{\overline{a}} \\
|\nabla y|^2 &= \tfrac{1}{\varepsilon^2} + y^2 \tfrac{|\nabla \overline{a}|^2}{\overline{a}^2} - 4\sigma_1 \overline{\varepsilon} y + 4\sigma_2 \overline{\varepsilon} y \tfrac{|\nabla \overline{a}|^2}{\overline{a}^2} + \mathcal{O}_{A,\sigma_i(T)}(\overline{\varepsilon}^2) \\
\partial_t y &= \tfrac{1}{\varepsilon} (\overline{\kappa^m} + \overline{g}) - y \tfrac{\partial_t \overline{a}}{\overline{a}} - 2H\sigma_2 \overline{\varepsilon} - 2\sigma_2 \overline{\varepsilon} \tfrac{\partial_t \overline{a}}{\overline{a}} + \mathcal{O}_{A,\sigma_i(T)}(\overline{\varepsilon}^2) \\
\Delta y &= \tfrac{1}{\varepsilon} \overline{\kappa^m} - y \overline{\kappa^s} - y \big(\tfrac{\Delta \overline{a}}{\overline{a}} - 2 \tfrac{|\nabla \overline{a}|^2}{\overline{a}^2} \big) - 2\sigma_1 \overline{\varepsilon} \\
&\quad - \sigma_2 \overline{\kappa^s} \overline{\varepsilon} - 2\sigma_2 \overline{\varepsilon} \big(\tfrac{\Delta \overline{a}}{\overline{a}} - \tfrac{|\nabla \overline{a}|^2}{\overline{a}^2} \big) + \mathcal{O}_{\|\kappa^s\|_{L^\infty(\Sigma)}}(\overline{\varepsilon}),
\end{aligned} \tag{5.10}$$

whence, from (2.3) and (2.2), we get, respectively,

$$\nabla y \cdot \nabla \overline{\varphi} = -y \frac{\nabla \overline{a}}{\overline{a}} \cdot \nabla \overline{\varphi} + \mathcal{O}_{A,\sigma_i(T)}(\overline{\varepsilon}), \tag{5.11}$$

and

$$\begin{aligned}
\nabla y \cdot \overline{(\nabla \varphi)} &= \tfrac{1}{\varepsilon} \overline{\varphi}' - y \tfrac{\nabla \overline{a}}{\overline{a}} \cdot \overline{(\nabla \varphi)} - 2\sigma_2 \overline{\varepsilon} \tfrac{\nabla \overline{a}}{\overline{a}} \cdot \overline{(\nabla \varphi)} + \mathcal{O}_{A,\sigma_i(T)}(\overline{\varepsilon}^2) \\
\nabla y \cdot \overline{(\nabla \varphi)}' &= \tfrac{1}{\varepsilon} \overline{\varphi}'' + \mathcal{O}_A(1).
\end{aligned} \tag{5.12}$$

We are now ready to define the subsolution for (2.10). To this end, we set

$$\begin{aligned}
\overline{\alpha}_1(\mathbf{x},t) &= \overline{g} \\
\overline{\alpha}_2(\mathbf{x},t) &= -\tfrac{\overline{a}'}{\overline{a}} \\
\overline{\beta}_1(\mathbf{x},t) &= \tfrac{\overline{(\partial_t a)}}{\overline{a}} - \tfrac{\pi^2}{8} \tfrac{\overline{a}'}{\overline{a}} (\overline{g} - \tfrac{\overline{a}'}{\overline{a}}) \\
\overline{\beta}_2(\mathbf{x},t) &= \overline{\kappa^s} + \tfrac{1}{2} \overline{g} (\overline{g} - \tfrac{\overline{a}'}{\overline{a}}) + \tfrac{\nabla \overline{a} \cdot \overline{(\nabla a)}}{\overline{a}^2} - \tfrac{(\partial_t - \Delta) \overline{a}}{\overline{a}} - 2 \tfrac{|\nabla \overline{a}|^2}{\overline{a}^2} - \tfrac{\overline{a}''}{\overline{a}} + (2 + \tfrac{\pi^2}{4}) \tfrac{(\overline{a}')^2}{\overline{a}^2} \\
\overline{\beta}_3(\mathbf{x},t) &= \tfrac{\overline{a}''}{\overline{a}} - \tfrac{7}{2} \tfrac{(\overline{a}')^2}{\overline{a}^2} + \tfrac{|\nabla \overline{a}|^2}{\overline{a}^2} + \tfrac{3}{2} \overline{g} \tfrac{\overline{a}'}{\overline{a}} \\
\overline{\beta}_4(\mathbf{x},t) &= -\tfrac{(\overline{a}')^2}{\overline{a}^2} \\
\overline{\beta}_5(\mathbf{x},t) &= -\tfrac{c_0}{2} \big(\tfrac{\overline{a}'}{\overline{a}} \overline{g} + \overline{g}' \big)
\end{aligned} \tag{5.13}$$

for all $(\mathbf{x},t) \in \mathcal{T}$. It is easy to check that

$$\|\overline{\alpha}_i\|_{W^{2,1}_\infty(\mathcal{T})} = \mathcal{O}_{A,G}(1), \qquad \|\overline{\beta}_j\|_{W^{2,1}_\infty(\mathcal{T})} = \mathcal{O}_{A,K,G}(1). \tag{5.14}$$

Having defined γ_- by (4.3) and truncation with -1, and recalling that $|y(\mathbf{x},t)| \le \frac{\pi}{2}$ implies $(\mathbf{x},t) \in \mathcal{T}$, the subsolution $v_\varepsilon = v_\varepsilon^-$ is given by

$$v_\varepsilon(\mathbf{x},t) = \gamma_- \big(y(\mathbf{x},t); \overline{\varepsilon}(\mathbf{x},t), \overline{\alpha}_1, \overline{\alpha}_2, \overline{\beta}_1 + 2\tfrac{\overline{a}'}{\overline{a}} \sigma_2, \overline{\beta}_2, \overline{\beta}_3, \overline{\beta}_4, \overline{\beta}_5 \big) \tag{5.15}$$

for all $(\mathbf{x},t) \in \mathcal{T}$, and is extended by ± 1 outside of $\mathcal{T}$ according to whether $\mathbf{x} \in O(t)$ or $\mathbf{x} \in I(t)$. Therefore, we get $v_\varepsilon \in W^{1,1}_\infty(\Omega \times (0,T))$ and $|v_\varepsilon| \le 1$ in $\Omega \times [0,T]$, together with

$$v_\varepsilon = \gamma(y) + \overline{\varepsilon}\textstyle\sum_{i=1}^2 \overline{\alpha}_i \widehat{\eta}_i(y) + 2\frac{\overline{a}'}{\overline{a}}\sigma_2 \overline{\varepsilon}^2 \xi_1^-(y) + \mathcal{O}_{A,K,G}(\overline{\varepsilon}^2) = \gamma(y) + \mathcal{O}_{A,G}(\overline{\varepsilon}) \tag{5.16}$$

in $\mathcal{T}$. It is convenient now to introduce the transition region $\mathcal{T}_\varepsilon(t) = \mathcal{T}_\varepsilon(t)^-$, namely

$$\mathcal{T}_\varepsilon(t) = \{\mathbf{x} \in \mathbb{R}^n : |v_\varepsilon(\mathbf{x},t)| < 1\} \qquad \forall\, t \in [0,T],$$

and note that $|y(\mathbf{x},t)| < \frac{\pi}{2}$ for $\mathbf{x} \in \mathcal{T}_\varepsilon(t)$.

We now intend to prove that v_ε is in fact a subsolution for the double obstacle problem (2.10). We first observe that

$$v_\varepsilon = 1 \qquad \text{on } \partial\Omega \times (0,T). \tag{5.17}$$

Then we check, for $t = 0$, that

$$v_\varepsilon(\mathbf{x},0) \le u^0_\varepsilon(\mathbf{x}) \qquad \forall\, \mathbf{x} \in \Omega. \tag{5.18}$$

Set $z(\mathbf{x}) = \frac{d(\mathbf{x},0)}{\overline{\varepsilon}(\mathbf{x},0)}$ and note that for $\mathbf{x} \in \mathcal{T}_\varepsilon(0)$, $y = z - C_2\overline{\varepsilon} + \mathcal{O}_{A,\sigma_i(T)}(\overline{\varepsilon}^3) < z - \frac{1}{2}C_2\overline{\varepsilon}$. Since we only have to consider $-\frac{\pi}{2} \le x_\varepsilon(\mathbf{x},0) < y < z < \frac{\pi}{2}$, we use the inequality [16]

$$\gamma(z) - \gamma(y) > \tfrac{1}{2\pi} C_2 \overline{\varepsilon}\Big(\tfrac{1}{2}C_2\overline{\varepsilon} + \min\left(\tfrac{\pi}{2} + y, \tfrac{\pi}{2} - z\right)\Big)$$

to conclude that, for C_2 sufficiently large but independent of ε,

$$\begin{aligned} v_\varepsilon(\mathbf{x},0) = \gamma_-(y) < \gamma(z) &- \overline{\varepsilon}\Big(\tfrac{1}{8\pi}C_2^2\overline{\varepsilon} + \tfrac{1}{2\pi}C_2 \min\left(\tfrac{\pi}{2}+y, \tfrac{\pi}{2}-z\right) - \textstyle\sum_{i=1}^2 \overline{\alpha}_i\widehat{\eta}_i(y)\Big) \\ &- \overline{\varepsilon}^2\Big(\tfrac{1}{8\pi}C_2^2 - \textstyle\sum_{j=1}^5 \overline{\beta}_j \xi_j^-(y) - 2\frac{\overline{a}'}{\overline{a}} C_2 \xi_1^-(y)\Big) \le \gamma(z) = u^0_\varepsilon(\mathbf{x}). \end{aligned}$$

We point out that the last inequality results from the property $\widehat{\eta}_i(y) = \mathcal{O}(\overline{\varepsilon})$ for $z = \frac{\pi}{2}$.

To demonstrate the first inequality in (3.1) we would like to integrate $\nabla \cdot (a\nabla v_\varepsilon)$ by parts. This would involve regularity of $\partial\mathcal{T}_\varepsilon(t)$, which is not a priori guaranteed. To circumvent such a difficulty, we introduce a set $\mathcal{T}^\rho_\varepsilon(t)$ slightly smaller than $\mathcal{T}_\varepsilon(t)$, namely,

$$\mathcal{T}^\rho_\varepsilon(t) = \{\mathbf{x} \in \mathcal{T}_\varepsilon(t) : -1 + \rho < v_\varepsilon(\mathbf{x},t) < 1\},$$

along with its boundaries

$$\partial^-\mathcal{T}^\rho_\varepsilon(t) = \{x \in \partial\mathcal{T}_\varepsilon(t) : v_\varepsilon(\mathbf{x},t) = -1 + \rho\} \qquad \partial^+\mathcal{T}^\rho_\varepsilon(t) = \partial\mathcal{T}_\varepsilon(t) \backslash \partial^-\mathcal{T}^\rho_\varepsilon(t),$$

where $0 < \rho \ll \varepsilon$ will tend to 0. Since $\widehat{\eta}_i(\frac{\pi}{2}) = \xi_j^-(\frac{\pi}{2}) = 0$, it turns out that $\partial^+\mathcal{T}_\varepsilon^\rho(t)$ is also the $\frac{\pi}{2}$–level set of $y(\cdot,t)$ which, in view of (5.10), is a smooth function with nonvanishing gradient. The Implicit Function Theorem yields the existence of an outward unit vector ν_ε^ρ (independent of ρ) normal to $\partial^+\mathcal{T}_\varepsilon^\rho(t)$. Moreover, the expression

$$(5.19)\quad \nabla v_\varepsilon = \gamma_-'(y)\nabla y + \varepsilon\textstyle\sum_{i=1}^2 \widehat{\eta}_i(y)\nabla(\overline{a\alpha_i}) + 2\varepsilon^2\xi_1^-(y)\nabla(\overline{aa}') + \varepsilon^2\sum_{j=1}^5 \xi_j^-(y)\nabla(\overline{a}^2\overline{\beta}_j)$$

in $\mathcal{T}_\varepsilon(t)$ shows that $\nabla v_\varepsilon = 0$ on $\partial^+\mathcal{T}_\varepsilon^\rho(t)$, because $\gamma_-'(\frac{\pi}{2}) = \widehat{\eta}_i(\frac{\pi}{2}) = \xi_j^-(\frac{\pi}{2}) = 0$. On the other hand, recalling that $\gamma_-(x_\varepsilon) = -1$, if $x_\varepsilon^\rho > x_\varepsilon$ is defined by $\gamma_-(x_\varepsilon^\rho) = -1+\rho$, then we infer that $\gamma_-'(x_\varepsilon^\rho) \geq C(x_\varepsilon^\rho - x_\varepsilon)$ with $C > 0$ independent of ε. This is certainly a consequence of the strict convexity of γ_- in a right neighborhood of x_ε. With the aid of (5.19), we deduce

$$\left|\nabla v_\varepsilon(\mathbf{x},t)\right| \geq \tfrac{1}{\varepsilon}C\big(x_\varepsilon^\rho(\mathbf{x},t) - x_\varepsilon(\mathbf{x},t)\big)\big(1-\mathcal{O}(\varepsilon)\big) \neq 0 \qquad \forall\, \mathbf{x} \in \partial^-\mathcal{T}_\varepsilon^\rho(t).$$

The Implicit Function Theorem applies again and shows the existence of a unit outward vector ν_ε^ρ orthogonal to $\partial^-\mathcal{T}_\varepsilon^\rho(t)$. We then realize that $\nabla v_\varepsilon \cdot \nu_\varepsilon^\rho = -|\nabla v_\varepsilon| < 0$ on $\partial^-\mathcal{T}_\varepsilon^\rho(t)$, because $\partial^-\mathcal{T}_\varepsilon^\rho(t)$ is the $(-1+\rho)$–level set of v_ε and $v_\varepsilon > -1+\rho$ in $\mathcal{T}_\varepsilon^\rho(t)$. Set

$$(5.20)\qquad f(\mathbf{x},t) = \begin{cases} \frac{1}{\varepsilon a}v_\varepsilon(\mathbf{x},t) + \frac{c_0}{2}g(\mathbf{x},t) & \text{if } v_\varepsilon(\mathbf{x},t) > -1 \\ 0 & \text{if } v_\varepsilon(\mathbf{x},t) = -1 \end{cases}$$

and observe that $f(\mathbf{x},t) > 0$ in $\{v_\varepsilon(\mathbf{x},t) = 1\}$ (for $\varepsilon \leq \mathcal{E}$). Integration by parts in $\mathcal{T}_\varepsilon^\rho(t)$ for all $t \in (0,T)$ leads to

$$(5.21)\qquad \begin{aligned} \int_\Omega \big(\varepsilon\partial_t(av_\varepsilon)\varphi + \varepsilon a\nabla v_\varepsilon\cdot\nabla\varphi - f\varphi\big) &= \int_{\mathcal{T}_\varepsilon(t)} \mathcal{J}v_\varepsilon\varphi + \varepsilon\int_{\partial\mathcal{T}_\varepsilon^\rho(t)} a\nabla v_\varepsilon\cdot\nu_\varepsilon^\rho\varphi \\ &\quad - \int_{\Omega\setminus\mathcal{T}_\varepsilon(t)} f\varphi + \varepsilon\int_{\mathcal{T}_\varepsilon(t)\setminus\mathcal{T}_\varepsilon^\rho(t)} \big(a\nabla v_\varepsilon\cdot\nabla\varphi + \nabla\cdot(a\nabla v_\varepsilon)\varphi\big) \\ &= \mathrm{I} + \mathrm{II}^\rho + \mathrm{III} + \mathrm{IV}^\rho \end{aligned}$$

for all $0 \leq \varphi \in H^1(\Omega)$, where $\mathcal{J}v_\varepsilon = \varepsilon\partial_t(av_\varepsilon) - \varepsilon\nabla\cdot(a\nabla v_\varepsilon) - \frac{1}{\varepsilon a}v_\varepsilon - \frac{c_0}{2}g$. In the sequel we examine each term separately. Since $\nabla v_\varepsilon\cdot\nu_\varepsilon^\rho \leq 0$ on $\partial\mathcal{T}_\varepsilon^\rho(t)$, we get $\mathrm{II}^\rho \leq 0$. From (5.20) we also have $\mathrm{III} \leq 0$. Hence, it only remains to establish that the first term is nonpositive, because $|\mathrm{IV}^\rho| \leq \frac{C}{\varepsilon}|\mathcal{T}_\varepsilon(t)\setminus\mathcal{T}_\varepsilon^\rho(t)|^{1/2}\|\varphi\|_{H^1(\Omega)} \downarrow 0$ as $\rho \downarrow 0$. Since

$$(5.22)\qquad \mathcal{J}v_\varepsilon = \varepsilon a\partial_t v_\varepsilon + \varepsilon v_\varepsilon\partial_t a - \varepsilon a\Delta v_\varepsilon - \varepsilon\nabla a\cdot\nabla v_\varepsilon - \tfrac{1}{\varepsilon a}v_\varepsilon - \tfrac{c_0}{2}g,$$

we will proceed to evaluate the above summands. We first use the definition (5.15) of v_ε, in conjunction with (5.14) and (5.11), to arrive at

$$(5.23)\qquad \begin{aligned} \nabla v_\varepsilon &= \gamma_-'(y)\nabla y + \mathcal{O}_{A,G}(\overline{\varepsilon}) \\ \partial_t v_\varepsilon &= \gamma_-'(y)\partial_t y + \mathcal{O}_{A,G}(\overline{\varepsilon}) \\ \Delta v_\varepsilon &= \gamma_-''(y)|\nabla y|^2 + \gamma_-'(y)\Delta y + \mathcal{O}_{A,G}(\overline{\varepsilon}), \end{aligned}$$

for all $\mathbf{x} \in \mathcal{T}_\varepsilon(t)$ and $t \in [0,T]$. Next, we use (5.8) and (5.9) to derive

$$
\begin{aligned}
a &= \overline{a}\Big(1 + \overline{\varepsilon} y \tfrac{\overline{a}'}{\overline{a}} + \tfrac{1}{2}\overline{\varepsilon}^2 y^2 \tfrac{\overline{a}''}{\overline{a}} + \sigma_2 \overline{\varepsilon}^2 \big(\tfrac{\overline{a}'}{\overline{a}} + \overline{\varepsilon} y \tfrac{\overline{a}''}{\overline{a}}\big) + \mathcal{O}_A(\overline{\varepsilon}^3)\Big) \\
&= \overline{a}\big(1 + \overline{\varepsilon} y \tfrac{\overline{a}'}{\overline{a}} + \sigma_2 \overline{\varepsilon}^2 \tfrac{\overline{a}'}{\overline{a}} + \mathcal{O}_A(\overline{\varepsilon}^2)\big) \\
\tfrac{1}{a} &= \tfrac{1}{\overline{a}}\Big(1 - \overline{\varepsilon} y \tfrac{\overline{a}'}{\overline{a}} + \overline{\varepsilon}^2 y^2 \big(\tfrac{(\overline{a}')^2}{\overline{a}^2} - \tfrac{1}{2}\tfrac{\overline{a}''}{\overline{a}}\big) \\
&\qquad - \sigma_2 \overline{\varepsilon}^2 \Big(\tfrac{\overline{a}'}{\overline{a}} - 2\overline{\varepsilon} y \big(\tfrac{(\overline{a}')^2}{\overline{a}^2} - \tfrac{1}{2}\tfrac{\overline{a}''}{\overline{a}}\big)\Big) + \mathcal{O}_A(\overline{\varepsilon}^3)\Big) \\
\nabla a &= \overline{(\nabla a)} + \overline{\varepsilon} y \overline{(\nabla a)}' + \sigma_2 \overline{\varepsilon}^2 \overline{(\nabla a)}' + \overline{a}\mathcal{O}_A(\overline{\varepsilon}^2) \\
\partial_t a &= \overline{(\partial_t a)} + \overline{a}\mathcal{O}_A(\overline{\varepsilon})
\end{aligned}
\tag{5.24}
$$

and

$$
g = \overline{g} + \overline{\varepsilon} y \overline{g}' + \sigma_2 \overline{\varepsilon}^2 \overline{g}' + \mathcal{O}_G(\overline{\varepsilon}^2). \tag{5.25}
$$

Tedious calculations based on (5.10), (5.12), (5.16), (5.23), (5.24), and (5.25) lead to the following asymptotic expressions for each of the summands in (5.22):

$$
\begin{aligned}
\varepsilon a \partial_t v_\varepsilon &= \overline{\varepsilon}\big(1 + \overline{\varepsilon} y \tfrac{\overline{a}'}{\overline{a}} + \sigma_2 \overline{\varepsilon}^2 \tfrac{\overline{a}'}{\overline{a}} + \mathcal{O}_A(\overline{\varepsilon}^2)\big)\big(\gamma_-'(y)\partial_t y + \mathcal{O}_{A,G}(\overline{\varepsilon})\big) \\
&= (\overline{\kappa^m} + \overline{g})\gamma'(y) \\
&\quad + \overline{\varepsilon}\big((\overline{\kappa^m} + \overline{g})\textstyle\sum_{i=1}^2 \overline{\alpha}_i \widehat{\eta}_i'(y) + (\overline{\kappa^m} + \overline{g}) y \gamma'(y) \tfrac{\overline{a}'}{\overline{a}} - y\gamma'(y)\tfrac{\partial_t \overline{a}}{\overline{a}}\big) \\
&\quad + \overline{\varepsilon}^2 \sigma_2 \big(\tfrac{\overline{a}'}{\overline{a}}(\overline{\kappa^m} + \overline{g})(2(\xi_1^-)'(y) + \gamma'(y)) - 2H\gamma'(y) - 2\gamma'(y)\tfrac{\partial_t \overline{a}}{\overline{a}}\big) + \mathcal{O}_{A,G}(\overline{\varepsilon}^2);
\end{aligned}
$$

$$
\varepsilon v_\varepsilon \partial_t a = \overline{\varepsilon}\big(\gamma(y) + \mathcal{O}_{A,G}(\overline{\varepsilon})\big)\big(\tfrac{\overline{(\partial_t a)}}{\overline{a}} + \mathcal{O}_A(\overline{\varepsilon})\big) = \overline{\varepsilon}\gamma(y)\tfrac{\overline{(\partial_t a)}}{\overline{a}} + \mathcal{O}_{A,G}(\overline{\varepsilon}^2);
$$

$$
\begin{aligned}
-\varepsilon a \Delta v_\varepsilon &= -\overline{\varepsilon}\big(1 + \overline{\varepsilon} y \tfrac{\overline{a}'}{\overline{a}} + \tfrac{1}{2}\overline{\varepsilon}^2 y^2 \tfrac{\overline{a}''}{\overline{a}} + \sigma_2 \overline{\varepsilon}^2 \big(\tfrac{\overline{a}'}{\overline{a}} + \overline{\varepsilon} y \tfrac{\overline{a}''}{\overline{a}}\big) + \mathcal{O}_A(\overline{\varepsilon}^3)\big) \times \\
&\qquad \big(\gamma_-''(y)|\nabla y|^2 + \gamma_-'(y)\Delta y + \mathcal{O}_{A,G}(\overline{\varepsilon})\big) \\
&= -\tfrac{1}{\overline{\varepsilon}}\gamma''(y) \\
&\quad - \big(\textstyle\sum_{i=1}^2 \overline{\alpha}_i \widehat{\eta}_i''(y) + y\gamma''(y)\tfrac{\overline{a}'}{\overline{a}} + \overline{\kappa^m}\gamma'(y)\big) \\
&\quad - \overline{\varepsilon}\Big(\textstyle\sum_{j=1}^5 \overline{\beta}_j (\xi_j^-)''(y) + 2\sigma_2 \tfrac{\overline{a}'}{\overline{a}}(\xi_1^-)''(y) + y\tfrac{\overline{a}'}{\overline{a}}\sum_{i=1}^2 \overline{\alpha}_i \widehat{\eta}_i''(y) \\
&\qquad + \overline{\kappa^m}\textstyle\sum_{i=1}^2 \overline{\alpha}_i \widehat{\eta}_i'(y) + y^2\gamma''(y)\big(\tfrac{|\nabla \overline{a}|^2}{\overline{a}^2} + \tfrac{1}{2}\tfrac{\overline{a}''}{\overline{a}}\big) \\
&\qquad + \sigma_2\gamma''(y)\tfrac{\overline{a}'}{\overline{a}} + \overline{\kappa^m} y\gamma'(y)\tfrac{\overline{a}'}{\overline{a}} - \overline{\kappa^s} y\gamma'(y) - y\gamma'(y)\big(\tfrac{\Delta \overline{a}}{\overline{a}} - 2\tfrac{|\nabla \overline{a}|^2}{\overline{a}^2}\big)\Big) \\
&\quad - \overline{\varepsilon}^2\Big(- 2\sigma_1\big(2y\gamma''(y) + \gamma'(y)\big) + \sigma_2\Big(y\gamma''(y)\big(4\tfrac{|\nabla \overline{a}|^2}{\overline{a}^2} + \tfrac{\overline{a}''}{\overline{a}}\big) - \overline{\kappa^s}\gamma'(y) \\
&\qquad + \overline{\kappa^m}\gamma'(y)\tfrac{\overline{a}'}{\overline{a}} - 2\gamma'(y)\big(\tfrac{\Delta \overline{a}}{\overline{a}} - \tfrac{|\nabla \overline{a}|^2}{\overline{a}^2}\big) + \tfrac{\overline{a}'}{\overline{a}}\textstyle\sum_{i=1}^2 \overline{\alpha}_i \widehat{\eta}_i''(y) \\
&\qquad + 2\tfrac{\overline{a}'}{\overline{a}}\overline{\kappa^m}(\xi_1^-)'(y) + 2y\tfrac{(\overline{a}')^2}{\overline{a}^2}(\xi_1^-)''(y)\Big)\Big) + \mathcal{O}_{A,K,G}(\overline{\varepsilon}^2);
\end{aligned}
$$

$$
\begin{aligned}
-\varepsilon\nabla a\cdot\nabla v_\varepsilon &= -\overline{\varepsilon}\Big(\frac{\overline{(\nabla a)}}{\overline{a}}+\overline{\varepsilon}y\frac{\overline{(\nabla a)}'}{\overline{a}}+\sigma_2\overline{\varepsilon}^2\frac{\overline{(\nabla a)}'}{\overline{a}}+\mathcal{O}_A(\overline{\varepsilon}^2)\Big)\big(\gamma_-'(y)\nabla y+\mathcal{O}_{A,G}(\overline{\varepsilon})\big)\\
&= -\gamma'(y)\tfrac{\overline{a}'}{\overline{a}}\\
&\quad-\overline{\varepsilon}\Big(y\gamma'(y)\big(\tfrac{\overline{a}''}{\overline{a}}-\tfrac{\nabla\overline{a}\cdot\overline{(\nabla a)}}{\overline{a}^2}\big)+\tfrac{\overline{a}'}{\overline{a}}\textstyle\sum_{i=1}^2\overline{\alpha}_i\widehat{\eta}_i'(y)\Big)\\
&\quad-\overline{\varepsilon}^2\sigma_2\Big(2\tfrac{(\overline{a}')^2}{\overline{a}^2}(\xi_1^-)'(y)+\gamma'(y)\big(\tfrac{\overline{a}''}{\overline{a}}-2\tfrac{\nabla\overline{a}\cdot\overline{(\nabla a)}}{\overline{a}^2}\big)\Big)+\mathcal{O}_{A,G}(\overline{\varepsilon}^2);
\end{aligned}
$$

$$
\begin{aligned}
-\tfrac{1}{\varepsilon a}v_\varepsilon &= -\tfrac{1}{\overline{\varepsilon}}\gamma(y)\\
&\quad-\big(\textstyle\sum_{i=1}^2\overline{\alpha}_i\widehat{\eta}_i(y)-y\gamma(y)\tfrac{\overline{a}'}{\overline{a}}\big)\\
&\quad-\overline{\varepsilon}\Big(\textstyle\sum_{j=1}^5\overline{\beta}_j\xi_j^-(y)+2\sigma_2\tfrac{\overline{a}'}{\overline{a}}\xi_1^-(y)-y\tfrac{\overline{a}'}{\overline{a}}\sum_{i=1}^2\overline{\alpha}_i\widehat{\eta}_i(y)\\
&\qquad+y^2\gamma(y)\big(\tfrac{(\overline{a}')^2}{\overline{a}^2}-\tfrac12\tfrac{\overline{a}''}{\overline{a}}\big)-\sigma_2\gamma(y)\tfrac{\overline{a}'}{\overline{a}}\Big)\\
&\quad-\overline{\varepsilon}^2\sigma_2\Big(2y\gamma(y)\big(\tfrac{(\overline{a}')^2}{\overline{a}^2}-\tfrac12\tfrac{\overline{a}''}{\overline{a}}\big)-2y\tfrac{(\overline{a}')^2}{\overline{a}^2}\xi_1^-(y)-\tfrac{\overline{a}'}{\overline{a}}\textstyle\sum_{i=1}^2\overline{\alpha}_i\widehat{\eta}_i(y)\Big)+\mathcal{O}_{A,K,G}(\overline{\varepsilon}^2).
\end{aligned}
$$

We now collect all the summands in (5.22) with the same power $\overline{\varepsilon}^i$, and label them S_i. The effect of shape corrections is now quite apparent in the vanishing of S_i for $i=-1,0,1$! In fact, by virtue of (2.8), (4.1), (4.2), and (5.13), we have:

Order $\overline{\varepsilon}^{-1}$:

$$S_{-1}=-\big(\gamma''(y)+\gamma(y)\big)=0;$$

Order $\overline{\varepsilon}^0$:

$$S_0=-\textstyle\sum_{i=1}^2\overline{\alpha}_i\big(\widehat{\eta}_i''(y)+\widehat{\eta}_i(y)\big)+\overline{g}\big(\gamma'(y)-\tfrac{c_0}{2}\big)-\tfrac{\overline{a}'}{\overline{a}}\big(\gamma'(y)-2y\gamma(y)\big)=0;$$

Order $\overline{\varepsilon}$:

$$
\begin{aligned}
S_1=&-\textstyle\sum_{j=1}^5\overline{\beta}_j\big((\xi_j^-)''(y)+\xi_j^-(y)\big)-2\sigma_2\tfrac{\overline{a}'}{\overline{a}}\big((\xi_1^-)''(y)+\xi_1^-(y)\big)+\gamma(y)\big(2\sigma_2\tfrac{\overline{a}'}{\overline{a}}+\tfrac{\overline{(\partial_t a)}}{\overline{a}}\big)\\
&+y\gamma'(y)\big(\overline{\kappa^s}+\tfrac{\overline{a}'}{\overline{a}}\overline{g}+\tfrac{\nabla\overline{a}\cdot\overline{(\nabla a)}}{\overline{a}^2}-\tfrac{(\partial_t-\Delta)\overline{a}}{\overline{a}}-2\tfrac{|\nabla\overline{a}|^2}{\overline{a}^2}-\tfrac{\overline{a}''}{\overline{a}}\big)\\
&+y^2\gamma(y)\big(\tfrac{|\nabla\overline{a}|^2}{\overline{a}^2}-\tfrac{(\overline{a}')^2}{\overline{a}^2}+\tfrac{\overline{a}''}{\overline{a}}\big)-\tfrac{c_0}{2}y\overline{g}'\\
&+\big(\overline{g}-\tfrac{\overline{a}'}{\overline{a}}\big)\textstyle\sum_{i=1}^2\overline{\alpha}_i\widehat{\eta}_i'(y)-\tfrac{\overline{a}'}{\overline{a}}\sum_{i=1}^2\overline{\alpha}_iy\big(\widehat{\eta}_i''(y)-\widehat{\eta}_i(y)\big)=0;
\end{aligned}
$$

Order $\overline{\varepsilon}^2$:

$$
\begin{aligned}
S_2=&-2\sigma_1\big(2y\gamma(y)-\gamma'(y)\big)-2\sigma_2H\gamma'(y)\\
&-\sigma_2\Big(\tfrac{c_0}{2}\overline{g}'-2\tfrac{\overline{a}'}{\overline{a}}\big(\overline{g}-\tfrac{\overline{a}'}{\overline{a}}\big)(\xi_1^-)'(y)+2\tfrac{(\overline{a}')^2}{\overline{a}^2}y\big((\xi_1^-)''(y)-\xi_1^-(y)\big)\\
&\quad+\tfrac{\overline{a}'}{\overline{a}}\textstyle\sum_{i=1}^2\overline{\alpha}_i\big(\widehat{\eta}_i''(y)-\widehat{\eta}_i(y)\big)-2y\gamma(y)\big(\tfrac{\overline{a}''}{\overline{a}}+2\tfrac{|\nabla\overline{a}|^2}{\overline{a}^2}-\tfrac{(\overline{a}')^2}{\overline{a}^2}\big)\\
&\quad-\gamma'(y)\big(\overline{\kappa^s}+\overline{g}\tfrac{\overline{a}'}{\overline{a}}-2\tfrac{(\partial_t-\Delta)\overline{a}}{\overline{a}}-2\tfrac{|\nabla\overline{a}|^2}{\overline{a}^2}+2\tfrac{\nabla\overline{a}\cdot\overline{(\nabla a)}}{\overline{a}^2}-\tfrac{\overline{a}''}{\overline{a}}\big)\Big)+\mathcal{O}_{A,K,G}(1)\\
=&-2\sigma_1\big(2y\gamma(y)-\gamma'(y)\big)-2\sigma_2H\gamma'(y)-\sigma_2\zeta(\cdot;A,K,G)+\mathcal{O}_{A,K,G}(1);
\end{aligned}
$$

here, in view of (2.9) and (5.14), $\zeta\big((\mathbf{x},t);A,K,G\big)$ is a function independent of a_0, ε, and C_1, C_2. Thus, on selecting $H = \|\zeta(\cdot;A,K,G)\|_{L^\infty(\Omega\times(0,T))}$, we clearly obtain

$$\mathcal{J} v_\varepsilon \le -\bar{\varepsilon}^2 \mathrm{e}^{2Ht}\Big(4C_1 y\gamma(y) + 2HC_2\gamma'(y) - 2C_1\gamma'(y) - C_2 H\Big) + \mathcal{O}_{A,K,G}(\bar{\varepsilon}^2),$$

and, owing to $y\gamma(y) + \gamma'(y) \ge 1$,

$$\mathcal{J} v_\varepsilon \le -\bar{\varepsilon}^2 \mathrm{e}^{2Ht}\Big(y\gamma(y)\big(4C_1 - C_2 H\big) + \gamma'(y)\big(C_2 H - 2C_1\big)\Big) + \mathcal{O}_{A,K,G}(\bar{\varepsilon}^2).$$

Recalling our convention (5.1), the choice $C_2 H = 3C_1$ then leads to

$$\mathcal{J} v_\varepsilon \le -\bar{\varepsilon}^2 e^{2Ht}\big(C_1 + \mathcal{O}_{A,K,G}(1) + \mathcal{O}_{A,K,G,\sigma_i(T)}(\bar{\varepsilon})\big) \le 0,$$

for C_1 sufficiently large but independent of $\varepsilon \le \mathcal{E}$ and $\mathcal{E} \ll 1$ depending on C_1, C_2 and exponentially on K, G, A, and T; thus $\mathrm{I} \le 0$. We point out that neither C_1, C_2 nor $\mathcal{E}$ depend on a_0, $\mathbf{x}$, or t.

Upon taking $\rho \downarrow 0$ in (5.21), and recalling (5.18) and (5.17), we conclude that $v_\varepsilon^- = v_\varepsilon$ satisfies (3.1). Finally, applying the Comparison Lemma of §3 we end up with

$$v_\varepsilon^-(\mathbf{x},t) \le u_\varepsilon(\mathbf{x},t) \qquad \forall\, (\mathbf{x},t) \in \Omega \times [0,T].$$

Based on v_ε^-, a supersolution v_ε^+ can be defined as follows. Let $\tilde{d}_\varepsilon^-$ and $\tilde{v}_\varepsilon^-$ indicate the modified distance function and the subsolution of the auxiliary problem corresponding to $\tilde{g} = -g$ and $\tilde{d} = -d$. We clearly obtain that $v_\varepsilon^+ = -\tilde{v}_\varepsilon^-$ is a supersolution and $d_\varepsilon^+ = -\tilde{d}_\varepsilon^-$ is the modified distance function for the original problem.

6. Local Interface Error Estimates. The proof of Theorem 2.1 is a simple consequence of the above construction and relies on the following properties of the function γ_- defined in §4: $\gamma_-'(x) > 0$ for all $x \in (x_\varepsilon, \frac{\pi}{2})$ and $\gamma_-'(0) > \frac{3}{4}$ for ε sufficiently small. These properties, indeed, combined with (5.19), (5.3), (5.4), (2.3), and (5.5) in this order, lead to

$$\begin{aligned}\big(\nabla v_\varepsilon^\mp \cdot \nabla d\big)(\mathbf{z},t) &= \gamma_\mp'\big(\tfrac{d_\varepsilon^\mp(\mathbf{z},t)}{\varepsilon a \mathbf{s}(\mathbf{z},t)}\big)\nabla\big(\tfrac{d_\varepsilon^\mp(\mathbf{z},t)}{\varepsilon a \mathbf{s}(\mathbf{z},t)}\big) \cdot \nabla d(\mathbf{z},t) \\ &= \gamma_\mp'\big(\tfrac{d_\varepsilon^\mp(\mathbf{z},t)}{\varepsilon a \mathbf{s}(\mathbf{z},t)}\big)\big(\tfrac{1}{\varepsilon a \mathbf{s}(\mathbf{z},t)} + \mathcal{O}(1)\big) > 0 \qquad \forall\, \mathbf{z} \in \mathcal{T}_\varepsilon(t),\end{aligned} \tag{6.1}$$

as well as to the crucial *nondegeneracy* property

$$\big(\nabla v_\varepsilon^\mp \cdot \nabla d\big)(\mathbf{x},t) \ge \tfrac{1}{2\varepsilon a(\mathbf{x},t)} \qquad \forall\, \mathbf{x} \in \Sigma(t). \tag{6.2}$$

For any $\mathbf{x} \in \Sigma(t)$ and $t \in [0,T]$, let $N(\mathbf{x},t)$ denote a straight segment of size $2D$ perpendicular to $\Sigma(t)$, namely $N(\mathbf{x},t) = \{\mathbf{z} = \mathbf{x} + s\nabla d(\mathbf{x},t) : |s| \le D\}$. For a given $\mathbf{y} \in$

$\Sigma_\varepsilon(t)$ and $t \in [0,T]$, we claim that $(\mathbf{y},t) \in \mathcal{T} = \{|d(\mathbf{z},t)| \le D\}$, whence $\mathbf{x} = \mathbf{s}(\mathbf{y},t) \in \Sigma(t)$ is uniquely defined. In fact, otherwise $(\mathbf{y},t) \notin \mathcal{T}$ and $-1 = v_\varepsilon^-(\mathbf{y},t) \le u_\varepsilon(\mathbf{y},t) = 0 \le v_\varepsilon^+(\mathbf{y},t) = 1$, which because of the definition of $v_\varepsilon^\mp$ yields the contradiction $\mathbf{y} \in I(t) \cap O(t)$. Consider now the problem of finding $\mathbf{y}^\mp \in N(\mathbf{x},t)$ such that $v_\varepsilon^\mp(\mathbf{y}^\mp,t) = 0$. In light of (5.3) and (5.16), we see that $v_\varepsilon^\mp(\mathbf{x},t) = \mathcal{O}(\varepsilon a(\mathbf{x},t))$ for $\mathbf{x} \in \Sigma(t)$. Upon invoking (6.2) we not only realize that such $\mathbf{y}^\mp$ do in fact exist but also that they satisfy

$$|\mathbf{x} - \mathbf{y}^\mp| = d(\mathbf{y}^\mp,t) = \mathcal{O}(\varepsilon^2 a^2(\mathbf{x},t)).$$

Moreover both $\mathbf{y}^\mp$ are unique in that (6.1) implies monotonicity of $v_\varepsilon^\mp$ on $N(\mathbf{x},t)$. With the aid of $v_\varepsilon^- \le u_\varepsilon \le v_\varepsilon^+$, we conclude that all the zeros of $u_\varepsilon(\cdot,t)$ on $N(\mathbf{x},t)$ must lie between $\mathbf{y}^-$ and $\mathbf{y}^+$, which in turn applies to the original $\mathbf{y} \in \Sigma_\varepsilon(t)$ because $|\mathbf{y}-\mathbf{x}| = |d(\mathbf{y},t)| \le D$. Therefore, we readily get

$$\operatorname{dist}\big(\mathbf{x},\Sigma_\varepsilon(t)\big), \operatorname{dist}\big(\mathbf{y},\Sigma(t)\big) \le |\mathbf{x}-\mathbf{y}| \le \max_{\mp} |\mathbf{x}-\mathbf{y}^\mp| \le C_*\varepsilon^2 a^2(\mathbf{x},t).$$

To conclude the argument we just have to notice that, by virtue of (2.9), $a^2(\mathbf{x},t) \le a^2(\mathbf{y},t) + A^3|\mathbf{x}-\mathbf{y}|$, which leads to the desired estimate for $\mathbf{y} \in \Sigma_\varepsilon(t)$ provided ε is sufficiently small depending on C_* and A.

Optimality of (2.11) has been already proved in [16] for the radially symmetric case with a vanishing forcing term $g = 0$ and $a = 1$. The proof of Theorem 2.1 is thus complete.

We finish our discussion with two comments about the structural assumption (2.9).

Remark6.1. If (2.9) is no longer valid but $\|a\|_{W^{4,2}_\infty(\Omega\times(0,T))} \le A_*$, then the above analysis clearly indicates that $H = \|\zeta\|_{L_\infty(\Omega\times(0,T))} \le CA_*^2/a_0^2$. In view of (5.3), (5.13), and (5.15) we infer that $v_\varepsilon^\mp(\mathbf{x},t) = \mathcal{O}(\exp(\frac{CA_*^2 t}{a_0^2})\varepsilon a(\mathbf{x},t))$ for $\mathbf{x} \in \Sigma(t)$. This leads to the following global interface error estimate in terms of the Hausdorff distance $\operatorname{dist}_H(\cdot,\cdot)$:

$$\operatorname{dist}_H\big(\Sigma_\varepsilon(t),\Sigma(t)\big) \le C_*\exp(\frac{CA_*^2 t}{a_0^2})\varepsilon^2 \qquad \forall\, t \in [0,T]. \tag{6.3}$$

This pessimistic result is in the same vein as those in [16,15] for $a = 1$, but the proof for variable a is certainly much more involved.

Remark6.2. It is relatively easy in practice to construct a density function a satisfying (2.9) if A is allowed to depend logarithmically on a_0; see §7.

7. Numerical Results. We present a numerical experiment for the motion by mean curvature ($g = 0$) that illustrates the significance of the local estimate of Theorem 2.1. We consider the evolution of the axially symmetric dumbbell studied in [1], with

$$\Sigma_c(0) = \{x_1^2 + x_2^2 = (1-x_3^2)(1-c+cx_3^2)^2\} \subset \mathbb{R}^3$$

being the initial front. We see that the surface $\Sigma_0(0)$ is a sphere whereas $\Sigma_1(0)$ consists of two tear shaped parts with a singularity at the origin. There exists a critical $0 < c^* < 1$ for which $\Sigma_{c^*}(t)$ remains regular and nonconvex until it shrinks to the origin [1]. For $c < c^*$, $\Sigma_c(t)$ becomes convex and shrinks to the origin at $t_c^\dagger$, whereas for $c > c^*$, $\Sigma_c(t)$ focuses at the origin and breaks at t_c^* and eventually disappears at $t_c^\dagger > t_c^*$. For the critical value c^* we thus have $t_{c^*}^\dagger = t_{c^*}^*$. The purpose of our simulation is to compute c^* along with Σ_{c^*}.

In view of the axial symmetry, we introduce the cylindrical coordinates $r = (x_1^2 + x_2^2)^{1/2}$ and $z = x_3$. Thus, we can use the 2D *dynamic mesh algorithm* of [14] over the domain $\Omega = \{(r,z) \in (0,4)^2\}$ with the time-independent density function

$$a(r,z) = \left(1 + \tfrac{1}{10}(r^3 + z^3)\right) \exp\left(-\tfrac{1}{4r^2+z^2+\alpha}\right), \qquad \text{where} \qquad \alpha = -\tfrac{1}{\log a_0},$$

where $a_0 = 0.05$. Note that this function satisfies the assumption (2.9) but with a constant A depending on $|\log a_0|$ to some power. It turns out that the local interface error estimates in Theorem 2.1 is quasi-optimal because of the logarithm.

Space discretization is based on piecewise linear finite elements over a graded mesh. Figure 7.1 shows the graded mesh corresponding to the mesh density $h(r,z) = 0.07a(r,z)$. It has to be stressed that the dynamic algorithm actually triangulates only the transition region, which is significantly smaller than a full mesh and the sole partition present at any time, and then updates the mesh to follow the transition region motion. Time stepping is explicit, but stability constraints force small time steps only when singularities develop, whereas relatively large time steps are allowed at the beginning or past singularities, when the evolution is smooth. Explicit time discretization guarantees that at most one layer of boundary elements of the transition region has to be either added or deleted, thereby making dynamic mesh updating feasible and overruling the use of implicit methods. This method thus retains the local character of the geometric problem at hand while taking advantage of the variational structure of (1.1), and exhibits potential to be exploited in connection with phase transitions.

Our numerical results, obtained with the mesh density $h(r,z) = 0.07a(r,z)$, suggest that $c^* = 0.654$. The sequences of interfaces for $c = 0.653$, $c^* = 0.654$, $c = 0.655$, and $c = 0.656$ are depicted in Figure 7.2a-d, where we observe the nonconvexity of $\Sigma_{c^*}(t)$ for the entire evolution. The boundary elements of the actual graded meshes for $c = c^*$ are drawn in Figure 7.3a-h together with the corresponding interfaces for a number of time steps. We point out that the thickness of the transition regions varies along the front and is dictated by the *local* size of a. This remarkable effect is of great importance in resolving singularities.

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FIGURE 7.1. Graded mesh over the whole domain, for $h(r, z) = 0.07a(r, z)$.

FIGURE 7.2. Front evolution for $c = 0.653$, $c^* = 0.654$, $c = 0.655$, and $c = 0.656$; $t = 0.025i$, $0 \leq i \leq 30$, and t_c^* (bold line).

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FIGURE 7.3. Graded meshes over the transition region at $t = 0.01i$, $0 \leq i \leq 7$, for $c^* = 0.654$.

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