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CONVERGENCE OF DOUBLE OBSTACLE PROBLEMS TO THE GENERALIZED GEOMETRIC MOTION OF FRONTS*

RICARDO H. NOCHETTO† AND CLAUDIO VERDI‡

Abstract. The connection between generalized geometric motion of interfaces, interpreted in the viscosity sense, and a singularly perturbed parabolic problem with double obstacle ± 1 and small parameter ε is examined. This approach retains the local character of the limit problem because the noncoincidence set, where all the action takes place, is a thin transition layer of thickness $\mathcal{O}(\varepsilon)$ irrespective of the forcing term. Zero level sets are shown to converge past singularities to the generalized motion by mean curvature with forcing, provided no fattening occurs. If the underlying viscosity solution satisfies a nondegeneracy property, namely its gradient does not vanish, then our results yield interface error estimates and layer width estimates of order $\mathcal{O}(\varepsilon)$. The proofs are based on constructing viscosity sub and supersolutions to the double obstacle problem in terms of the signed distance function and approximate traveling waves dictated by formal asymptotics.

Key words. reaction-diffusion, double obstacle, generalized curvature driven motion, interface convergence, interface error estimates

AMS(MOS) subject classifications. 35B25, 35K55, 35K57, 35K85, 49Q05

1. Introduction. In this paper we investigate the relation between the generalized curvature driven motion of interfaces and a singularly perturbed parabolic double obstacle problem. The zero level set of u_ε, solution to the singularly perturbed reaction-diffusion PDE

$$\varepsilon \partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon + \frac{1}{2\varepsilon}\Psi'(u_\varepsilon) = \frac{c_0}{2} g \qquad \text{in } \Omega \times (0,T), \tag{1.1}$$

with quartic-like double equal well potential Ψ and $c_0 := \int_{-1}^{1} \sqrt{\Psi(s)}ds$, is known to converge to an interface $\Sigma(t)$ that moves (formally) in the normal direction with velocity

$$V(\mathbf{x},t) = \kappa(\mathbf{x},t) + g(\mathbf{x},t) \qquad \forall\, \mathbf{x} \in \Sigma(t); \tag{1.2}$$

hereafter $\kappa(\mathbf{x},t)$ indicates the sum of the principal curvatures of $\Sigma(t)$ at $\mathbf{x}$ and g is a forcing term defined in $\Omega \times (0,T)$. This connection has been rigorously established in [3], [9] in case of no fattening, that is provided $\Sigma(t)$ has empty interior. Equation (1.1), in turn, arises in the Landau-Ginzburg theory of phase transitions [1]. A typical example of potential is the quartic $\Psi(s) = (s^2-1)^2$, but there is no physical reason why Ψ has to be of that form, or even smooth. The key condition on Ψ, to achieve the geometric law (1.2) in the limit, is that the two wells possess equal depth. Exploiting such a freedom of choice, we consider the double obstacle potential

$$\Psi(s) := \begin{cases} 1 - s^2 & \text{if } s \in [-1,1] \\ +\infty & \text{if } s \notin [-1,1]. \end{cases} \tag{1.3}$$

In contrast to the usual reaction-diffusion approach with a quartic-like nonlinearity, the coupling of (1.1) and (1.3) exhibits a narrow transition region of order $\mathcal{O}(\varepsilon)$, outside

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† Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA.

‡ Dipartimento di Matematica, Università di Milano, 20133 Milano, Italy.

of which $u_\varepsilon = \pm 1$ irrespective of g. Since the resulting problem has to be solved within such a thin noncoincidence set, where all the action takes place, we realize that this approach retains the geometric (or local) structure of the original problem while being insensitive to singularity formation. This property is essential for numerical purposes, and it bears some intrinsic interest as well. There is, however, no complete theoretical justification of this formulation because the *optimal* interface error estimates of [14]–[16], of order $\mathcal{O}(\varepsilon^2)$, are only valid before the onset of singularities, that is provided the motion is smooth. The convergence results past singularities of [3], [9], on the other hand, apply to *smooth* quartic-type potentials but not to the (singular) double obstacle (1.3).

In this paper we demonstrate that the zero level sets $\Sigma_\varepsilon(t)$ of u_ε converge to $\Sigma(t)$ past singularities provided no fattening occurs. The proof is based, as in [3], [9], on constructing viscosity supersolutions to the reaction-diffusion PDE in terms of the signed distance function d to suitable level sets of the viscosity solution ω to the generalized geometric motion (1.2) [7], [10], [12]. The novelties here are the presence of obstacles, that entail lack of regularity of u_ε even for smooth initial data, and the use of an explicit traveling wave dictated by formal asymptotics. Inspired by [19], the first issue is tackled with a suitable notion of viscosity supersolution and corresponding comparison principle. On the other hand dealing with an approximate explicit traveling wave with fixed transition layer width $\pi\varepsilon$, thereby independent of g, makes the construction of supersolutions simpler than in [3], [9] and certainly more transparent. It also avoids considering generalized flows (1.2) corresponding to perturbed forcing terms $g \pm o(1)$, as in [3], and yields a new (local) linear rate of convergence $\mathcal{O}(\varepsilon)$ for interfaces, along with layer width estimates, under a further *nondegeneracy* assumption on ω. In fact, if $|\nabla\omega(\mathbf{x},t)| > 0$ for $\mathbf{x} \in \Sigma(t)$, then

$$\operatorname{dist}\big(\mathbf{x}, \Sigma_\varepsilon(t)\big) \le C|\nabla\omega(\mathbf{x},t)|^{-1}\varepsilon.$$

Since this requirement is always valid before the onset of singularities, our results extend those in [6] past singularities. It is remarkable that the linear rate is preserved between singularities, which applies to a number of geometric flows [2], [18].

The local character of the double obstacle formulation, together with its convergence properties even beyond singularities, leads to a robust but effective numerical tool: the *dynamic mesh algorithm* of [13]. This is a finite element solver that solely triangulates the transition layer and then updates it, after having solved the discrete problem, to advance the algorithm in time. Such a simple but crucial idea results in savings of computing time and memory allocation, along with enhanced singularity resolution via a space-time dependent relaxation parameter $\varepsilon(\mathbf{x},t)$. This claim has been confirmed both theoretically and numerically [13]–[17], and is further supported here with a rigorous convergence result and error estimates past singularities.

This paper is organized as follows. We recall in §2 key properties of d and introduce the notion of viscosity supersolutions to the double obstacle problem. We show in §3 how to obtain an explicit representation of an approximate traveling wave via formal asymptotics. With the candidate for supersolution at hand, we perform a formal calculation in §4 that presumes regularity of d and confirms the desired properties of our supersolution. This is useful in understanding the main idea behind the quite technical rigorous discussion of §5. A (viscosity) comparison lemma, adequate for obstacle problems, is fully discussed in §6. Both the explicit form of supersolutions and the comparison principle are key ingredients in proving convergence of $\Sigma_\varepsilon(t)$ to $\Sigma(t)$ together with interface and layer width estimates; this is carried out in §7.

2. Viscosity solutions. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and set $Q = \Omega \times (0,T)$ for $T > 0$. Let the forcing term g satisfy

$$g \in W^{2,1}_\infty(Q). \tag{2.1}$$

Remark 2.1. A weaker condition, meaningful whenever $g = g_\varepsilon$ depends on ε, reads

$$\varepsilon \|\partial_t g_\varepsilon - \Delta g_\varepsilon\|_{L^\infty(Q)} + \|\nabla g_\varepsilon\|_{L^\infty(Q)} \leq C. \tag{2.2}$$

This could happen if g_ε is the solution of another PDE, as in the phase field model for solidification [5], or just a regularized version of a rougher g.

Let $\Sigma_0 \subset \Omega$ be a closed oriented manifold of codimension 1 and class C^1, and let d_0 denote the signed distance function to Σ_0 that is positive outside Σ_0. Let ω denote the (continuous) viscosity solution of the nonlinear degenerate parabolic PDE

$$\partial_t \omega - \Big(\delta_{ij} - \frac{\partial_{x_i}\omega \; \partial_{x_j}\omega}{|\nabla \omega|^2}\Big)\partial_{x_i x_j}\omega - g|\nabla\omega| = 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \tag{2.3}$$

satisfying $\omega(\cdot,0) = d_0(\cdot)$ [7], [8], [10], [12]. Such an expression says that level sets of ω evolve formally in their normal direction with velocity $V = \frac{\partial_t \omega}{|\nabla\omega|} = \kappa + g$, as stated in (1.2). Let $\Sigma(t)$ indicate the zero level set of ω. Since $\Sigma(t)$ is independent of the special form of $\omega(\cdot,0)$, provided $\{\omega(\cdot,0) < 0\} = \text{Interior}\,(\Sigma_0)$, $\Sigma(t)$ is called a generalized geometric evolution for (1.2) [7], [10], [12]. Let $I(t)$ be the inside and $O(t)$ the outside of $\Sigma(t)$ defined in terms of ω by

$$I(t) := \{\mathbf{x} \in \Omega : \omega(\mathbf{x},t) < 0\}, \qquad O(t) := \{\mathbf{x} \in \Omega : \omega(\mathbf{x},t) > 0\}.$$

Assume $\Sigma(t)$ remains within Ω for all $t \in [0,T]$, whence $\Omega = \Sigma(t) \cup I(t) \cup O(t)$. This condition is guaranteed, for $g = 0$, whenever Ω contains the convex hull of Σ_0 [10]. Let $d(\cdot,t)$ denote the signed distance function to the front $\Sigma(t)$ that is positive in $O(t)$; thus $d(\cdot,0) = d_0(\cdot)$. Such a function d satisfies the following property in the viscosity sense [3, p.446]:

$$\partial_t d - \Delta d - g\big(\mathbf{x} - d(\mathbf{x},t)\nabla d(\mathbf{x},t), t\big) \geq 0 \qquad \text{in } \{d > 0\}, \tag{2.4}$$

at least provided $\Sigma(t)$ has empty interior (no fattening). When $\Sigma(t)$ has a nonempty interior, $d(\mathbf{x},t)$ must be replaced by the distance between $\mathbf{x}$ and $\Omega^-(t) := \{\omega(\cdot,t) < 0\}$ for all $\mathbf{x} \in \Omega \backslash \overline{\Omega^-(t)}$. Equation (2.4) is also valid, but with ≤ 0, for $-\text{dist}\big(\mathbf{x}, \Omega^+(t)\big)$ for all $\mathbf{x} \in \Omega \backslash \overline{\Omega^+(t)}$. In what follows we will use $d(\mathbf{x},t)$ instead of $\text{dist}\big(\mathbf{x}, \Omega^-(t)\big)$ or $-\text{dist}\big(\mathbf{x}, \Omega^+(t)\big)$ for notational simplicity. Inequality (2.4) means that whenever $\varphi \in C^\infty(Q)$ *is so that* $d - \varphi$ *has a minimum at* $(\mathbf{x}_0, t_0) \in Q$ *where* $d(\mathbf{x}_0,t_0) = \varphi(\mathbf{x}_0,t_0) > 0$, *then*

$$\partial_t \varphi(\mathbf{x}_0,t_0) - \Delta\varphi(\mathbf{x}_0,t_0) - g\big(\mathbf{x}_0 - \varphi(\mathbf{x}_0,t_0)\nabla\varphi(\mathbf{x}_0,t_0), t_0\big) \geq 0. \tag{2.5}$$

In addition the following regularity properties of d hold [9, p.1101]:

(2.6) *d is lower semicontinuous in $\{d > 0\}$, upper semicontinuous in $\{d < 0\}$, continuous from below in time, and Lipschitz continuous in space.*

The variational approximation to $\Sigma(t)$ via a singularly perturbed double obstacle problem reads as follows. We consider the (singular) potential Ψ defined in (1.3) and let $c_0 = \int_{-1}^{1} \sqrt{\Psi(s)}ds = \frac{\pi}{2}$. The subdifferential Ψ' of Ψ is the maximal graph

$$\tfrac{1}{2}\Psi'(s) := \mathrm{sign}^{-1}(s) - s = \begin{cases} (-\infty, 1] & \text{if } s = -1 \\ -s & \text{if } s \in (-1, 1) \\ [-1, +\infty) & \text{if } s = 1. \end{cases}$$

Let u_ε be the solution to the parabolic double obstacle PDE

$$\varepsilon\partial_t u_\varepsilon - \varepsilon\Delta u_\varepsilon - \tfrac{1}{\varepsilon}u_\varepsilon + \mathrm{sign}^{-1}(u_\varepsilon) \ni \tfrac{c_0}{2}g \qquad \text{in } Q, \tag{2.7}$$

subject to the initial and boundary conditions $u_\varepsilon(\mathbf{x}, 0) = \mathrm{sign}\big(d_0(\mathbf{x})\big)$ for $\mathbf{x} \in \Omega$ and $u_\varepsilon = 1$ on $\partial\Omega \times (0, T)$, respectively. Although this problem has both a variational [11] and viscosity interpretation [8], [19], the latter turns out to be more convenient in this setting.

We thus conclude this section with a definition of viscosity solutions for obstacle problems [8], [19]. We say that a function u_ε^+ *is a viscosity supersolution to the double obstacle problem* (2.7) *if and only if* $u_\varepsilon^+ \geq -1$ *and, if* $u_\varepsilon^+ - \varphi$ *attains a minimum at* $(\mathbf{x}_0, t_0) \in Q$ *for* $\varphi \in C^\infty(Q)$ *and* $u_\varepsilon^+(\mathbf{x}_0, t_0) = \varphi(\mathbf{x}_0, t_0) < 1$*, then*

$$\mathcal{J}\varphi := \varepsilon\partial_t\varphi - \varepsilon\Delta\varphi - \tfrac{1}{\varepsilon}\varphi - \tfrac{c_0}{2}g \geq 0 \qquad \text{at } (\mathbf{x}_0, t_0). \tag{2.8}$$

Similarly we can define a viscosity subsolution. A function u_ε is called *viscosity solution* of (2.7) if it is both a super and a subsolution.

3. Traveling waves and asymptotics. We study the traveling wave q_α, solution of the boundary value problem: *given* α*,* $|\alpha| \ll 1$*, we seek* $v_\alpha, x_\alpha \in \mathbb{R}$ *and* $q_\alpha \in C^{1,1}(\mathbb{R})$ *such that* $|q_\alpha(x)| = 1$ *for all* $x \in \mathbb{R}\backslash(-x_\alpha, x_\alpha)$*,* $q_\alpha'(x) > 0$ *for all* $x \in (-x_\alpha, x_\alpha)$*, and*

$$q_\alpha''(x) + q_\alpha(x) - v_\alpha q_\alpha'(x) = -\alpha \qquad \text{in } (-x_\alpha, x_\alpha);$$

v_α is the velocity of the traveling wave. It is easy to see that there exists only one v_α

$$v_\alpha = \tfrac{1}{x_\alpha}\log\left(\tfrac{1+\alpha}{1-\alpha}\right) = \tfrac{2}{c_0}\alpha + \mathcal{O}(\alpha^3)$$

such that $x_\alpha = \frac{\pi}{2\sqrt{1-v_\alpha^2/4}} > \frac{\pi}{2}$, and the corresponding explicit expression of q_α is

$$q_\alpha(x) = \exp\left(v_\alpha\tfrac{x+x_\alpha}{2}\right)(1-\alpha)\Big(\sin\left(\sqrt{1-v_\alpha^2/4}\; x\right) + \tfrac{v_\alpha}{2\sqrt{1-v_\alpha^2/4}}\cos\left(\sqrt{1-v_\alpha^2/4}\; x\right)\Big) - \alpha.$$

Note that $q_\alpha(\mp x_\alpha) = \mp 1$, $q_\alpha'(\mp x_\alpha) = 0$, $q_\alpha''(\mp x_\alpha^\pm) = \pm 1 - \alpha$.

In the sequel we will choose $\alpha(\mathbf{x}, t) := \varepsilon\frac{c_0}{2}g(\mathbf{x}, t)$, and denote the corresponding traveling waves by $q_\varepsilon(x; \mathbf{x}, t)$. We stress that $v_{\alpha(\mathbf{x},t)}$ and $x_{\alpha(\mathbf{x},t)}$ depend implicitly on $(\mathbf{x}, t)$ via $g(\mathbf{x}, t)$. An explicit and simpler representation of q_ε can be obtained as follows by means of asymptotics [14], [15], [17].

The unique absolute minimizer γ of the functional $\mathcal{F}(\varphi) := \int_\mathbb{R} \left(|\varphi'(x)|^2 + \Psi(\varphi(x))\right) dx$, such that $\gamma(0) = 0$, $|\gamma(x)| \leq 1$ in $\mathbb{R}$, and $\lim_{x\to\pm\infty}\gamma(x) = \pm 1$, is given by

$$\gamma(x) := \begin{cases} -1 & \text{if } x < -\frac{\pi}{2} \\ \sin x & \text{if } x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ +1 & \text{if } x > \frac{\pi}{2}. \end{cases}$$

Hence, $\gamma \in C^{1,1}(\mathbb{R})$ and solves the (elliptic) double obstacle problem $\gamma''(x) - \frac{1}{2}\Psi'\big(\gamma(x)\big) \ni 0$ in $\mathbb{R}$. In particular,

$$\gamma''(x) + \gamma(x) = 0 \qquad \forall\, x \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}). \tag{3.1}$$

In addition, let us define the function $\eta \in C^{1,1}(\mathbb{R})$

$$\eta(x) := \begin{cases} \frac{1}{2}\big(x\gamma(x) - c_0 + \gamma'(x)\big) & \text{if } |x| \le \frac{\pi}{2} \\ 0 & \text{if } |x| > \frac{\pi}{2}, \end{cases}$$

which satisfies $|\eta(x)|, |\eta'(x)| < \gamma'(x)$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and solves the problem

$$\eta''(x) + \eta(x) = \gamma'(x) - \tfrac{c_0}{2} \qquad \forall\ x \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}). \tag{3.2}$$

Asymptotics suggests the following approximation Γ of the traveling wave q_ε [14], [15], [17]:

$$\Gamma(x; \mathbf{x}, t) := \gamma(x) + \varepsilon g(\mathbf{x}, t)\eta(x) \qquad \forall\ x \in \mathbb{R},\ (\mathbf{x}, t) \in Q; \tag{3.3}$$

Γ is written without subscript ε for notational simplicity. Note that $\Gamma(\cdot; \mathbf{x}, t) \in C^{1,1}(\mathbb{R})$ is strictly increasing in $(-\frac{\pi}{2}, \frac{\pi}{2})$, because $\Gamma'(x) > \frac{1}{2}\gamma'(x)$ for small ε, and satisfies

$$\Gamma''(x) + \Gamma(x) - \varepsilon g\big(\Gamma'(x) - \tfrac{c_0}{2}\big) = -\varepsilon^2 g^2 \eta'(x) = \gamma'(x)\mathcal{O}(\varepsilon^2) \qquad \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}), \tag{3.4}$$

as results from (3.1) and (3.2); hereafter we use the notation $\Gamma^{(k)} = \partial_x^k \Gamma$. In view of (2.2), the following (formal) property of Γ holds in Q:

$$\varepsilon(\partial_t \Gamma - \Delta_{\mathbf{x}} \Gamma) - 2\nabla_{\mathbf{x}} \Gamma' \cdot \nabla_{\mathbf{x}} d = \varepsilon^2 \eta(\partial_t g - \Delta_{\mathbf{x}} g) - 2\varepsilon\eta' \nabla_{\mathbf{x}} g \cdot \nabla_{\mathbf{x}} d = \gamma' \mathcal{O}(\varepsilon). \tag{3.5}$$

This illustrates the advantage of having the explicit expression (3.3) and corresponding uniform transition region $(-\frac{\pi}{2}, \frac{\pi}{2})$ independent of $(\mathbf{x}, t) \in Q$. Compare with [3].

4. Supersolution: formal discussion. We intend to use (2.4) to motivate the construction of supersolutions u_ε^+ to the double obstacle problem (2.7). Set

$$G := \|\nabla g\|_{L^\infty(Q)} + 1 \qquad \text{and} \qquad \sigma(t) := \Lambda e^{2G(T-t)},$$

where $\Lambda > 0$ is to be selected. With Γ as in (3.3), our candidate for supersolution reads

$$u_\varepsilon^+(\mathbf{x}, t) := \Gamma\big(y(\mathbf{x}, t); \mathbf{x}, t\big) = \gamma\big(y(\mathbf{x}, t)\big) + \varepsilon g(\mathbf{x}, t)\eta\big(y(\mathbf{x}, t)\big) \qquad \forall\ (\mathbf{x}, t) \in Q, \tag{4.1}$$

where y denotes the stretched variable, with time-dependent shift $\frac{\pi}{2} + \sigma(t)$,

$$y(\mathbf{x}, t) := \tfrac{d(\mathbf{x}, t)}{\varepsilon} - \tfrac{\pi}{2} - \sigma(t). \tag{4.2}$$

Lemma 4.1. *For $\Lambda > 0$ sufficiently large, u_ε^+ is a (formal) supersolution to* (2.7).

Proof. Since no confusion is possible, we set $u := u_\varepsilon^+$. We have to show that $\mathcal{J}u = \varepsilon\partial_t u - \varepsilon\Delta u - \frac{1}{\varepsilon}u - \frac{c_0}{2}g \ge 0$. Since

$$\begin{aligned} \varepsilon\partial_t u &= \Gamma'\partial_t d - \varepsilon\sigma'(t)\Gamma' + \varepsilon^2\eta\partial_t g, \\ \varepsilon\Delta u &= \tfrac{1}{\varepsilon}\Gamma'' + \Gamma'\Delta d + 2\varepsilon\eta'\nabla g \cdot \nabla d + \varepsilon^2\eta\Delta g, \end{aligned}$$

we have

$$\begin{aligned} \mathcal{J}u = &-\tfrac{1}{\varepsilon}\big(\Gamma'' + \Gamma - \varepsilon g(\Gamma' - \tfrac{c_0}{2})\big) + \Gamma'\big(\partial_t d - \Delta d - g\big) \\ &- \varepsilon\sigma'(t)\Gamma' - 2\varepsilon\eta'\nabla g \cdot \nabla d + \varepsilon^2\eta(\partial_t g - \Delta g). \end{aligned}$$

We only have to examine the case $-1 \le u(\mathbf{x}, t) < 1$, that is $y < \frac{\pi}{2}$. If $y < -\frac{\pi}{2}$, then $\Gamma = -1$ and $\eta = \eta' = \Gamma' = \Gamma'' = 0$, whence $\mathcal{J}u = \frac{1}{\varepsilon} - \frac{c_0}{2}g > 0$ for ε sufficiently small. If $-\frac{\pi}{2} < y < \frac{\pi}{2}$, instead, then $0 < d(\mathbf{x}, t) < \varepsilon(\sigma(t) + \pi)$ and

$$\big|g(\mathbf{x}, t) - g\big(\mathbf{x} - d(\mathbf{x}, t)\nabla d(\mathbf{x}, t), t\big)\big| \le \|\nabla g\|_{L^\infty(Q)} d(\mathbf{x}, t) \le \varepsilon G\big(\sigma(t) + \pi\big). \tag{4.3}$$

In view of (2.4), we infer that

$$\Gamma'(\partial_t d - \Delta d - g) \geq -\varepsilon G \Gamma'\big(\sigma(t) + \pi\big).$$

Owing to the definition of $\sigma(t)$, (3.4), (3.5) and property $\Gamma' > \frac{1}{2}\gamma'$, we thus have

$$\begin{aligned} \mathcal{J}u &\geq \varepsilon\Gamma'\big(- G\sigma(t) - \sigma'(t) + \mathcal{O}(1)\big) \\ &= \varepsilon\Gamma'\big(G\sigma(t) + \mathcal{O}(1)\big) \geq \varepsilon\Gamma'\big(G\Lambda + \mathcal{O}(1)\big) \geq 0, \end{aligned} \tag{4.4}$$

for a suitable choice of $\Lambda > 0$. Since $\Gamma(\cdot;\mathbf{x},t) \in C^{1,1}(\mathbb{R})$ by construction, we conclude that $\Gamma''\big(y(\mathbf{x},t);\mathbf{x},t\big)$ is a bounded function and so a distribution without mass concentrated on $\big\{y(\mathbf{x},t) = -\frac{\pi}{2}\big\}$. The proof is thus complete. □

Remark 4.1. We see that the effect of shift σ is the term $G\sigma(t)$ that controls $\mathcal{O}(1)$ in (4.4). A similar effect is obtained in [3] upon considering the distance to a perturbed motion $V = \kappa + g + o(1)$, instead of (1.2), but with a time-independent shift. This is only feasible in proving convergence without error estimates; see Remark 7.4 below.

Remark 4.2. It is apparent from (4.3) that the exponential form of σ cannot be avoided unless g is independent of $\mathbf{x}$. In such a case, we could consider the linear shift $\sigma(t) := \Lambda(T-t)$, with $\Lambda > 0$ to be chosen sufficiently large for u_ε^+ to be again a formal supersolution. When $g = 0$, the shift becomes constant, namely $\sigma(t) = 0$. The above argument in Lemma 4.1 still applies and is much simpler than those in [3], [9].

Remark 4.3. A subsolution can be constructed along the same lines, namely

$$u_\varepsilon^- := \Gamma\big(\tfrac{d(\mathbf{x},t)}{\varepsilon} + \tfrac{\pi}{2} + \sigma(t);\mathbf{x},t\big) \qquad \forall\ (\mathbf{x},t) \in Q. \tag{4.5}$$

5. Supersolution: rigorous discussion. We want to prove that $u := u_\varepsilon^+$ defined in (4.1) is a viscosity supersolution of (2.7). Since $u \geq -1$, we only have to demonstrate (2.8). That is, suppose $\varphi \in C^\infty(Q)$ is such that $u - \varphi$ attains a minimum at $(\mathbf{x}_0,t_0) \in Q$ and $u(\mathbf{x}_0,t_0) = \varphi(\mathbf{x}_0,t_0) < 1$. Hence, in view of (4.1) and (4.2),

$$y_0 := y(\mathbf{x}_0,t_0) < \tfrac{\pi}{2} \qquad \text{that is} \qquad d(\mathbf{x}_0,t_0) < \varepsilon\big(\sigma(t_0) + \pi\big).$$

We have to show that

$$\mathcal{J}\varphi = \varepsilon\partial_t\varphi - \varepsilon\Delta\varphi - \tfrac{1}{\varepsilon}\varphi - \tfrac{c_0}{2}g \geq 0 \qquad \text{at } (\mathbf{x}_0,t_0), \tag{5.1}$$

which, in turn, we split into several steps.

1. First of all we point out that we can always assume that the minimum of $u - \varphi$ at $(\mathbf{x}_0,t_0)$ is strict, that is

$$U(\mathbf{x},t) := (u - \varphi)(\mathbf{x},t) > 0 \qquad \text{for } (\mathbf{x},t) \neq (\mathbf{x}_0,t_0). \tag{5.2}$$

In fact, for $0 < \alpha \ll 1$, we have

$$\varphi_\alpha(\mathbf{x},t) := \varphi(\mathbf{x},t) - \alpha\big(|\mathbf{x}-\mathbf{x}_0|^2 + |t-t_0|^2\big) < \varphi(\mathbf{x},t) \qquad \text{for } (\mathbf{x},t) \neq (\mathbf{x}_0,t_0);$$

thus $u - \varphi_\alpha > U \geq 0$. If we obtain (5.1) for φ_α, then

$$\varepsilon\partial_t\varphi - \varepsilon\Delta\varphi - \tfrac{1}{\varepsilon}\varphi - \tfrac{c_0}{2}g \geq 2n\alpha \qquad \text{at } (\mathbf{x}_0,t_0).$$

Taking now $\alpha \downarrow 0$ leads to (5.1) for φ.

2. Let $y_0 < -\frac{\pi}{2}$. By virtue of (4.2) and the last two properties in (2.6), there exists $\alpha > 0$ such that, for all $|\mathbf{x}-\mathbf{x}_0| \leq \alpha$ and $0 \leq t_0 - t \leq \alpha$,

$$y(\mathbf{x},t) < -\tfrac{\pi}{2} \qquad \text{whence} \qquad -1 \leq u(\mathbf{x},t) = \Gamma\big(y(\mathbf{x},t);\mathbf{x},t\big) \leq -1.$$

The facts that u is constant for all those $(\mathbf{x},t)$, and U attains a minimum at $(\mathbf{x}_0,t_0)$ imply

$$-\Delta\varphi \geq 0, \qquad \partial_t\varphi \geq 0, \qquad \text{at } (\mathbf{x}_0,t_0).$$

Therefore $\varphi(\mathbf{x}_0,t_0)=-1$ yields

$$\varepsilon\partial_t\varphi - \varepsilon\Delta\varphi - \tfrac{1}{\varepsilon}\varphi - \tfrac{c_0}{2}g \geq \tfrac{1}{\varepsilon} - \tfrac{c_0}{2}g \geq 0 \qquad \text{at } (\mathbf{x}_0,t_0),$$

for ε sufficiently small, and (5.1) follows.

3. Let $y_0 \geq -\frac{\pi}{2}$, that is $d(\mathbf{x}_0,t_0) \geq \varepsilon\sigma(t_0) > 0$. We first point out that, because of the lower semicontinuity of d (see (2.6)), for $\beta > 0$ sufficiently small there is a neighborhood of $(\mathbf{x}_0,t_0)$ where $y(\mathbf{x},t) > -\frac{\pi}{2} - \beta$ and $d(\mathbf{x},t) > \varepsilon\big(\sigma(t)-\beta\big) > 0$; in addition, $y_0 < \frac{\pi}{2} - \beta$. We would like to use the fact that d satisfies (2.4). To do so, we must be able to construct a smooth function $\delta(\mathbf{x},t)$, close to $d(\mathbf{x},t)$, via inversion of Γ. Since Γ is not strictly increasing we define δ as follows. Let γ_α and η_α be regularizations of γ and η by convolution with a smooth kernel ζ_α, whose support is contained in $(-\alpha,\alpha)$, and $0<\alpha\leq\beta$ is sufficiently small; thus $|\eta_\alpha|, |\eta'_\alpha| \leq \gamma'_\alpha$ in $\mathbb{R}$. Similarly $g_\alpha \in C^\infty(Q)$ is a smooth approximation of g verifying $\|g_\alpha - g\|_{L^\infty(Q)} \leq C\frac{\alpha}{\varepsilon}$, which is consistent with (2.2). Let

$$\Gamma_\alpha(x;\mathbf{x},t) := \gamma_\alpha(x) + \alpha x + \varepsilon g_\alpha(\mathbf{x},t)\eta_\alpha(x) \tag{5.3}$$

and observe that

$$\Gamma'_\alpha(x;\mathbf{x},t) \geq \tfrac{1}{2}\gamma'_\alpha(x) + \alpha \geq \alpha > 0 \qquad \text{for all } x\in\mathbb{R},\ (\mathbf{x},t)\in Q, \tag{5.4}$$

$$\Gamma_\alpha(x;\mathbf{x},t) \to_{\alpha\downarrow 0} \Gamma(x;\mathbf{x},t) \text{ uniformly for } x \text{ in compact sets of } \mathbb{R},\ (\mathbf{x},t)\in Q. \tag{5.5}$$

Consider the function

$$U_\alpha(\mathbf{x},t) := \Gamma_\alpha\big(y(\mathbf{x},t);\mathbf{x},t\big) - \varphi(\mathbf{x},t).$$

Using again that d is lower semicontinuous, in conjunction with (5.2), (5.4), and (5.5), we infer the existence of a point $(\mathbf{x}_\alpha,t_\alpha)$ where U_α attains a local minimum, say ν_α, such that $(\mathbf{x}_\alpha,t_\alpha) \to_{\alpha\downarrow 0} (\mathbf{x}_0,t_0)$ and

$$-\tfrac{\pi}{2} - \beta < y_\alpha := y(\mathbf{x}_\alpha,t_\alpha) < \tfrac{\pi}{2} - \beta. \tag{5.6}$$

The rightmost inequality in (5.6) is a consequence of $y_\alpha \to_{\alpha\downarrow 0} y_0 < \frac{\pi}{2} - \beta$. If this were not true, then there would exist $0 < \xi \leq \beta$ and a subsequence, still labeled y_α, satisfying $y_\alpha \geq y_0 + \xi$ and, in view of the minimality of $(\mathbf{x}_\alpha,t_\alpha)$ and (5.4),

$$\Gamma_\alpha(y_0;\mathbf{x}_0,t_0) - \varphi(\mathbf{x}_0,t_0) = U_\alpha(\mathbf{x}_0,t_0) \geq U_\alpha(\mathbf{x}_\alpha,t_\alpha) \geq \Gamma_\alpha(y_0+\xi;\mathbf{x}_\alpha,t_\alpha) - \varphi(\mathbf{x}_\alpha,t_\alpha).$$

Upon taking $\alpha\downarrow 0$ and using (5.5), $(\mathbf{x}_\alpha,t_\alpha) \to_{\alpha\downarrow 0} (\mathbf{x}_0,t_0)$, $-\frac{\pi}{2} < y_0+\xi < \frac{\pi}{2}$ and that Γ is strictly increasing in $(-\frac{\pi}{2},\frac{\pi}{2})$, we derive the contradiction

$$\Gamma(y_0;\mathbf{x}_0,t_0) \geq \Gamma(y_0+\xi;\mathbf{x}_0,t_0) > \Gamma(y_0;\mathbf{x}_0,t_0).$$

Since $\Gamma_\alpha(\cdot;\mathbf{x},t)$ is strictly increasing, the following relation uniquely defines $\delta\in C^\infty(Q)$:

$$\varphi(\mathbf{x},t) = \Gamma_\alpha\big(z(\mathbf{x},t);\mathbf{x},t\big) - \nu_\alpha, \qquad \text{where} \qquad z(\mathbf{x},t) = \tfrac{\delta(\mathbf{x},t)}{\varepsilon} - \tfrac{\pi}{2} - \sigma(t).$$

Moreover

$$\Gamma_\alpha\big(y(\mathbf{x},t);\mathbf{x},t\big) - \varphi(\mathbf{x},t) = U_\alpha(\mathbf{x},t) \geq \nu_\alpha = \Gamma_\alpha\big(z(\mathbf{x},t);\mathbf{x},t\big) - \varphi(\mathbf{x},t),$$

and equality holds at $(\mathbf{x}_\alpha, t_\alpha)$. We thus deduce that $(\mathbf{x}_\alpha, t_\alpha)$ is also a minimum for $d-\delta$ and $0 < d(\mathbf{x}_\alpha, t_\alpha) = \delta(\mathbf{x}_\alpha, t_\alpha)$. Consequently,

$$\partial_t \delta(\mathbf{x}_\alpha, t_\alpha) - \Delta\delta(\mathbf{x}_\alpha, t_\alpha) - g\big(\mathbf{x}_\alpha - \delta(\mathbf{x}_\alpha, t_\alpha)\nabla\delta(\mathbf{x}_\alpha, t_\alpha), t_\alpha\big) \geq 0, \tag{5.7}$$

as results from (2.5). Since $\varphi, \delta \in C^\infty$, the following pointwise calculation makes sense:

$$\partial_t \varphi = \Gamma'_\alpha\big(\tfrac{1}{\varepsilon}\partial_t\delta - \sigma'(t)\big) + \partial_t \Gamma_\alpha,$$
$$\Delta\varphi = \tfrac{1}{\varepsilon}\Gamma'_\alpha\Delta\delta + \tfrac{1}{\varepsilon^2}\Gamma''_\alpha|\nabla\delta|^2 + \tfrac{2}{\varepsilon}\nabla_{\mathbf{x}}\Gamma'_\alpha\cdot\nabla\delta + \Delta_{\mathbf{x}}\Gamma_\alpha.$$

Property $\sigma'(t) = -2G\sigma(t)$ thus yields

$$\begin{aligned}\varepsilon\partial_t\varphi - \varepsilon\Delta\varphi - \tfrac{1}{\varepsilon}\varphi - \tfrac{c_0}{2}g &= \Gamma'_\alpha\big(\partial_t\delta - \Delta\delta - g(\cdot - \delta\nabla\delta, \cdot)\big)\\ &- \tfrac{1}{\varepsilon}\big(\Gamma''_\alpha|\nabla\delta|^2 + \Gamma_\alpha - \varepsilon g(\Gamma'_\alpha - \tfrac{c_0}{2})\big) + 2\varepsilon G\sigma(t)\Gamma'_\alpha\\ &+ \varepsilon^2\eta_\alpha\big(\partial_t g_\alpha - \Delta g_\alpha\big) - 2\varepsilon\eta'_\alpha\nabla g_\alpha\cdot\nabla\delta + \Gamma'_\alpha\big(g(\cdot - \delta\nabla\delta, \cdot) - g\big).\end{aligned} \tag{5.8}$$

We now intend to show, upon suitably selecting $\Lambda > 0$, that the right hand side of (5.8) is almost nonnegative at $(\mathbf{x}_\alpha, t_\alpha)$. With the aid of (5.4) and (5.7), we readily get

$$\Gamma'_\alpha\big(\partial_t\delta - \Delta\delta - g(\cdot - \delta\nabla\delta, \cdot)\big) \geq 0 \qquad \text{at } (\mathbf{x}_\alpha, t_\alpha).$$

Since $(\mathbf{x}_\alpha, t_\alpha)$ is a minimum for $d-\delta$ we have $|\nabla\delta(\mathbf{x}_\alpha, t_\alpha)| = 1$ [9, p.1104]. In light of (5.6), we distinguish two cases according to whether $-\frac{\pi}{2} - \beta < y_\alpha < -\frac{\pi}{2} + \beta$ or not.

4. Let $-\frac{\pi}{2} + \beta < y_\alpha < \frac{\pi}{2} - \beta$. We invoke (3.4) along with (2.2) and (5.3), to arrive at

$$\begin{aligned}&\Gamma''_\alpha(y_\alpha; \mathbf{x}_\alpha, t_\alpha) + \Gamma_\alpha(y_\alpha; \mathbf{x}_\alpha, t_\alpha) - \varepsilon g(\mathbf{x}_\alpha, t_\alpha)\big(\Gamma'_\alpha(y_\alpha; \mathbf{x}_\alpha, t_\alpha) - \tfrac{c_0}{2}\big)\\ &\quad = \int_{-\pi/2}^{\pi/2}\Big(\Gamma''(s; \mathbf{x}_\alpha, t_\alpha) + \Gamma(s; \mathbf{x}_\alpha, t_\alpha) - \varepsilon g(\mathbf{x}_\alpha, t_\alpha)\big(\Gamma'(s; \mathbf{x}_\alpha, t_\alpha) - \tfrac{c_0}{2}\big)\Big)\\ &\qquad \times \zeta_\alpha(y_\alpha - s)ds + \mathcal{O}(\alpha) = \gamma'_\alpha\mathcal{O}(\varepsilon^2) + \mathcal{O}(\alpha).\end{aligned}$$

Hereafter functions are evaluated at $y_\alpha, \mathbf{x}_\alpha$, and t_α. Owing to (2.2) we get

$$\varepsilon^2\eta_\alpha\big(\partial_t g_\alpha - \Delta g_\alpha\big) - 2\varepsilon\eta'_\alpha\nabla g_\alpha\cdot\nabla\delta = \gamma'_\alpha\mathcal{O}(\varepsilon).$$

Since $|y_\alpha| = \big|\frac{\delta(\mathbf{x}_\alpha, t_\alpha)}{\varepsilon} - \frac{\pi}{2} - \sigma(t_\alpha)\big| < \frac{\pi}{2}$, we have $0 < \delta(\mathbf{x}_\alpha, t_\alpha) < \varepsilon(\sigma(t_\alpha) + \pi)$ and

$$\Gamma'_\alpha\big(g(\cdot - \delta\nabla\delta, \cdot) - g\big) \leq G\delta\Gamma'_\alpha \leq \varepsilon G\Gamma'_\alpha\big(\sigma(t_\alpha) + \pi\big).$$

Therefore, owing to (5.4), we can choose Λ sufficiently large in such a fashion that (5.8) becomes

$$\varepsilon\partial_t\varphi - \varepsilon\Delta\varphi - \tfrac{1}{\varepsilon}\varphi - \tfrac{c_0}{2}g \geq \varepsilon\Gamma'_\alpha\big(G\Lambda + \mathcal{O}(1)\big) + \mathcal{O}(\tfrac{\alpha}{\varepsilon}) \geq -C\tfrac{\alpha}{\varepsilon} \qquad \text{at } (\mathbf{x}_\alpha, t_\alpha). \tag{5.9}$$

5. Let $-\frac{\pi}{2} - \beta < y_\alpha \leq -\frac{\pi}{2} + \beta$. The fact $\Gamma(\cdot; \mathbf{x}, t) \in C^{1,1}(\mathbb{R})$ can be used to write again the above integral representation of the second term in (5.8), but now computed in $(-\frac{\pi}{2} - 2\beta, -\frac{\pi}{2} + 2\beta)$. We easily see that such an integral is of order $\mathcal{O}(\beta)$. Choosing $\Lambda = \mathcal{O}(1)$ appropriately, we again conclude (5.9) but with β instead of α.

The assertion (5.1) is just a consequence of $\varphi \in C^\infty(Q), (\mathbf{x}_\alpha, t_\alpha) \to (\mathbf{x}_0, t_0)$ as $\alpha \leq \beta \downarrow 0$, and (5.9). We finally summarize the preceding derivation as follows. Note that we can construct a subsolution along the same lines.

THEOREM 5.1. *The function u^+_ε defined in* (4.1) *is a viscosity supersolution of the double obstacle problem* (2.7).

6. Comparison. The following result is a crucial tool in comparing viscosity barriers of (2.7), which in particular implies uniqueness for (2.7).

COMPARISON LEMMA 6.1. *Let u_ε^+ be a lower semicontinuous viscosity supersolution and u_ε^- be an upper semicontinuous viscosity subsolution. If $u_\varepsilon^+ \geq u_\varepsilon^-$ at $t=0$ and on $\partial\Omega \times (0,T)$, then $u_\varepsilon^+ \geq u_\varepsilon^-$ for all $(\mathbf{x},t) \in Q$.*

Proof. We split the proof into several steps.

1. Let $\hat{u}_\varepsilon^\pm := \exp(-\lambda t) u_\varepsilon^\pm$, $\hat{g} := \exp(-\lambda t) g$, and $\hat{\lambda} := \varepsilon\lambda - \frac{1}{\varepsilon} > 0$ for $\lambda > 0$ to be chosen later. Then $\hat{u}_\varepsilon^+$ satisfies

$$\varepsilon \partial_t \hat{u}_\varepsilon^+ - \varepsilon \Delta \hat{u}_\varepsilon^+ + \hat{\lambda} \hat{u}_\varepsilon^+ - \tfrac{c_0}{2} \hat{g} \geq 0 \qquad \text{in } \left\{ \hat{u}_\varepsilon^+ < \exp(-\lambda t) \right\},$$

in the viscosity sense [9, p.1116], and so does $\hat{u}_\varepsilon^-$ but with reversed inequalities. Since no confusion is possible, we set $u^\pm = \hat{u}_\varepsilon^\pm$ and $g = \hat{g}$.

2. We argue by contradiction. Suppose $u^+ < u^-$ somewhere and set

$$\nu := \sup_{(\mathbf{x},t)\in Q} (u^- - u^+) > 0.$$

Since $u^- - u^+$ is upper semicontinuous and Q is bounded, such a sup is attained in $\overline{Q}$, say at $(\mathbf{x}_0, t_0)$. The fact that $u^- - u^+ \leq 0$ on $\partial\Omega \times (0,T)$ and $\Omega \times \{0\}$, implies $\mathbf{x}_0 \in \Omega$ and $t_0 > 0$. We do not know whether u^- and u^+ are smooth at $(\mathbf{x}_0, t_0)$ (not even the solution u_ε of (2.7) because u_ε is not better than $W_\infty^{2,1}(Q)$ [11]). We use then a standard argument in the theory of viscosity solutions [8], namely we double the number of variables and at the same time penalize the doubling. Consider the upper semicontinuous function

$$U(\mathbf{x},\mathbf{y},t,s) := u^-(\mathbf{x},t) - u^+(\mathbf{y},s) - \tfrac{1}{\alpha}\left(|\mathbf{x}-\mathbf{y}|^2 + |t-s|^2\right)$$

for $\alpha \downarrow 0$. Since $U(\mathbf{x},\mathbf{x},t,t) = u^-(\mathbf{x},t) - u^+(\mathbf{x},t)$, we see that

$$2 \geq \nu_\alpha := \sup_{(\mathbf{x},\mathbf{y},t,s)\in Q^2} U(\mathbf{x},\mathbf{y},t,s) \geq \nu > 0.$$

Since Q is bounded, we conclude that there exists a point $(\mathbf{x}_\alpha, \mathbf{y}_\alpha, t_\alpha, s_\alpha) \in \overline{Q}^2$ at which the sup is attained, $U(\mathbf{x}_\alpha, \mathbf{y}_\alpha, t_\alpha, s_\alpha) = \nu_\alpha$. An application of Lemma 3.1 in [8, p.15] yields

$$\tfrac{1}{\alpha}\left(|\mathbf{x}_\alpha - \mathbf{y}_\alpha|^2 + |t_\alpha - s_\alpha|^2\right) \to_{\alpha\downarrow 0} 0, \tag{6.1}$$

$$\nu_\alpha \downarrow_{\alpha\downarrow 0} \nu = (u^- - u^+)(\mathbf{x}_0, t_0). \tag{6.2}$$

We report here the argument for completeness. As $\alpha \downarrow 0$ we clearly have that $\{\nu_\alpha\}$ is decreasing, which, in turn, shows the existence of $\lim_{\alpha\downarrow 0} \nu_\alpha$. Since

$$\begin{aligned}\nu_{2\alpha} &\geq u^-(\mathbf{x}_\alpha, t_\alpha) - u^+(\mathbf{y}_\alpha, s_\alpha) - \tfrac{1}{2\alpha}\left(|\mathbf{x}_\alpha - \mathbf{y}_\alpha|^2 + |t_\alpha - s_\alpha|^2\right) \\ &= \nu_\alpha + \tfrac{1}{2\alpha}\left(|\mathbf{x}_\alpha - \mathbf{y}_\alpha|^2 + |t_\alpha - s_\alpha|^2\right),\end{aligned}$$

we get

$$\tfrac{1}{\alpha}\left(|\mathbf{x}_\alpha - \mathbf{y}_\alpha|^2 + |t_\alpha - s_\alpha|^2\right) \leq 2(\nu_{2\alpha} - \nu_\alpha) \to_{\alpha\downarrow 0} 0.$$

This proves (6.1). Since $(\mathbf{x}_\alpha, \mathbf{y}_\alpha, t_\alpha, s_\alpha)$ belongs to a compact set, we can extract a subsequence, still labeled as before, such that

$$(\mathbf{x}_\alpha, \mathbf{y}_\alpha, t_\alpha, s_\alpha) \to_{\alpha\downarrow 0} (\overline{\mathbf{x}}, \overline{\mathbf{y}}, \overline{t}, \overline{s}) = (\overline{\mathbf{x}}, \overline{\mathbf{x}}, \overline{t}, \overline{t}).$$

Since $u^- - u^+$ is upper semicontinuous we deduce

$$\nu \geq u^-(\overline{\mathbf{x}},\overline{t}) - u^+(\overline{\mathbf{x}},\overline{t}) \geq \limsup_{\alpha\downarrow 0} U(\mathbf{x}_\alpha,\mathbf{y}_\alpha,t_\alpha,s_\alpha) = \lim_{\alpha\downarrow 0}\nu_\alpha \geq \nu.$$

So (6.2) follows, and $(\overline{\mathbf{x}},\overline{t})$ can be identified with $(\mathbf{x}_0,t_0)$ above; thus $\overline{\mathbf{x}} \in \Omega, \overline{t} > 0$. Moreover, from (6.1) we see that

$$\mathbf{x}_\alpha,\mathbf{y}_\alpha \in \Omega, \qquad t_\alpha, s_\alpha > 0, \tag{6.3}$$

$$u^-(\mathbf{x}_\alpha,t_\alpha) > -\exp(-\lambda t_\alpha), \qquad u^+(\mathbf{y}_\alpha,s_\alpha) < \exp(-\lambda s_\alpha). \tag{6.4}$$

3. Set

$$U^+(\mathbf{x},t) := u^+(\mathbf{y}_\alpha,s_\alpha) + \tfrac{1}{\alpha}\big(|\mathbf{x}-\mathbf{y}_\alpha|^2 + |t-s_\alpha|^2\big) + \nu_\alpha$$

and note that $U(\mathbf{x},\mathbf{y}_\alpha,t,s_\alpha) = u^-(\mathbf{x},t) - U^+(\mathbf{x},t) + \nu_\alpha$. We then realize that $u^- - U^+$ attains a maximum at $(\mathbf{x}_\alpha,t_\alpha)$, and from (2.8), (6.3), and (6.4) we infer that

$$\varepsilon\partial_t U^+ - \varepsilon\Delta_{\mathbf{x}}U^+ + \hat{\lambda}U^+ - \tfrac{c_0}{2}g \leq 0 \qquad \text{at } (\mathbf{x}_\alpha,t_\alpha);$$

thus

$$\tfrac{2\varepsilon}{\alpha}(t_\alpha - s_\alpha) - \tfrac{2n\varepsilon}{\alpha} + \hat{\lambda}\Big(u^+(\mathbf{y}_\alpha,s_\alpha) + \tfrac{1}{\alpha}\big(|\mathbf{x}_\alpha-\mathbf{y}_\alpha|^2 + |t_\alpha-s_\alpha|^2\big) + \nu_\alpha\Big) \leq \tfrac{c_0}{2}g(\mathbf{x}_\alpha,t_\alpha). \tag{6.5}$$

4. Set

$$U^-(\mathbf{y},s) := u^-(\mathbf{x}_\alpha,t_\alpha) - \tfrac{1}{\alpha}\big(|\mathbf{x}_\alpha-\mathbf{y}|^2 + |t_\alpha-s|^2\big) - \nu_\alpha$$

and note that $U(\mathbf{x}_\alpha,\mathbf{y},t_\alpha,s) = U^-(\mathbf{y},s) - u^+(\mathbf{y},s) + \nu_\alpha$. The function $u^+ - U^-$ attains a minimum at $(\mathbf{y}_\alpha,s_\alpha)$, which in view of (2.8), (6.3), and (6.4) yields

$$\varepsilon\partial_s U^- - \varepsilon\Delta_{\mathbf{y}}U^- + \hat{\lambda}U^- - \tfrac{c_0}{2}g \geq 0 \qquad \text{at } (\mathbf{y}_\alpha,s_\alpha).$$

This can be written as

$$\tfrac{2\varepsilon}{\alpha}(t_\alpha - s_\alpha) + \tfrac{2n\varepsilon}{\alpha} + \hat{\lambda}\Big(u^-(\mathbf{x}_\alpha,t_\alpha) - \tfrac{1}{\alpha}\big(|\mathbf{x}_\alpha-\mathbf{y}_\alpha|^2 + |t_\alpha-s_\alpha|^2\big) - \nu_\alpha\Big) \geq \tfrac{c_0}{2}g(\mathbf{y}_\alpha,s_\alpha). \tag{6.6}$$

5. Subtract now (6.5) from (6.6) to get

$$\begin{aligned} \tfrac{4n\varepsilon}{\alpha} + \hat{\lambda}\Big(U(\mathbf{x}_\alpha,\mathbf{y}_\alpha,t_\alpha,s_\alpha) - \tfrac{1}{\alpha}\big(|\mathbf{x}_\alpha-\mathbf{y}_\alpha|^2 + |t_\alpha-s_\alpha|^2\big) - 2\nu_\alpha\Big)& \\ + \tfrac{c_0}{2}\big(g(\mathbf{x}_\alpha,t_\alpha) - g(\mathbf{y}_\alpha,s_\alpha)\big) \geq 0.& \end{aligned} \tag{6.7}$$

In view of (6.1) and (6.2), we can choose α so small that

$$U(\mathbf{x}_\alpha,\mathbf{y}_\alpha,t_\alpha,s_\alpha) - \tfrac{1}{\alpha}\big(|\mathbf{x}_\alpha-\mathbf{y}_\alpha|^2 + |t_\alpha-s_\alpha|^2\big) - 2\nu_\alpha = -\nu_\alpha + o(1) < -\tfrac{1}{2}\nu$$

and $\frac{c_0}{2}\big|g(\mathbf{x}_\alpha,t_\alpha) - g(\mathbf{y}_\alpha,s_\alpha)\big| \leq \frac{4n\varepsilon}{\alpha}$. Select finally $\lambda > \frac{16n}{\alpha\nu} + \frac{1}{\varepsilon^2}$, which yields $\frac{1}{2}\hat{\lambda}\nu = \frac{1}{2}\big(\varepsilon\lambda - \frac{1}{\varepsilon}\big)\nu > \frac{8n\varepsilon}{\alpha}$. Then (6.7) reads

$$0 < \tfrac{8n\varepsilon}{\alpha} - \tfrac{1}{2}\big(\varepsilon\lambda - \tfrac{1}{\varepsilon}\big)\nu < 0,$$

which is a contradiction. This proves the assertion $u^+ \geq u^-$ in Q. $\square$

7. Convergence and error estimates. Our goal is to prove convergence of the zero level sets $\Sigma_\varepsilon(t)$ to the generalized geometric motion $\Sigma(t)$. Key ingredients are the explicit form of supersolutions of §5 and the comparison lemma of §6.

Let $l_\varepsilon := (\Lambda e^{2GT}+\pi)\varepsilon = (\sigma(0)+\pi)\varepsilon$ and $\Sigma^{\pm}_{0,\varepsilon} := \{\mathbf{x}\in\Omega : d_0(\mathbf{x}) = \mp l_\varepsilon\}$, which is of class C^1. We designate by $\Sigma^{\pm}_\varepsilon(t)$ the generalized evolving fronts that starting from $\Sigma^{\pm}_{0,\varepsilon}$ are governed by (1.2), namely

$$\Sigma^{\pm}_\varepsilon(t) = \{\mathbf{x}\in\Omega : \omega(\mathbf{x},t) = \mp l_\varepsilon\},$$

where ω is the (unique) continuous viscosity solution of (2.3). Let $d^{\pm}_\varepsilon(\cdot,t)$ be the corresponding signed distance functions to $\Sigma^{\pm}_\varepsilon(t)$, and $u^{\pm}_\varepsilon$ be the barriers just constructed in §4 in terms of $d^{\pm}_\varepsilon$ (see (4.1) and (4.5)). If $\Sigma^{\pm}_\varepsilon(t)$ develops interior, then $d^{+}_\varepsilon(\cdot,t)$ must be replaced by $\mathrm{dist}(\cdot,\{\omega(\cdot,t)<-l_\varepsilon\})$ and $d^{-}_\varepsilon(\cdot,t)$ by $-\mathrm{dist}(\cdot,\{\omega(\cdot,t)>l_\varepsilon\})$ for the argument below to apply (see (2.4)). We would like to prove that

$$u^-_\varepsilon(\mathbf{x},t) \le u_\varepsilon(\mathbf{x},t) \le u^+_\varepsilon(\mathbf{x},t) \qquad \forall\ (\mathbf{x},t)\in Q. \tag{7.1}$$

In light of Lemma 6.1 and the fact that u^+_ε is lower semicontinuous and u^-_ε is upper semicontinuous, because of (2.6), we solely have to show that (7.1) is valid on the parabolic boundary of Q. This is certainly the case on $\partial\Omega\times(0,T)$, simply because $u^-_\varepsilon = u^+_\varepsilon = u_\varepsilon = 1$ in this set. In addition, since $d^{\pm}_\varepsilon(\mathbf{x},0) = d_0(\mathbf{x}) \pm l_\varepsilon$, we have for $t=0$ and $\mathbf{x}\in\Omega$:

$$u^+_\varepsilon(\mathbf{x},0) = \Gamma\big(\tfrac{d^+_\varepsilon(\mathbf{x},0)}{\varepsilon} - \tfrac{\pi}{2} - \sigma(0)\big) = \Gamma\big(\tfrac{d_0(\mathbf{x})}{\varepsilon} + \tfrac{\pi}{2}\big) \ge \mathrm{sign}\big(d_0(\mathbf{x})\big) = u_\varepsilon(\mathbf{x},0) \ge u^-_\varepsilon(\mathbf{x},0).$$

The desired inequality (7.1) then follows immediately from Lemma 6.1.

THEOREM 7.1. *For $\mathbf{x}\in I(t)$ (resp. $\mathbf{x}\in O(t)$), there exists $\varepsilon_0(\mathbf{x},t)>0$ such that*

$$u_\varepsilon(\mathbf{x},t) = -1 \qquad (\text{resp. } u_\varepsilon(\mathbf{x},t) = +1) \qquad \forall\ \varepsilon \le \varepsilon_0(\mathbf{x},t).$$

Proof. Let $\mathbf{x}\in I(t) = \{\omega(\cdot,t)<0\}$. For ε sufficiently small, $\omega(\mathbf{x},t) < -l_\varepsilon$, whence $d^+_\varepsilon(\mathbf{x},t) < 0$. This implies $\frac{d^+_\varepsilon(\mathbf{x},t)}{\varepsilon} - \frac{\pi}{2} - \sigma(t) < -\frac{\pi}{2}$ and therefore $-1 \le u_\varepsilon(\mathbf{x},t) \le u^+_\varepsilon(\mathbf{x},t) = -1$, because of (7.1). Similar reasoning for u^-_ε completes the proof. □

Remark 7.1. If $\Sigma(t)$ has empty interior, then Theorem 7.1 establishes convergence of $\Sigma_\varepsilon(t)$ to $\Sigma(t)$. As far as we know, whether such a condition is always valid for the evolution of smooth initial surfaces by mean curvature, namely $g=0$, remains a conjecture [10]. If $g\neq 0$, instead, then $\Sigma(t)$ may develop interior [4].

In order to derive interface error estimates we are forced to assume more regularity of $\Sigma(t)$. We say that $\mathbf{x}\in\Sigma^*(t)$, the *regular* part of $\Sigma(t)$, if $\mathbf{x}\in\Sigma(t)$ and $\omega(\cdot,t)$ is of class C^1 in a neighborhood of $\mathbf{x}$ and satisfies the nondegeneracy condition $|\nabla\omega(\mathbf{x},t)|>0$. Note that ω is only known to be Lipschitz continuous in Q and also that $\Sigma(t) = \Sigma^*(t)$ for as long as the motion is classical, that is before the onset of singularities. Nevertheless, $\Sigma(t) = \Sigma^*(t)$ is known to hold between consecutive singularities for a number of flows, e.g., for surfaces of rotation [2], [18].

Let $\mathrm{thick}\big(\mathcal{T}_\varepsilon(t);\mathbf{x},\mathbf{n}\big)$ denote the thickness of the transition region $\mathcal{T}_\varepsilon(t) := \{\mathbf{x}\in\Omega : |u_\varepsilon(\mathbf{x},t)|<1\}$ in the normal direction $\mathbf{n} := \nabla\omega(\mathbf{x},t)/|\nabla\omega(\mathbf{x},t)|$ across $\mathbf{x}\in\Sigma(t)$.

THEOREM 7.2. *For $\mathbf{x}\in\Sigma^*(t)$, there exists $\varepsilon_0(\mathbf{x},t)>0$ such that*

$$\mathrm{dist}\big(\mathbf{x},\Sigma_\varepsilon(t)\big),\ \tfrac{1}{2}\mathrm{thick}\big(\mathcal{T}_\varepsilon(t);\mathbf{x},\mathbf{n}\big) \le 2(\Lambda e^{2GT}+\pi)|\nabla\omega(\mathbf{x},t)|^{-1}\varepsilon \quad \forall\ \varepsilon\le\varepsilon_0(\mathbf{x},t). \tag{7.2}$$

Proof. Since $u^{\pm}_\varepsilon(\cdot,t) = \mp 1$ on $\Sigma^{\pm}_\varepsilon(t)$ and (7.1) is valid, it suffices to estimate the distance between $\Sigma^{\pm}_\varepsilon(t)$ and $\mathbf{x}\in\Sigma^*(t)$. Using Taylor's formula about $\mathbf{x}$, we obtain for $\mathbf{x}+\delta\mathbf{n}\in\Sigma^-_\varepsilon(t)$, and so $\delta>0$, and ε sufficiently small depending on $(\mathbf{x},t)$,

$$l_\varepsilon = \omega\big(\mathbf{x}+\delta\mathbf{n},t\big) = \omega(\mathbf{x},t) + |\nabla\omega(\mathbf{x},t)|\delta + o(\delta) \ge \tfrac{1}{2}|\nabla\omega(\mathbf{x},t)|\delta.$$

Hence $\delta \le 2l_\varepsilon|\nabla\omega(\mathbf{x},t)|^{-1}$, as asserted in (7.2). □

Remark 7.2. The exponential blow-up of the constant in (7.2) can only be avoided if g is independent of $\mathbf{x}$ (see Remark 4.2). In fact, (7.2) cannot be improved without additional assumptions, as the following radially symmetric flow in two dimensions reveals. Let $g(r,t) = r$ and consider the initial condition $r_\delta(0) = 1 + \delta$. The corresponding evolution is given by $r_\delta(t) = \big(1 + (r_\delta^2(0) - 1)e^{2t}\big)^{1/2}$, which yields $r_\delta(t) - r_0(t) = \mathcal{O}\big(\delta e^{2t}\big)$.

Remark 7.3. Chen and Elliott [6] derived a linear rate of convergence for interfaces before the onset of singularities, say in $[0,t^*)$, valid for the mean curvature flow ($g = 0$):

$$\operatorname{dist}_H\big(\Sigma(t), \Sigma_\varepsilon(t)\big) \le C_{t^\#}\varepsilon \qquad \forall\ t \le t^\# < t^*;$$

here dist_H stands for the Hausdorff distance. In such a regime, however, optimal error estimates of order $\mathcal{O}(\varepsilon^2)$ were established in [14], [15]. The virtue of (7.2) is thus its validity even beyond singularities if the motion is locally smooth.

Remark 7.4. We stress that perturbing the original motion (1.2) as in [3], that is replacing g by $g \pm o(1)$, would not immediately lead to (7.2). This is because $d_\varepsilon^\pm$ would depend on the viscosity solutions $\omega_\varepsilon^\pm$ of the resulting perturbed problems rather than on ω. Even though convergence of $\omega_\varepsilon^\pm$ to ω is known as $\varepsilon \downarrow 0$, we would also need uniform nondegeneracy of $\omega_\varepsilon^\pm$ to hold, which does not seem to be available.

Remark 7.5. We can summarize Theorem 7.2 by simply saying that $\Sigma_\varepsilon(t)$ lies between the surfaces $\omega(\cdot,t) = \pm l_\varepsilon = \pm\mathcal{O}(\varepsilon)$. It is then the profile of ω near a singularity, or equivalently how far the level sets $\Sigma_\varepsilon^\pm(t)$ may become apart in the vicinity of a singular point, that determines the rate of convergence.

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