SYLLABUS: MATH 808F
REPRESENTATIONS OF HECKE ALGEBRAS
SPRING 2021

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1. Course details

This is Math 808F. We will meet MWF 1:00 - 1:50 pm, online. Zoom meetings are accessed through Canvas.

First Lecture: Monday, Jan. 25, 2021.

Last Lecture: Monday, May 10, 2021.

2. Goal of the course

The goal of this course is to prove the Deligne-Langlands conjecture, which classifies Iwahori-spherical irreducible representations of a p-adic group in terms of certain Langlands parameters. This is perhaps the first interesting result in the direction of the local Langlands correspondence for p-adic groups which applies to general groups (not just to specific examples such as GL_n) but the price we pay is that this result only treats a special class of representations: those with fixed vectors under an Iwahori subgroup.

We will focus on split groups, although the theorem is known more generally, for many non-split groups as well. Originally it was proved for split groups with connected center by Kazhdan and Lusztig using equivariant K-theory. Lusztig discovered that in order to handle non-split groups (and those whose center is disconnected) it is better to work with equivariant homology, and we shall follow this approach even though we will focus on split groups. In particular, we plan to make no use of K-theory, but only equivariant (co)homology. The proof involves the reduction to the classification of irreducible representations of graded affine Hecke algebras, which is carried out with the help of equivariant homology of the Steinberg variety.

3. Some topics in the course

- Iwahori subgroups, the Iwahori decomposition.
- Smooth/admissible representations of p-adic groups, Iwahori-Hecke algebras.
- Passing from groups to Hecke algebras via the theory of types.
- Affine Weyl groups, apartments and buildings.
- Some more decompositions: Cartan, Bruhat-Tits, Iwasawa.
- Iwahori-Matsumoto presentation and affine Hecke algebras (AHAs).
- Bernstein presentation and description of the center.
- Graded affine Hecke algebras (GAHAs): definitions and properties.
- Geometric construction of irreducible representations of GAHAs (heart of the course).
- How to recover representations of AHAs from GAHAs (another longish section).
- IF TIME: discuss recent paper of Eugen Hellmann in references below.

- Insert somewhere: Kazhdan-Lusztig polynomials and purity theorem; connection to Schubert varieties; and construction of some representations via (left) cells.

4. Prerequisites

I will assume some familiarity with schemes and varieties as in the first two chapters of Hartshorne's *Algebraic Geometry*. I will assume knowledge of the basic language of categories. I will assume the theory of linear algebraic groups, although students not familar with the general theory can take $G = \mathrm{GL}_n$ throughout the course.

5. List of references

5.1. Main references.

- T. Haines, R. Kottwitz, A. Prasad, *Iwahori-Hecke algebras*, J. Ramanujan Math. Soc. **25**, No.2 (2010), 113-145.
- G. Lusztig. *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. vol. **2**, no. 3 (1989), 599-635.
- G. Lusztig, *Cuspidal local systems and graded Hecke algebras, I*, Publ. Math. de l'IHÉS, t. **67**, (1988), 145-202.
- M. Solleveld, *Affine Hecke algebras and their representations*, arXiv:2009.03007.
- A.-M. Aubert, A. Moussaoui, M. Solleveld, *Graded Hecke algebras for disconnected reductive groups*, In: Geometric aspects of the trace formula, 23–84, Simons Symp., Springer, Cham, 2018.
- E. Hellmann, *On the derived category of the Iwahori-Hecke algebra*, arXiv:2006.03013.

These references will be made available on the Canvas course page.

5.2. Other references.

- D. Kazhdan, G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87**, (1987), 153-215.
- N. Chriss, V. Ginzburg, *Representation Theory and Complex Geometry*, Birkäuser, 1997.
- T. Haines, *On Hecke algebra isomorphisms and types for depth-zero principal series*, available at math.umd.edu/∼tjh.
- T. Haines, *A proof of the Kazhdan-Lusztig purity theorem via the decomposition theorem of BBD*, available at math.umd.edu/∼tjh.
- M. A. de Cataldo, T. Haines, L. Li, *Frobenius semisimplicity for convolution morphisms*, Math. Zeitschrift **289**, Issue 1-2 (2018), 119-169.

6. What is expected of those taking this course for a grade

Mostly, I hope you will attend the lectures and ask a lot of questions. In addition, I exptect to assign a few homework problems which will be aimed at verifying some statements made in class, or at working out some specific example of something we've discussed.

There will be no exams in this course.