ON THE NORMALITY OF SCHUBERT VARIETIES: REMAINING CASES IN POSITIVE CHARACTERISTIC

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Abstract. We study the geometry of equicharacteristic partial affine flag varieties associated to tamely ramified groups G in characteristics $p > 0$ dividing the order of the fundamental group $\pi_1(G_{\mathrm{der}})$. We obtain that most Schubert varieties are not normal and provide an explicit criterion for when this happens. Apart from this, we show, on the one hand, that loop groups of semisimple groups satisfying $p \mid |\pi_1(G_{\mathrm{der}})|$ are not reduced, and on the other hand, that their integral realizations are ind-flat. Our methods allow us to classify all tamely ramified Pappas-Zhu local models of Hodge type which are normal.

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1. Introduction

Partial affine flag varieties are important objects in arithmetic algebraic geometry for their intimate relation to local models of Shimura varieties and moduli stacks of shtukas. They first appeared extensively in the realm of Kac-Moody theory by means of (integral) representation theory of Kac-Moody algebras. They were later reinterpreted via the theory of affine Grassmannians as parametrizing torsors under parahoric group schemes over the formal disk equipped with a trivialization over the punctured one.

In the seminal works of Faltings ([Fal03]), Pappas-Rapoport ([PR08]), Zhu ([Zhu14]) and Pappas-Zhu ([PZ13]), the authors establish several geometric properties of affine flag varieties, such as normality of Schubert varieties or reducedness of the special fiber of local models, under the following working hypothesis: the reductive group G over the non-archimedean local base field is *tamely ramified*, and the residue characteristic $p > 0$ *does not divide* the order of $\pi_1(G_{\mathrm{der}})$, that is, the simply connected cover $G_{\mathrm{sc}} \to G_{\mathrm{der}}$ is an étale isogeny. The first type of restriction has been substantially

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lifted in the work of Levin [Lev16] for Weil-restricted groups, and in [Loua] for absolutely almost simple, wildly ramified groups. The second type of restriction is dealt with in this paper, whose main finding can be summarized as follows: Let $F = k(\!(t)\!)$ be the Laurent series field in the formal variable t with algebraically closed residue field k of characteristic $p > 0$. Let G be an absolutely almost simple, semi-simple, tamely ramified, connected reductive group, $\mathbf{f}$ a facet of its Bruhat-Tits building and $\mathbf{a}$ an alcove containing $\mathbf{f}$ in its closure. For each class $w \in W/W_{\mathbf{f}}$ in the Iwahori-Weyl group quotient, let $S_w = S_w(\mathbf{a}, \mathbf{f})$ be the associated Schubert variety in the partial affine flag variety $\mathcal{F}\ell_{G,\mathbf{f}}$.

Theorem 1.1 (Thm. 2.5, Prop. 6.5)**.** *If p divides $|\pi_1(G)|$, then there are only finitely many Schubert varieties S_w, $w \in W/W_{\mathbf{f}}$ in the partial affine flag variety $\mathcal{F}\ell_{G,\mathbf{f}}$ which are normal. The non-normal Schubert varieties are geometrically unibranch, but neither weakly normal, nor Cohen-Macaulay, nor Frobenius split.*

The existence of non-normal Schubert varieties in bad residue characteristics was first observed by the second named author. This came as a total surprise to us as these seem to be the very first examples of badly behaved, that is, non-normal Schubert varieties in the literature. The easiest such example occcurs for the quasi-minuscule Schubert variety inside the affine Grassmannian for $G = \mathrm{PGL}_2$ in residue characteristic $p = 2$: the complete local ring at the singular point is isomorphic to the k-algebra

$$k[\![x, y, v, w]\!]/(vw + x^2y^2, v^2 + x^3y, w^2 + xy^3, xw + yv).$$

This is a surface singularity which is not weakly normal. Its (weak) normalization morphism identifies with the inclusion map of the subalgebra of $k[\![x, y, z]\!]/(z^2+xy)$ generated by $x, y, v = xz, w = yz$ (see Appendix B).

The reason why non-normal Schubert varieties must exist can be summarized in a few lines. Up to translation by a suitable element in $G(F)$ which stabilizes $\mathbf{a}$, we may assume that S_w lies in the neutral component of $\mathcal{F}\ell_{G,\mathbf{f}}$. Then one has a map

$$S_{\mathrm{sc},w} = S_{\mathrm{sc},w}(\mathbf{a}, \mathbf{f}) \longrightarrow S_w(\mathbf{a}, \mathbf{f}) = S_w \tag{1.1}$$

where $S_{\mathrm{sc},w}$ is the Schubert variety for w inside $\mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}}$ and $G_{\mathrm{sc}} \to G$ is the simply connected cover. The Schubert variety $S_{\mathrm{sc},w}$ is known to be normal by [PR08, Thm. 8.4], and the map (1.1) can be shown to be finite, birational and a universal homeomorphism by using Demazure resolutions. In other words, the map (1.1) is the (weak) normalization morphism of S_w, just as in the example of the quasi-minuscule Schubert variety above. On the other hand, the affine flag variety $\mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}}$ is reduced as an ind-scheme by [PR08, Thm. 6.1], that is, equals the colimit of its Schubert varieties. If all Schubert varieties in $\mathcal{F}\ell_{G,\mathbf{f}}$ were normal, then these two facts would imply the map $\mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}} \to \mathcal{F}\ell_{G,\mathbf{f}}$ is a monomorphism. By looking at tangent spaces, this is clearly not true as soon as the kernel of $G_{\mathrm{sc}} \to G$ is non-étale, or equivalently, as soon as p divides $|\pi_1(G)|$. Exploiting tangent spaces a bit further, we show that the normality of S_w is equivalent to the injectivity of the induced map $T_eS_{\mathrm{sc},w} \to T_eS_w$ on tangent spaces, which yields the following key observation:

Lemma 1.2 (Cor. 2.2)**.** *Let $w \in W/W_{\mathbf{f}}$.*

1. *If S_w is normal, then S_v is normal for all $v \leq w$.*
2. *If S_w is not normal, then S_v is not normal for all $v \geq w$.*

We should however stress that the above reasoning only shows that there are infinitely many non-normal Schubert varieties in $\mathcal{F}\ell_{G,\mathbf{f}}$. In order to give an effective normality criterion, we are led to a deeper study of tangent spaces of Schubert varieties for simply connected groups. In this, we recast in §4 old results of Kumar ([Ku96]), Mathieu ([Ma89]), Ramanathan ([Ra87]) and Polo ([Po94]) in the following fashion.

We lift our whole setting to the Witt vectors $W(k)$ as in [PR08, §§7–9], and denote by $\underline{S}_{\mathrm{sc},w} \subset \underline{\mathcal{F}\ell}_{G_{\mathrm{sc}},\mathbf{f}}$ the lift to $W(k)$ of $S_{\mathrm{sc},w} \subset \mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}}$ which comes equipped with a section $e\colon \mathrm{Spec} W(k) \to \underline{S}_{\mathrm{sc},w}$ given by the base point. Given any ample line bundle $\mathcal{L}$ on $\underline{\mathcal{F}\ell}_{G_{\mathrm{sc}},\mathbf{f}}$, we obtain the Kac-Moody action of $T_e\underline{\mathcal{F}\ell}_{G_{\mathrm{sc}},\mathbf{f}}$ on $\Gamma(\underline{\mathcal{F}\ell}_{G_{\mathrm{sc}},\mathbf{f}}, \mathcal{L})^\vee$, see §5.3.

Theorem 1.3 (Lem. 5.9)**.** *For any $W(k)$-algebra R, the R-valued tangent space*

$$T_e\underline{S}_{\mathrm{sc},w}(R) \;=\; \mathrm{Hom}_{W(k)}(e^*\Omega_{\underline{S}_{\mathrm{sc},w}/W(k)}, R)$$

identifies with the submodule of $T_e\underline{\mathcal{F}\ell}_{G_{\mathrm{sc}},\mathbf{f}}(R)$ consisting of those X such that $X\Theta_{\mathcal{L}}^\vee$ lies in $\Gamma(\underline{S}_w,\mathcal{L})^\vee\otimes R$, where $\Theta_{\mathcal{L}}\in\Gamma(\underline{\mathcal{F}\ell}_{G_{\mathrm{sc}},\mathbf{f}},\mathcal{L})$ is the usual theta divisor attached to $\mathcal{L}$.

This formula can in principle be used to determine whether a given Schubert variety is normal or not (see Corollary 5.12). We also think that it is of independent interest to have a good source for this material (some of which was known before in related contexts), and that it would surely help in a future classification of all normal Schubert varieties when $p \mid |\pi_1(G_{\mathrm{der}})|$.

It is not clear to us whether tangent spaces of Schubert varieties can be computed in a characteristic-independent way determined by the characteristic 0 description, see Remark 4.4 which comments on the argument in [Po94, Cor. 4.1].

The key to Theorem 1.1 is to show that the tangent spaces of quasi-minuscule Schubert varieties in twisted affine Grassmannians for absolutely special vertices in characteristic $p>0$ are big enough, see Proposition 6.1. Thanks to some elementary observations (see Lemmas 5.5 and 5.6) the calculation can be reduced to characteristic 0 where we identify the tangent spaces with tangent spaces at minimal nilpotent orbits, see Appendix C. This uses the exponential map and representation-theoretic methods. For split groups, this relation to minimal nilpotent orbits is well known and easy [MOV05, §2.10]. For twisted groups, our method extends the method from [HRb, §8] and requires a fine analysis of twisted root systems. As a consequence, quasi-minuscule Schubert varieties are never normal if $p \mid |\pi_1(G_{\mathrm{der}})|$. From here we use our key observation in Lemma 1.2 along with combinatorial methods to finish the proof of Theorem 1.1. In particular, we reprove in Proposition 6.3 below some recent results from [BH, Thm. 4.1] for split groups and extend these to the case of twisted groups.

Let us mention two other contributions of this paper to the understanding of the geometry of affine flag varieties: reducedness and ind-flatness. For the rest of the introduction, let G be an arbitrary tamely ramified, reductive F-group. As we stated earlier, simply connected affine flag varieties are reduced and a similar result holds for all semisimple groups G such that $p\nmid|\pi_1(G_{\mathrm{der}})|$ by [PR08, Thm. 6.1]. On the other hand, affine flag varieties of non-semisimple reductive groups are non-reduced. In [PR08, Ex. 6.5], the same authors suggest that the same should hold for the group PGL_2 in characteristic 2.

Theorem 1.4 (Prop. 7.7, Prop. 7.10)**.** *The partial affine flag variety $\mathcal{F}\ell_{G,\mathbf{f}}$ is reduced if and only if G is semisimple and $p\nmid|\pi_1(G_{\mathrm{der}})|$.*

We give two different proofs of this result. If G is split, we contemplate the module of so-called distributions, that is, higher differential operators of $\mathcal{F}\ell_{G,\mathbf{f}}$ supported at the origin e and prove that the homomorphism $\mathrm{Dist}(\mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}},e)\to\mathrm{Dist}(\mathcal{F}\ell_{G,\mathbf{f}},e)$ is not surjective in bad characteristic, implying non-reducedness, by essentially analyzing the effect of the multiplication-by-p map on Grassmannians. If G is tamely ramified, we factor the homomorphism $\mathrm{Res}_{F/F^p}G_{\mathrm{sc}}\to\mathrm{Res}_{F/F^p}G$ of pseudo-reductive groups as an epimorphism to a pseudo-reductive group $\overline{G}$ and a closed immersion whose image is strictly smaller than $\mathrm{Res}_{F/F^p}G$ - this works under the hypothesis that G is semisimple and p divides $|\pi_1(G)|$. Then we use the recently developed Bruhat-Tits theory for pseudo-reductive groups from [Loub] to prove that $\mathcal{F}\ell_{\overline{G},\mathbf{f}}\to\mathcal{F}\ell_{\mathrm{Res}_{F/F^p}G,\mathbf{f}}=\mathcal{F}\ell_{G,\mathbf{f}}$ is a closed immersion, but not an isomorphism, for Lie-algebraic reasons.

Another natural question concerns the behavior of the integral realizations $\underline{\mathcal{F}\ell}_{G,\mathbf{f}}$ of the affine flag varieties over the Witt vectors (or just affine Grassmannians of split groups over integers). We are able to show:

Theorem 1.5 (Prop. 3.4, Prop. 8.8, Prop. 8.9)**.** *The ind-scheme $\underline{\mathcal{F}\ell}_{G,\mathbf{f}}$ is ind-flat over $W(k)$. It is reduced if and only if G is semisimple. In general, the reduced locus $(\underline{\mathcal{F}\ell}_{G,\mathbf{f}})_{\mathrm{red}}$ coincides with the union of the integral Schubert varieties $\underline{S}_w=\underline{S}_w(\mathbf{a},\mathbf{f})$, $w\in W/W_{\mathbf{f}}$. Furthermore, for fixed $w\in W/W_{\mathbf{f}}$, the following are equivalent:*

(1) *The Schubert variety S_w over k is normal;*
(2) *its integral realization $\underline{S}_w$ over $W(k)$ is normal;*
(3) *the special fibre $\underline{S}_w \otimes k$ is reduced, hence equals S_w.*

The proof of ind-flatness relies on computing the formal completion of the affine flag variety along the identity section, similar to Faltings' work [Fal03]. For this, we compare the affine flag variety to its flat closure and it suffices, as both are ind-Noetherian, to show that their functors restricted to strictly Henselian Artinian local rings coincide, see Lemma 8.6. This can be achieved by translating with the positive loop group and representative $W(k)$-sections $\dot{w}$ of the Iwahori-Weyl group, so that those rings are supported at the identity section. Here we invoke the fake open cell to reduce the ind-flatness to the cases of tori and unipotent groups where it is easy to check. The determination of the reduced locus is an immediate consequence because partial affine flag varieties for semi-simple groups in characteristic 0 are reduced. Finally, the equivalent conditions characterizing normal Schubert varieties are easily deduced by standard methods, see Proposition 3.4. We also refer the reader to Appendix B for the calculation of an integral Schubert variety whose reduction to characteristic 2 is not reduced.

Theorem 1.5 is strongly connected with the theory of local models as follows. Let F temporarily denote a discretely valued, complete field of characteristic 0 with algebraically closed residue field k of characteristic $p > 0$, G a tamely ramified reductive F-group and S a maximal F-split torus of G. For each facet $\mathbf{f} \subset \mathscr{A}(G, S, F)$, Pappas-Zhu [PZ13] have constructed a parahoric group scheme $\underline{\mathcal{G}}_{\mathbf{f}}$ over $\mathcal{O}_F[t]$ lifting $\mathcal{G}_{\mathbf{f}}$ along the specialization $\mathcal{O}_F[t] \to \mathcal{O}_F$, sending t to a preferred choice of uniformizer $\varpi \in \mathcal{O}_F$, see [PZ13, Thm. 4.1]. We note that the construction of this group scheme for split groups is easy and that the essential difficulty lies in its construction for twisted groups, see [MRR, Exam. 3.3]. This group scheme is then used together with the Beilinson-Drinfeld affine Grassmannian [BD99] to construct so-called Pappas-Zhu local models $\mathbb{M} = \mathbb{M}(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$ where $\{\mu\}$ is a geometric conjugacy class of cocharacters of G. Note that the notation $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$ seems to have first appeared in the survey article of Pappas-Rapoport-Smithling, see [PRS13], and first termed LM triple by He-Pappas-Rapoport in [HPR20, §2.1] (but always under the assumption that $\{\mu\}$ is minuscule). Recall that the reduction of the special fiber of a PZ local model is always given by the admissible locus $\mathcal{A}(G, \{\mu\})$ ([HRa, Thm. 6.12]), that is, by a certain explicit union of Schubert varieties in the partial affine flag variety over k.

Corollary 1.6 (Cor. 9.2)**.** *Assume p divides the order of $\pi_1(G_{\mathrm{der}})$.*

(1) *If every Schubert variety in the admissible locus $\mathcal{A}(G, \{\mu\})$ is normal, then $\mathbb{M}$ is normal and its special fiber is reduced. This is the case when $\bar{\mu}$ is minuscule for the échelonnage roots and $\mathbf{f}$ contains a special vertex in its closure.*
(2) *If any $(\mathbf{f}, \mathbf{f})$-Schubert variety inside the admissible locus $\mathcal{A}(G, \{\mu\})$ is not normal, then $\mathbb{M}$ is not normal and its special fiber is not reduced.*

For details on part (1) we refer to Proposition 9.1 below and [HRc, Thm. 2.1, Rem. 2.2]. For (2), suppose one of the Schubert varieties inside $\mathcal{A}(G, \{\mu\})$ is not normal. Then the irreducible component containing this Schubert variety is not normal as well by our key observation in Lemma 1.2. By comparing the Pappas-Zhu local model with its normalization (which is the Pappas-Zhu local model of some z-extension of G), we see that the special fiber cannot be reduced: compute global sections of line bundles and compare with the generic fiber by flatness. Hence, the Pappas-Zhu local model itself is not normal and its special fiber is not reduced, see also [HRc, Rem. 2.4]. In fact, this nuisance appears even if we assume that $\{\mu\}$ is minuscule but $\bar{\mu}$ is sufficiently large for the échelonnage root system (which is possible if the ramification degree of G is sufficiently large). More concretely, we give examples with restriction of scalars along ramified extensions or for unitary groups along ramified extensions, see Examples 9.3 and 9.4.

Finally, we use Corollary 1.6 to classify in Proposition 9.7 all tamely ramified PZ local models of Hodge type which are normal. We refer to §9 for the definition of Hodge and of abelian type.

Proposition 1.7 (Prop. 9.7)**.** *Let $(G, \{\mu\})$ be of abelian type with a Hodge central lift $(G_1, \{\mu_1\})$, and let $\mathbb{M}_1$ be the PZ local model attached to $(G_1, \{\mu_1\}, \mathcal{G}_{\mathbf{f},1})$. Then the following properties hold:*

(1) *If $p > 2$ or G_{ad} has no D-factors, then $\mathbb{M}_1$ is normal.*
(2) *If $p = 2$ and $(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\})$ is $\breve{F}$-simple of type $D_n^{\mathbb{H}}$, $n \geq 5$, then $\mathbb{M}_1$ is non-normal for all sufficiently large $\bar{\mu}$.*
(3) *If $p = 2$ and $(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\})$ is $\breve{F}$-simple of type $D_{2m+1}^{\mathbb{R}}$, $m \geq 2$, then $\mathbb{M}_1$ is normal.*
(4) *If $p = 2$ and $(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\})$ is $\breve{F}$-simple of type $D_{2m}^{\mathbb{R}}$, $m \geq 2$, and $\bar{\mu}$ is sufficiently large, then $\mathbb{M}_1$ is normal if and only if $G_{1,\mathrm{der}} = G_{1,\mathrm{sc}}$.*

These realizations are viewed via a Hodge embedding as flat closed subschemes of ordinary partial affine flag varieties for GL_n, as was done in [KP18]. The upshot is that, for $(G, \{\mu\})$ of abelian type, the Hodge embedding can always be arranged to give normal PZ local models except if $p = 2$ and $(G, \{\mu\})$ defines sufficiently ramified PEL data of type $D_n^{\mathbb{H}}$.

Let us finish with a side remark that has minimal relevance for our paper. Recall that Scholze-Weinstein construct in [SW20] a certain v-sheaf $\mathrm{Gr}^{\mathrm{dR}}_{\mathcal{G}_{\mathbf{f}}, \leq\{\mu\}}$ over $\mathcal{O}_E$, see [SW20, §21], and conjecture that for minuscule $\{\mu\}$ it is representable by a unique flat projective $\mathcal{O}_E$-scheme $\mathbb{M}^{\mathrm{loc}}_{\mathrm{SW}}$ with reduced special fiber, see [SW20, Conj. 21.4.1]; they prove this for EL and PEL cases (when $p > 2$), see [SW20, Cor. 21.6.10]. Motivated by [SW20], He-Pappas-Rapoport associated in [HPR20] to any tamely ramified LM triple $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$ (with $\{\mu\}$ assumed to be minuscule) the local model $\mathbb{M}^{\mathrm{loc}}_K(G, \{\mu\})$ given as the (weak) normalization of the (crude) PZ local model $\mathbb{M}$. They showed in [HPR20, Thm. 2.14, Cor. 2.15] that this satisfies the characterization of the conjectural local model of [SW20] in almost all tamely ramified abelian type cases, and their constraints were subsequently removed by the second author, see [Loua, Prop. 1.8]. Furthermore, the work of Levin (see [Lev16]) and the second author ([Loua], [Louc]) yield also local models with reduced special fiber for wildly ramified LM triples, satisfying the description in [SW20, Conj. 21.4.1] in the abelian type case.

In some sense, our results can be seen as vindicating the point of view of building normality into the construction of local models, since we show that non-normal affine Schubert varieties are abundant and unavoidable in nature, when p divides $|\pi_1(G_{\mathrm{der}})|$. From this point of view, Proposition 9.7 asserts that a Hodge embedding almost always induces a corresponding closed immersion of the normalized local models, but that it can also fail to do so if $p = 2$, G is even orthogonal and $\bar{\mu}$ is large. Note however that the corresponding morphism of v-sheaves is a closed immersion, as exploited in [Loua, §3.11.].

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1.2. **Conventions on ind-schemes.** We recall some basic results on ind-schemes, see [Ri, §1] for details. An ind-scheme is a functor $X\colon \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ from the category of affine schemes such that there exists a presentation as functors $X = \mathrm{colim} X_i$ where $\{X_i\}_{i\in I}$ is a filtered system of schemes X_i with transition maps being closed immersions. Maps of ind-schemes are natural transformations of functors. We denote by IndSch the category of ind-schemes which is locally small. It contains the category of schemes as a full subcategory, is closed under fibre products and has $\mathrm{Spec}(\mathbb{Z})$ as final object. We identify $\mathrm{AffSch}^{\mathrm{op}}$ with the category of Rings whenever convenient. Note that every ind-scheme defines an fpqc sheaf on the category of affine schemes. Moreover, if $X = \mathrm{colim}_i X_i$, $Y = \mathrm{colim}_j Y_j$ where each X_i is quasi-compact, then

$$\mathrm{Hom}_{\mathrm{IndSch}}(X, Y) \;=\; \mathrm{colim}_i \lim_j \mathrm{Hom}_{\mathrm{Sch}}(X_i, Y_j).$$

If S is a scheme, then an S-ind-scheme X is an ind-scheme X together with a map of functors $X \to S$. If $S = \mathrm{Spec}(R)$ is affine, we also use the term R-ind-scheme.

2. A normality criterion

Let k be an algebraically closed field, and let $F = k(\!(t)\!)$ denote the Laurent series field. Let G be a (connected) reductive F-group which splits over a tamely ramified extension of F.

Let $\mathbf{f} \subset \mathscr{B}(G, F)$ be a facet in the Bruhat-Tits building, and denote by $\mathcal{G}_{\mathbf{f}}$ the associated parahoric $\mathcal{O}_F$-group scheme. The loop group LG (resp. $L^+\mathcal{G}_{\mathbf{f}}$) is the functor on the category of k-algebras R defined by $LG(R) = G(R(\!(t)\!))$ (resp. $L^+\mathcal{G}_{\mathbf{f}}(R) = \mathcal{G}_{\mathbf{f}}(R[\![t]\!])$). Then $L^+\mathcal{G}_{\mathbf{f}} \subset LG$ is a subgroup functor, and the *twisted affine flag variety* is the étale quotient

$$\mathcal{F}\ell_{G,\mathbf{f}} \overset{\text{def}}{=} LG/L^+\mathcal{G}_{\mathbf{f}},$$

which is representable by an ind-projective k-ind-scheme.

Let $S \subset G$ be a maximal F-split torus whose apartment $\mathscr{A} = \mathscr{A}(G, S, F)$ contains $\mathbf{f}$. We fix an alcove $\mathbf{a} \subset \mathscr{A}$ which contains $\mathbf{f}$ in its closure. Fixing also a special vertex $\mathbf{0}$ in the closure of $\mathbf{a}$, we may identify $\mathscr{A}$ with the vector space $X_*(S)_{\mathbb{R}}$, and, following Bruhat-Tits, we obtain an action of the Iwahori-Weyl group $W = W(G, S, F)$ on $\mathscr{A}$ and thus an isomorphism $W \xrightarrow{\sim} W_{\text{aff}} \rtimes \Omega_{\mathbf{a}}$ where W_{aff} denotes the affine Weyl group and where $\Omega_{\mathbf{a}}$ is an abelian group isomorphic to the subgroup of W preserving $\mathbf{a}$. The left-$L^+\mathcal{G}_{\mathbf{a}}$-orbits inside $\mathcal{F}\ell_{\mathbf{f}}$ are enumerated by the quotient $W/W_{\mathbf{f}}$, where $W_{\mathbf{f}} \subset W_{\text{aff}}$ is the subgroup of the affine Weyl group of reflections fixing $\mathbf{f}$. For each class $w \in W/W_{\mathbf{f}}$, we define the Schubert variety

$$S_w = S_w(\mathbf{a}, \mathbf{f}) \subset \mathcal{F}\ell_{G,\mathbf{f}}$$

as the reduced $L^+\mathcal{G}_{\mathbf{a}}$-orbit closure of $\dot{w} \cdot e$ where $e \in \mathcal{F}\ell_{G,\mathbf{f}}(k)$ is the base point and $\dot{w} \in LG(k)$ is any representative of the class w. Then S_w is a projective k-variety. The choice of $\mathbf{a}$ equips the quotient $W/W_{\mathbf{f}}$ with a length function $l = l(\mathbf{a}, \mathbf{f})$ and a Bruhat partial order $\leq$ satisfying $\dim(S_w) = l(w)$, and $S_v \subset S_w$ if and only if $v \leq w$ for $v, w \in W/W_{\mathbf{f}}$, see [Ri13, Prop. 2.8].

Let $\phi \colon G_{\text{sc}} \to G_{\text{der}} \subset G$ be the simply connected cover. Then $S_{\text{sc}} := \phi^{-1}(S)^o \subset \phi^{-1}(T)^o =: T_{\text{sc}}$ is a maximal F-split torus contained in a maximal torus. This induces a map on apartments $\mathscr{A}(G_{\text{sc}}, S_{\text{sc}}, F) \to \mathscr{A}(G, S, F)$ under which the facets correspond bijectively to each other. We denote the preimage of $\mathbf{f}$ by the same letter. The map $G_{\text{sc}} \to G$ extends to a map on parahoric group schemes $\mathcal{G}_{\text{sc},\mathbf{f}} \to \mathcal{G}_{\text{sc},\mathbf{f}}$, and hence to a map on twisted partial affine flag varieties $\mathcal{F}\ell_{G_{\text{sc}},\mathbf{f}} \to \mathcal{F}\ell^o_{G,\mathbf{f}}$ onto the neutral component. We are interested in comparing their Schubert varieties.

The natural map on Iwahori-Weyl groups

$$W_{\text{sc}} = W(G_{\text{sc}}, S_{\text{sc}}, F) \longrightarrow W(G, S, F) = W,$$

is injective and its image identifies with the affine Weyl group W_{aff} compatibly with the subgroup $W_{\mathbf{f}}$. Thus, for each class $w \in W_{\text{aff}}/W_{\mathbf{f}}$ we get a map of projective k-varieties

$$S_{\text{sc},w} = S_{\text{sc},w}(\mathbf{a}, \mathbf{f}) \longrightarrow S_w(\mathbf{a}, \mathbf{f}) = S_w. \tag{2.1}$$

Proposition 2.1. *For each class $w \in W_{\text{aff}}/W_{\mathbf{f}}$, the following statements are equivalent:*

1. *The Schubert variety S_w is normal.*
2. *The map (2.1) is an isomorphism.*
3. *The map (2.1) induces an injective map on tangent spaces at the base points.*

Proof. (1)⇒(2): The map (2.1) is a finite birational universal homeomorphism by [HRc, Prop. 3.5], and thus is an isomorphism whenever S_w is normal.

(2)⇒(1): Since G splits over a tamely ramified extension of F, the Schubert variety $S_{\text{sc},w} \subset \mathcal{F}\ell_{G_{\text{sc}},\mathbf{f}}$ is normal by [PR08, Thm. 0.2], and so is S_w whenever (2.1) is an isomorphism.

(2)⇒(3): This is trivial.

(3)⇒(2): The locus in $S_{\text{sc},w}$, where (2.1) is an isomorphism, is non-empty, open and $L^+\mathcal{G}_{\text{sc},\mathbf{a}}$-invariant. Thus, it suffices to show that the map of local rings at the base points

$$\mathcal{O} := \mathcal{O}_{S_w,e} \longrightarrow \mathcal{O}_{S_{\text{sc},w},e} =: \mathcal{O}_{\text{sc}}$$

is an isomorphism. Here e denotes the base point of both $\mathcal{F}\ell_{G_{\text{sc}},\mathbf{f}}$ and $\mathcal{F}\ell_{G,\mathbf{f}}$. As (2.1) is a finite birational map between integral schemes, the map $\mathcal{O} \hookrightarrow \mathcal{O}_{\text{sc}}$ is a finite ring extension which induces an isomorphism on fraction fields. If (3) holds, then the map (2.1) is unramified at the base points by [StaPro, 0B2G] so that $\mathfrak{m}\mathcal{O}_{\mathrm{sc}} = \mathfrak{m}_{\mathrm{sc}}$ for the maximal ideals. An application of Nakayama's Lemma [StaPro, 00DV (6)] to the finite map $\mathcal{O} \hookrightarrow \mathcal{O}_{\mathrm{sc}}$ (both viewed as $\mathcal{O}$-modules) shows that this map is surjective as well. This finishes the proof of the proposition. □

Corollary 2.2. *Let $w \in W_{\mathrm{aff}}/W_{\mathbf{f}}$.*

(1) *If S_w is normal, then S_v is normal for all $v \leq w$.*
(2) *If S_w is not normal, then S_v is not normal for all $v \geq w$.*

Proof. Both parts (1) and (2) are immediate from Proposition 2.1 (3). □

Let us also record the following two useful results. We have an isomorphism $W \xrightarrow{\sim} W_{\mathrm{aff}} \rtimes \Omega_{\mathbf{a}}$, where $\Omega_{\mathbf{a}}$ is the subgroup of W preserving $\mathbf{a}$, see §6.1. The properties of Schubert varieties indexed by W reduce to those indexed by W_{aff}, as follows: for any $w \in W_{\mathrm{aff}}$ and $\tau \in \Omega_{\mathbf{a}}$, there is an isomorphism

$$S_{w\tau}(\mathbf{a}, \mathbf{f}) \cong S_w(\mathbf{a},\, \tau\mathbf{f}),$$

where $\tau\mathbf{f}$ is the image of $\mathbf{f}$ under the action of τ on facets in the boundary of $\mathbf{a}$.

Proposition 2.3. *Let $w \in W_{\mathrm{aff}}$ and $\tau \in \Omega_{\mathbf{a}}$. The following are equivalent:*

(1) *$S_{w\tau\eta}(\mathbf{a}, \mathbf{a})$ is normal for all $\eta \in W_{\mathbf{f}}$;*
(2) *$S_{w\tau\eta_0}(\mathbf{a}, \mathbf{a})$ is normal for $\eta_0 \in W_{\mathbf{f}}$ such that $w\tau\eta_0$ is right $\mathbf{f}$-maximal;*
(3) *$S_{w\tau}(\mathbf{a}, \mathbf{f})$ is normal.*

Proof. By the above discussion, we immediately reduce to the case $\tau = e$. The implication (1)⇒(2) is obvious, and the opposite implication follows from Corollary 2.2. For (2)⇔(3) use the fact that the inverse image of $S_w(\mathbf{a}, \mathbf{f})$ under the smooth surjective morphism $\mathcal{F}\ell_{\mathbf{a}} \to \mathcal{F}\ell_{\mathbf{f}}$ is the Schubert variety $S_{w\eta_0}(\mathbf{a}, \mathbf{a})$ We conclude by observing that normality is local for the smooth topology, see [StaPro, Tag 034F]. □

Lemma 2.4. *Let $\tau \in \Omega_{\mathbf{a}}$. For each class $w \in W_{\mathrm{aff}}/W_{\mathbf{f}}$, the $(\mathbf{a}, \mathbf{f})$-Schubert variety $S_w \subset \mathcal{F}\ell_{G,\mathbf{f}}$ is normal (resp. smooth) if and only if the $(\mathbf{a}, \tau\mathbf{f})$-Schubert variety $S_{\tau w \tau^{-1}} \subset \mathcal{F}\ell_{G,\tau\mathbf{f}}$ is normal (resp. smooth).*

Proof. First note that the class of $\dot{\tau} \cdot \dot{w} \cdot \dot{\tau}^{-1}$ inside $W_{\mathrm{aff}}/W_{\tau\mathbf{f}}$ is well-defined where $\dot{\tau}, \dot{w} \in LG(k)$ are any representatives. Thus, the $(\mathbf{a}, \tau\mathbf{f})$-Schubert variety $S_{\tau w\tau^{-1}}$ is well-defined. Further, the isomorphism $LG \to LG, g \mapsto \dot{\tau} g \dot{\tau}^{-1}$ descends to an isomorphism $\mathcal{F}\ell_{G,\mathbf{f}} \to \mathcal{F}\ell_{G,\tau\mathbf{f}}$ mapping the $(\mathbf{a}, \mathbf{f})$-Schubert variety S_w isomorphically onto the $(\mathbf{a}, \tau\mathbf{f})$-Schubert variety $S_{\tau w\tau^{-1}}$. This proves the lemma. □

Let us state one of the main results of the paper, which will be proved as soon as we have built upon our knowledge of tangent spaces of Schubert varieties. In this paper, a group will be termed *absolutely almost simple* if its absolute Dynkin diagram is connected, and it will be called *absolutely simple* if it is absolutely almost simple and adjoint.

Theorem 2.5. *Suppose G is an absolutely almost simple, semisimple group such that its simply connected cover is a non-étale isogeny. Then $\mathcal{F}\ell_{G,\mathbf{f}}$ contains only finitely many Schubert varieties which are normal.*

The proof is given in §6 below, see Proposition 6.5. Notice that we can easily find non absolutely almost simple semisimple groups having infinitely many normal Schubert varieties. However, it is still true that the great majority of them are not normal: indeed, as soon as all of their projections to the partial affine flag variety of the adjoint factors of G have sufficiently large dimension, then the Schubert varieties will not be normal.

3. Tame liftings and negative loop groups

In this section, we explain how to lift Schubert varieties from characteristic $p > 0$ to characteristic 0, and set the stage for the calculation of tangent spaces.

3.1. **Tame liftings of groups.** Let k be an algebraically closed field, and let $F = k(\!(t)\!)$ denote the Laurent series field. Let G be an absolutely almost simple, tamely ramified reductive F-group, and assume that G is it has the same splitting field as its simply connected group (equivalently, as its adjoint group). We follow the presentation in [PR08, §7][1]. Let F'/F be the tamely ramified splitting field of G. The extension F'/F is a cyclic Galois extension of degree $e = 1$, $e = 2$ or $e = 3$, cf. [PR08, §7]. We fix a uniformizer $u \in F'$ such that $u^e = t$, and a generator $\langle\tau\rangle = \mathrm{Gal}(F'/F)$. We have $\tau u = \zeta \cdot u$ where ζ is a primitive e-th root of unity.

Fix a Chevalley group H over $\mathbb{Z}$ together with an isomorphism $G \otimes_F F' = H \otimes_{\mathbb{Z}} F'$ compatible with splittings on both sides. The splitting for G over F is denoted (G, B, T, X), and for H over $\mathbb{Z}$ it is denoted (H, B_H, T_H, X_H). Here $T \subset G$ is the centralizer of the maximal F-split torus S as above. Recall that we fixed an alcove $\mathbf{a} \subset \mathscr{A}(G, S, F)$ containing a facet $\mathbf{f}$ in its closure.

The automorphism $\mathrm{id}\otimes\tau$ on $\mathrm{Res}_{F'/F}(G_{F'}) = \mathrm{Res}_{F'/F}(H_{F'})$ induces an automorphism σ on $H \otimes_{\mathbb{Z}} F'$ which can be written in the form $\sigma = \sigma_0 \otimes \tau$ where $\sigma_0 \in \mathrm{Aut}(X^*(T_H), \Delta_H, X_*(T_H), \Delta_H^\vee)$ is viewed as an order e automorphism of H. Here Δ_H (resp. $\Delta_H^\vee$) denotes the simple roots (resp. coroots) for (H, B_H, T_H). The parahoric group scheme can now be written in the form

$$\mathcal{G}_{\mathbf{f}} \;=\; \mathrm{Res}_{k[\![u]\!]/k[\![t]\!]}(\mathcal{H}_{\mathbf{f}})^{\sigma,o}, \tag{3.1}$$

where $\mathcal{H}_{\mathbf{f}}$ is the parahoric group scheme associated with the image of the facet $\mathbf{f} \subset \mathscr{B}(G, F) \subset \mathscr{B}(H, F')$. Here $(\text{-})^o$ denotes the fiberwise neutral component which only plays a role if G is not simply connected. This leads to the identifications of loop groups

$$LG \;=\; (LH_{k(\!(u)\!)})^\sigma \quad \text{and} \quad L^+\mathcal{G} \;=\; (L^+\mathcal{H})^{\sigma,o}, \tag{3.2}$$

where we refer to the discussion below (3.5) for the second equality.

We now lift (3.1) and (3.2) to the Witt vectors. For this, assume that k is of characteristic $p > 0$, and denote by $W = W(k)$ the ring of Witt vectors[2] together with the natural map $W \to k$. Let $K = \mathrm{Frac}(W)$ be the field of fractions. Following the arguments in [PR08, §7] (for Iwahori group schemes), or [PZ13, §4.2.2 (a)], we have the 'parahoric group scheme' $\underline{\mathcal{H}}_{\mathbf{f}}$ over the ring $W[\![u]\!]$ such that $\underline{\mathcal{H}}_{\mathbf{f}} \otimes k[\![u]\!] = \mathcal{H}_{\mathbf{f}}$. The group $\underline{\mathcal{H}}_{\mathbf{f}}$ is by construction 'horizontal along the W-direction', so that $\underline{\mathcal{H}}_{\mathbf{f}} \otimes K[\![u]\!]$ is an Iwahori group scheme of the same type as $\mathcal{H}_{\mathbf{f}}$ (but now the residue field K is of characteristic zero). Note that $\underline{\mathcal{H}}_{\mathbf{f}} \otimes W(\!(u)\!) = H \otimes_{\mathbb{Z}} W(\!(u)\!)$. Further, the automorphism τ lifts to the automorphism $\tau\colon W[\![u]\!] \to W[\![u]\!]$, $u \mapsto [\zeta] \cdot u$ where [-] denotes the Teichmüller lift. Again we have the automorphism $\sigma = \sigma_0 \otimes \tau$ on $\mathrm{Res}_{W[\![u]\!]/W[\![t]\!]}(\underline{\mathcal{H}}_{\mathbf{f}})$ so that we can define the fiberwise neutral component (cf. [SGA 3, VI_B, §3] for general base schemes)

$$\underline{\mathcal{G}}_{\mathbf{f}} \;\overset{\mathrm{def}}{=}\; \mathrm{Res}_{W[\![u]\!]/W[\![t]\!]}(\underline{\mathcal{H}}_{\mathbf{f}})^{\sigma,o}. \tag{3.3}$$

By [SGA3, VI_B, Thm. 3.10], this is a smooth $W[\![t]\!]$-group scheme with connected fibers such that $\underline{\mathcal{G}}_{\mathbf{f}} \otimes k[\![t]\!] = \mathcal{G}_{\mathbf{f}}$, and Lemma 3.1 below shows that it is affine as well. We define $\underline{G} := \underline{\mathcal{G}}_{\mathbf{f}} \otimes W(\!(t)\!)$, and $\underline{H} := \underline{\mathcal{H}}_{\mathbf{f}} \otimes W(\!(u)\!) = H \otimes_{\mathbb{Z}} W(\!(u)\!)$. We have by base change

$$\underline{G} \;=\; \mathrm{Res}_{W(\!(u)\!)/W(\!(t)\!)}(\underline{H})^\sigma, \tag{3.4}$$

and since $W(\!(u)\!)/W(\!(t)\!)$ is étale the latter is a reductive group scheme over $W(\!(t)\!)$ (with connected fibers).

Lemma 3.1. *The $W[\![t]\!]$-group scheme $\underline{\mathcal{G}}_{\mathbf{f}}$ is a Bruhat-Tits group scheme for $\underline{G}$ in the sense[3] of* [PZ13, Thm. 4.1]. *In particular, it is smooth affine with connected fibers.*

Proof. If $\underline{G} = \underline{H}$ is split, then by construction $\underline{\mathcal{G}}_{\mathbf{f}} = \underline{\mathcal{H}}_{\mathbf{f}}$ is such a Bruhat-Tits group scheme, cf. [PZ13, §4.2.2 (a)]. This is the first step in showing that (3.3) agrees with the construction in [PZ13, p. 180, middle] in general: starting from $\underline{G}$, we may follow [PZ13, §4.2] and construct the group scheme analogous to the one Pappas-Zhu denotes as $\mathcal{G}_\Omega = (\mathcal{G}'_\Omega)^o$, where $\mathcal{G}'_\Omega$ is defined on the

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[1] In *loc. cit.*, the group G is also assumed to be simply connected, but our weaker hypothesis suffices.

[2] The notation for the Witt vectors should not be confused with the notation for the Iwahori-Weyl group.

[3] As in [PZ13, Thm. 4.1], the reductive group scheme $\underline{G}$ is defined over $W[t, t^{-1}]$. It is more convenient for us to consider the base change along $W[t, t^{-1}] \to W(\!(t)\!)$.

bottom of p. 187. We observe the following: if we start from $\mathcal{G}_{\mathbf{f}}^{\#} := \underline{\mathcal{G}}_{\mathbf{f}} \otimes_{W[\![t]\!],t\mapsto p} W$, which is a parahoric group scheme in mixed characteristic, and apply the construction of Pappas-Zhu [PZ13, Thm. 4.1] to it, then we recover the group scheme $\underline{\mathcal{G}}_{\mathbf{f}}$. We use along the way that there is a canonical identification

$$\mathcal{G}_{\mathbf{f}}^{\#} \otimes_W K[p^{\frac{1}{e}}] \;=\; H \otimes_{\mathbb{Z}} K[p^{\frac{1}{e}}]$$

coming from (3.3) or (3.4), that is, under $\mathrm{Gal}(F'/F) = \mathrm{Gal}(K[p^{\frac{1}{e}}]/K)$ the Galois actions on the group $\mathrm{Res}_{W[\![u]\!]/W[\![t]\!]}(\underline{\mathcal{H}})$ induced from $\mathcal{G}_{\mathbf{f}}$ resp. $\mathcal{G}_{\mathbf{f}}^{\#}$ agree. □

We define the loop groups as the functor on the category of W-algebras R given by $L_W\underline{G}(R) = \underline{G}(R(\!(t)\!))$ (resp. $L_W^+\underline{\mathcal{G}}_{\mathbf{f}} = \underline{\mathcal{G}}_{\mathbf{f}}(R[\![t]\!])$), and $L_W\underline{H}(R) = \underline{H}(R(\!(u)\!))$ (resp. $L_W^+\underline{\mathcal{H}}_{\mathbf{f}} = \underline{\mathcal{H}}_{\mathbf{f}}(R[\![u]\!])$). This leads to the identifications

$$L_W\underline{G} \;=\; (L_W\underline{H})^{\sigma} \quad\text{and}\quad L_W^+\underline{\mathcal{G}}_{\mathbf{f}} \;=\; (L_W^+\underline{\mathcal{H}}_{\mathbf{f}})^{\sigma,o}. \tag{3.5}$$

For the second equality, we note that $L_W^+\mathrm{Res}_{W[\![u]\!]/W[\![t]\!]}(\underline{\mathcal{H}}_{\mathbf{f}})^{\sigma} \;=\; (L_W^+\underline{\mathcal{H}}_{\mathbf{f}})^{\sigma}$ which is a countably infinite successive extension of

$$\mathrm{Res}_{W[\![u]\!]/W[\![t]\!]}(\underline{\mathcal{H}}_{\mathbf{f}})^{\sigma} \otimes_{W[\![t]\!],t\mapsto 0} W$$

by vector groups which only depend on a neighborhood of the unit section and so are the same for $L_W^+\underline{\mathcal{G}}_{\mathbf{f}}$, cf. [RS20, Prop. A.4.9, (A.4.11)]. Since vector groups are fiberwise connected, we obtain the desired equality using that taking fiberwise connected components commutes with base change, cf. [SGA 3, VI_B, Prop. 3.3].

Corollary 3.2. *As group ind-schemes $L_W\underline{G} \otimes_W k = LG$ compatible with the subgroup schemes $L_W^+\underline{\mathcal{G}}_{\mathbf{f}} \otimes_W k = L^+\mathcal{G}_{\mathbf{f}}$.*

Proof. This is immediate from (3.5) and (3.2). □

3.2. **Tame liftings of Schubert varieties.** Being ind-schemes, the loop groups (3.5) define fpqc (in particular étale) sheaves on the category AffSch/W of affine W-schemes.

Lemma 3.3. *The étale quotient $\underline{\mathcal{F}\ell}_{G,\mathbf{f}} := L_W\underline{G}/L_W^+\underline{\mathcal{G}}_{\mathbf{f}}$ is an fpqc sheaf on AffSch/W which is represented by an ind-projective W-ind-scheme. There is an identification*

$$\underline{\mathcal{F}\ell}_{G,\mathbf{f}} \otimes_W k = \mathcal{F}\ell_{G,\mathbf{f}}.$$

Proof. The proof is the same as in [RS20, Lem. 5.3.2 (i)]: Let $T' \to T$ be a faithfully flat map in AffSch/W. Let $T' \leftarrow P' \to L_W\underline{G}$ be an object in $\underline{\mathcal{F}\ell}_{G,\mathbf{f}}(T')$ together with a descent datum along $T' \to T$. By effectivity of descent for affine schemes [StaPro, 0244], the torsor $T' \leftarrow P'$ descends to a fpqc-locally trivial torsor $T \leftarrow P$ represented by affine schemes. The map $P' \to L_W\underline{G}$ descends as well because every ind-scheme is an fpqc-sheaf. By [RS20, Prop. A.4.9, Exam. A.4.12 iii.(a)] every fpqc-locally trivial $L_W^+\underline{\mathcal{G}}_{\mathbf{f}}$-torsor is étale-locally trivial. Thus, $T \leftarrow P \to L_W\underline{G}$ is an object of $\underline{\mathcal{F}\ell}_{G,\mathbf{f}}(T)$. Now the representability of $\underline{\mathcal{F}\ell}_{G,\mathbf{f}}$ is a special case of [PZ13, Prop. 6.5] in view of Lemma 3.1. Finally, the displayed formula is immediate from Corollary 3.2 noting that sheafification commutes with fiber products. □

We can also provide something close to a lift of Schubert varieties. First, it is well known that the Iwahori-Weyl group not only does not depend on p but it admits an integral realization (see the discussion in [Loua, §2.6.]). Indeed, we have a canonical isomorphism:

$$\underline{N}(W(k)(\!(t)\!))/\underline{T}(W(k)[\![t]\!]) \xrightarrow{\sim} N(k(\!(t)\!))/T(k[\![t]\!]) = W$$

where N is the normalizer of S in G and its underlined counterpart is its canonical lift to a closed subgroup of $\underline{G}$. For any representative $\dot{w} \in \underline{N}(W(k)(\!(t)\!))$ of $w \in W/W_{\mathbf{f}}$, we denote by $\underline{S}_w = \underline{S}_w(\mathbf{a},\mathbf{f})$ the scheme-theoretic image of the morphism $L^+\underline{\mathcal{G}}_{\mathbf{a}} \to \underline{\mathcal{F}\ell}_{G,\mathbf{f}}, g \mapsto g \cdot \dot{w} \cdot e$.

Proposition 3.4. *For any $w \in W/W_{\mathbf{f}}$, the $W(k)$-scheme $\underline{S}_w = \underline{S}_w(\mathbf{a},\mathbf{f})$ is projective, integral, geometrically unibranch and flat over the base. Its special fiber $\underline{S}_w \otimes k$ coincides with S_w precisely when the former is reduced or the latter is normal.*

The latter situation always occurs whenever $p \nmid |\pi_1(G_{\mathrm{der}})|$ and only very rarely otherwise.

Proof. We may and do assume that $w \in W_{\mathrm{aff}}$. Projectivity follows from the existence of Demazure resolutions (see [PR08, Eq. (9.18)]), whereas being integral and flat over the base follows from the similar properties for the smooth finite type quotients of the positive loop group $L^+\underline{\mathcal{G}}_{\mathbf{a}}$. For the remaining claims, we consider the morphism

$$\underline{S}_{\mathrm{sc},w} = \underline{S}_{\mathrm{sc},w}(\mathbf{a},\mathbf{f}) \longrightarrow \underline{S}_w(\mathbf{a},\mathbf{f}) = \underline{S}_w. \tag{3.6}$$

As for $S_{\mathrm{sc},w} \to S_w$, this can be shown to be finite, birational, and a universal homeomorphism (see [HRc, pf. of Prop. 3.5]). Moreover, we know that $\underline{S}_{\mathrm{sc},w}$ is geometrically normal over $W(k)$ and that its special fiber is nothing more than $S_{\mathrm{sc},w}$, by [PR08, §9]. Hence $\underline{S}_w$ is at least geometrically unibranch.

Suppose now that S_w is normal. The scheme $\underline{S}_w$ is reduced, hence has a normalization, which can be identified with the canonical morphism $c\colon \underline{S}_{\mathrm{sc},w} \to \underline{S}_w$ from (3.6). The canonical closed immersion $S_w \to \underline{S}_w \otimes k$ fits into a commutative diagram

$$\begin{tikzcd} S_{\mathrm{sc},w} \arrow[r, "a"] \arrow[d, "\wr"'] & S_w \arrow[d, hook] \\ \underline{S}_{\mathrm{sc},w} \otimes k \arrow[r, "b"] & \underline{S}_w \otimes k. \end{tikzcd}$$

The map a is an isomorphism since S_w is normal, hence b is a closed immersion. The map c is therefore fiberwise a closed immersion and a homeomorphism, hence is by Nakayama's lemma a closed immersion of reduced schemes. It follows that c is an isomorphism, and then the diagram shows that $S_w = \underline{S}_w \otimes k$.

On the other hand, if $\underline{S}_w \otimes k = S_w$, equivalently, the special fiber $\underline{S}_w \otimes k$ is reduced, then we have an equality

$$\dim_k H^0(S_w, \mathcal{L}^N) = \dim_k H^0(S_{\mathrm{sc},w}, \mathcal{L}^N)$$

for any ample line bundle $\mathcal{L}$ on $\underline{S}_w$ and $N > 0$ sufficiently large, by transporting the claim to the generic fiber using flatness. This implies that the map $S_{\mathrm{sc},w} \to S_w$ is an isomorphism and thus S_w is normal, so that we are done. □

Remark 3.5. Assume S_w is normal, so that, as above, $\underline{S}_w$ is normal. In this case, one can show more generally that the formation of $\underline{S}_w$ is compatible with arbitrary base change, in the following sense. Let Z be any $W(k)$-scheme. Then $\underline{S}_{w,Z} := \underline{S}_w \times_{\mathrm{Spec}(W(k))} Z$ is equal to the scheme theoretic image of the map

$$L^+_W \underline{\mathcal{G}}_{\mathbf{f},Z} \longrightarrow \underline{\mathcal{F}\ell}_{G,\mathbf{f},Z}, \quad b \mapsto b \cdot \dot{w} \cdot e_Z, \tag{3.7}$$

where e_Z denotes the base point in $\underline{\mathcal{F}\ell}_{G,\mathbf{f},Z} := \underline{\mathcal{F}\ell}_{G,\mathbf{f}} \times_{\mathrm{Spec}(W(k))} Z$. The main ingredient in the essential case of $\mathbf{f} = \mathbf{a}$ is the fact that, if $\underline{\pi}_w : \underline{D}(\tilde{w}) \to \underline{S}_w$ is the Demazure resolution, the formation of $\underline{\pi}_{w,*}\mathcal{O}_{\underline{D}(\tilde{w})}$ commutes with arbitrary base change, cf. [Fal03], [Go03, Lem. 3.13, Prop. 3.15 ff.].

3.3. **Negative loop groups.** We continue with the same notation and assumptions. The base point $\mathbf{0} \in \mathscr{A}(H, T_H, F')$ defined by $H \otimes_{\mathbb{Z}} \mathcal{O}_{F'}$ is invariant under the Galois group, and defines a special point also denoted $\mathbf{0} \in \mathscr{A}(G, S, F)$ (because by construction σ_0 preserves the splitting (G, B, T, X)). After conjugation by an element in W_{aff}, we may assume that the alcove $\mathbf{a}$ contains $\mathbf{0}$ in its closure, and lies in the chamber defined by the Borel B_H.

We adapt the notion of the negative loop group from [dHL18, §3.6] to our set-up as follows: the Iwahori $\mathcal{H}_{\mathbf{a}}$ corresponds now to the Borel subgroup $B_H \subset H$, more precisely, $\mathcal{H}_{\mathbf{a}}$ is the Néron blow up (resp. dilatation) of $H \otimes_{\mathbb{Z}} \mathcal{O}_{F'}$ in $B \otimes_{\mathbb{Z}} k$, cf. [MRR, Exam. 3.3]. We let $B_H^{\mathrm{op}} = T_H \ltimes U_H^{\mathrm{op}}$ denote the opposite Borel subgroup. The negative loop group is the functor on the category of W-algebras

R given by $L^-_W H(R) := H(R[u^{-1}])$. We define $L^{--}_W H := \ker(L^-_W H \to H_W)$, $u^{-1} \mapsto 0$, and further we define *strictly negative loop group*

$$L^{--}_W \underline{\mathcal{H}}_{\mathbf{a}} := L^{--}_W H \rtimes U^{\mathrm{op}}_{H,W},$$

which is a subgroup ind-scheme of the ind-affine ind-scheme $L_W \underline{H}$ over W. Finally, for the facet $\mathbf{f}$ contained in the closure of $\mathbf{a}$, we define the *strictly negative loop group*

$$L^{--}_W \underline{\mathcal{H}}_{\mathbf{f}} \overset{\mathrm{def}}{=} \bigcap_{w \in W_{H,\mathbf{f}}} {}^{w}\big(L^{--}_W \underline{\mathcal{H}}_{\mathbf{a}}\big), \tag{3.8}$$

where $W_{H,\mathbf{f}}$ denotes the subgroup of the affine Weyl group $W_{H,\mathrm{aff}}$ corresponding to the unique facet containing $\mathbf{f} \subset \mathscr{A}(H, T_H, F')$. As H is split, each element $w \in W_{H,\mathrm{aff}}$ has a representative in $\dot{w} \in H(W(\!(u)\!))$. We set

$${}^{w}\big(L^{--}_W \underline{\mathcal{H}}_{\mathbf{a}}\big) := \dot{w} \cdot \big(L^{--}_W \underline{\mathcal{H}}_{\mathbf{a}}\big) \cdot \dot{w}^{-1} \ \subset \ L_W \underline{H},$$

and the intersection (3.8) is taken inside $L_W \underline{H}$. The strictly negative loop group has the following key property.

Lemma 3.6. *The map $L^{--}_W \underline{\mathcal{H}}_{\mathbf{f}} \to L_W \underline{H}/L^+_W \underline{\mathcal{H}}_{\mathbf{f}}$, $h^- \mapsto h^- \cdot e$ is representable by a quasi-compact open immersion.*

Proof. Equivalently, we have to show that the multiplication map

$$L^{--}_W \underline{\mathcal{H}}_{\mathbf{f}} \times L^+_W \underline{\mathcal{H}}_{\mathbf{f}} \to L_W \underline{H} \tag{3.9}$$

is a quasi-compact open immersion (to check the equivalence we use that $L^+_W \underline{\mathcal{H}}_{\mathbf{f}} \to \mathrm{Spec}(W)$ is fpqc, and that quasi-compact immersions are of effective fpqc descent [StaPro, 02JR, 0246]).

This in turn is equivalent to being representable in schemes by a finitely presented étale monomorphism (see [EGAIV4, Thm. 17.9.1]). This was already known in the case $\mathbf{f} = 0$ (see [HRa, pf. of Lem. 3.1]) or working over a field (see [dHL18, pf. of Prop. 3.7.1]).

The representability follows by writing the morphism as a composition of a closed immersion

$$L^{--}_W \underline{\mathcal{H}}_{\mathbf{f}} \times L^+_W \underline{\mathcal{H}}_{\mathbf{f}} \to L_W \underline{H} \times L^+_W \underline{\mathcal{H}}_{\mathbf{f}}$$

followed by the group multiplication

$$L_W \underline{H} \times L^+_W \underline{\mathcal{H}}_{\mathbf{f}} \to L_W \underline{H}$$

which is representable, because the functor $L^+_W \underline{\mathcal{H}}_{\mathbf{f}}$ also is. For finite presentedness, we simply observe that both $L^{--}_W \underline{\mathcal{H}}_{\mathbf{f}}$ and $L_W \underline{H}/L^+_W \underline{\mathcal{H}}_{\mathbf{f}}$ are of ind-finite type.

Next we show that the map is a monomorphism, that is, that the finite type W-group subscheme

$$L^{--}_W \underline{\mathcal{H}}_{\mathbf{f}} \cap L^+_W \underline{\mathcal{H}}_{\mathbf{f}}$$

of $L_W H$ equals its unit section. We can do this in two different ways: either check it on both fibers, see [dHL18, Prop. 3.7.1], which implies that the defining ideal is p-divisible and p-power torsion, hence trivial; or we check that its field valued points are trivial, again by [dHL18, Prop. 3.7.1], and that its Lie algebra with coefficients in any W-algebra R vanishes.

Actually, we are going to show more generally that we have a triangular decomposition

$$\mathrm{Lie}\, L^{--}_W \underline{\mathcal{H}}_{\mathbf{f}} \oplus \mathrm{Lie}\, L^+_W \underline{\mathcal{H}}_{\mathbf{f}} = \mathrm{Lie}\, L_W \underline{H}.$$

This can be easily calculated using our choice of pinning (especially if $\mathbf{f} = \mathbf{a}$ or 0); comp. [dHL18, Prop. 3.6.4]. Alternatively, we can observe that we have equalities

$$\mathrm{Lie}\, L^+_W \underline{\mathcal{H}}_{\mathbf{f}} = \sum_{w \in W_{H,\mathbf{f}}} {}^{w}\big(\mathrm{Lie}\, L^+_W \underline{\mathcal{H}}_{\mathbf{a}}\big) \tag{3.10}$$

$$\mathrm{Lie}\, L^{--}_W \underline{\mathcal{H}}_{\mathbf{f}} = \bigcap_{w \in W_{H,\mathbf{f}}} {}^{w}\big(\mathrm{Lie}\, L^{--}_W \underline{\mathcal{H}}_{\mathbf{a}}\big) \tag{3.11}$$

almost by definition. This reduces the decomposition to the case $\mathbf{f} = \mathbf{a}$, where it is clear. Note that formation of the Lie algebra commutes with (completed) base change for all functors under consideration. Finally, it is enough to remark that this decomposition implies étaleness of the original map, via translating back to the origin (here we must use that all functors are formally smooth). □

Remark 3.7. We could have also argued via a bundle interpretation as in [HRa, Lem. 3.1] by constructing an appropiate opposite parahoric group scheme over $W[t^{-1}]$. All the necessary ideas can be found in [Loua, §3.2.-3.3., §3.5.].

We now want to descend the result.

Lemma 3.8. *The subgroup ind-scheme $L_W^{--}\underline{\mathcal{H}}_{\mathbf{f}} \subset L_W\underline{\mathcal{H}}_{\mathbf{f}}$ is σ-invariant.*

Proof. As the base point 0 is σ-invariant, one finds that the subgroup $L_W^{--}H$ is σ-invariant. The σ-invariance of the alcove $\mathbf{a} \subset \mathscr{A}(H, T_H, F')$ implies that the opposite unipotent radical $U_{H,W}^{\mathrm{op}}$ is σ-invariant. Note that σ acts on $U_{H,W}^{\mathrm{op}}$ through the automorphism σ_0. The lemma follows from the definition (3.8) using the σ-invariance of $\mathbf{f}$. □

We define the *twisted strictly negative loop group* as

$$L_W^{--}\underline{\mathcal{G}}_{\mathbf{f}} \overset{\mathrm{def}}{=} (L_W^{--}\underline{\mathcal{H}}_{\mathbf{f}})^{\sigma,o} \tag{3.12}$$

Corollary 3.9. *The map $L_W^{--}\underline{\mathcal{G}}_{\mathbf{f}} \to L_W\underline{G}/L_W^{+}\underline{\mathcal{G}}_{\mathbf{f}} = \underline{\mathcal{F}\ell}_{G,\mathbf{f}}$, $g^- \mapsto g^- \cdot e$ is representable by a quasi-compact open immersion.*

Proof. As in the proof of Lemma 3.6, it is enough to show that the multiplication map $L_W^{--}\underline{\mathcal{G}}_{\mathbf{f}} \times_W L_W^{+}\underline{\mathcal{G}}_{\mathbf{f}} \to L_W\underline{G}$ is a quasi-compact open immersion. There is a Cartesian diagram

$$\begin{array}{ccc} L_W^{--}\underline{\mathcal{H}}_{\mathbf{f}} \times_W L_W^{+}\underline{\mathcal{H}}_{\mathbf{f}} & \longrightarrow & L_W\underline{H} \\ \uparrow & & \uparrow \\ (L_W^{--}\underline{\mathcal{H}}_{\mathbf{f}})^{\sigma} \times_W (L_W^{+}\underline{\mathcal{H}}_{\mathbf{f}})^{\sigma} & \longrightarrow & (L_W\underline{H})^{\sigma}, \end{array}$$

where the horizontal maps are given by multiplication, and the vertical maps are the canonical closed immersions. As the top arrow is an open immersion by Lemma 3.6, the bottom arrow is an open immersion as well. The corollary now follows from (3.5), (3.12) by passing to neutral components. □

4. Kac-Moody flag varieties and projective embeddings

In this subsection, we aim at generalizing Ramanathan's methods [Ra87] via Frobenius splitting to gather information on the homogeneous ideals that define Kac-Moody Schubert varieties inside projective spaces or their Schubert overvarieties. We will also follow the treatment of Mathieu and use some ideas of [Ma89]. An important result for us is Corollary 4.3 which gives a formula for the tangent spaces of Schubert varieties in arbitrary characteristic. All Kac-Moody algebras below are assumed to be symmetrizable.

Let us start by recalling the definition of a (symmetrizable) Kac-Moody algebra. These are (mostly infinite-dimensional) Lie algebras $\mathfrak{g}$ over $\mathbb{C}$ associated to symmetrizable generalized Cartan matrices, i.e. finite integer-valued square matrices $A = (a_{ij})$ with $a_{ii} = 2$ and $a_{ij} \leq 0$, $i \neq j$, which become symmetric after multiplication by an invertible diagonal matrix, see [Kac90, §1.1]. To that end, one starts with the notion of a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of the given generalized Cartan matrix, consisting of a finite dimensional $\mathbb{C}$-vector space $\mathfrak{h}$, a linearly independent set of roots $\alpha_i \in \mathfrak{h}^\vee$, $i = 1, \ldots, n$ and coroots $h_i := \alpha_i^\vee \in \mathfrak{h}$ such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ and $\dim \mathfrak{h} = n + \mathrm{corank} A$, see [Kac90, §1.1]. Then, we extend the abelian Lie algebra $\mathfrak{h}$ to a Kac-Moody algebra $\mathfrak{g}$ by freely adding generators e_i, f_i for each positive simple root α_i and then by imposing the relations: $[h, e_i] = \alpha_i(h)e_i$, $[h, f_i] = -\alpha_i(h)f_i$, $[e_i, f_j] = \delta_{ij}h_i$, $\mathrm{ad}^{-a_{ij}+1}(e_i)(e_j) = 0$ and similarly for f_i, f_j, cf. [Ma88, p. 16–17].

We have root and coroot lattices $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i \subseteq \mathfrak{h}^\vee$, $Q^\vee = \sum_{i=1}^n \mathbb{Z}\alpha_i^\vee \subseteq \mathfrak{h}$. It turns out that $\mathfrak{g}_\alpha$ factors into a sum of weight spaces

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi \cup 0} \mathfrak{g}_\alpha$$

for the adjoint action of $\mathfrak{h}$. Here $\Phi \subset Q$ denotes the root system of $\mathfrak{g}$, for which the α_i form a basis (see [Kac90, Thm. 1.2] for these assertions). It admits a natural partition into real roots, that is, those that are conjugate to a positive simple root under the Weyl group, and imaginary roots, that is, the rest of them. Note that there is a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{n}_+$, resp. $\mathfrak{n}_-$ denotes the sum of all positive resp. negative weight spaces; we denote by $\mathfrak{b}_+$ the positive Borel subalgebra. Finally, we fix once and for all a weight lattice $P \subseteq \mathfrak{h}^\vee$ and a coweight lattice $P^\vee \subseteq \mathfrak{h}$ given by taking $\mathbb{Z}$-duals, such that there are saturated (that is, with flat cokernel) inclusions of free abelian groups $Q \subseteq P$, $Q^\vee \subseteq P^\vee$ (compare also with [Ma88, p. 16]).

Even though the category of arbitrary $\mathfrak{g}$-modules is not so well behaved, we still obtain, up to adding some finiteness conditions, a good notion of highest weight module $V(\lambda)$ with maximal dominant weight $\lambda \in P$ for the Bruhat order. These arise as the only irreducible quotient of the universal Verma module $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \mathbb{C}_\lambda$. The extremal weights of $V(\lambda)$ are the conjugates $w\lambda$ of the highest weight under the Weyl group action and they have multiplicity 1. Demazure modules are the cyclic $\mathfrak{b}_+$-modules generated by $V(\lambda)_{w\lambda}$.

In order to study arithmetic related to Kac-Moody algebras and groups, Tits introduced a $\mathbb{Z}$-form $\mathfrak{U}_\mathbb{Z}(\mathfrak{g})$ of the universal enveloping algebra and a fortiori a $\mathbb{Z}$-form $\mathfrak{g}_\mathbb{Z}$ of the Lie algebra. In [Ma88] and [Ma89], Mathieu uses this to define a certain ind-affine $\mathbb{Z}$-group ind-scheme $\mathfrak{G}$ whose Hopf algebra of distributions supported at the origin (also known as hyperalgebra) is given by the completion $\widehat{\mathfrak{U}}_\mathbb{Z}(\mathfrak{g})$ of $\mathfrak{U}_\mathbb{Z}(\mathfrak{g})$ for the obvious descending filtration (compare with [Ma89, Lem. 2, Lem. 3]). It comes equipped with a canonical maximal split torus $\mathfrak{T}$ corresponding to $\mathfrak{h}_\mathbb{Z}$, as well as a positive Borel subgroup $\mathfrak{B}^+ = \mathfrak{T} \ltimes \mathfrak{U}^+$ containing it. Let us mention that $\mathfrak{B}^+$ is an affine, non-finitely presented, flat, closed subgroup scheme of $\mathfrak{G}$ with underlying hyperalgebra given by $\widehat{\mathfrak{U}}_\mathbb{Z}(\mathfrak{b}^+)$. The fppf quotient $\mathfrak{F} := \mathfrak{G}/\mathfrak{B}^+$ is representable by a reduced ind-projective $\mathbb{Z}$-ind-scheme. It is known as Mathieu's flag variety associated to the Kac-Moody algebra $\mathfrak{g}$ (together with the rest of the chosen data). Given an admissible set $J \subseteq I$ of positive simple roots (i.e. such that the subgroup W_J generated by the corresponding reflections is finite), we can associate to it a standard parabolic subgroup $\mathfrak{P}_J^+ = \mathfrak{L}_J \ltimes \mathfrak{U}_J^+$ containing $\mathfrak{B}$ as a closed subgroup scheme. We have a partial flag variety $\mathfrak{F}_J := \mathfrak{G}/\mathfrak{P}_J^+$, which is still representable by an ind-projective $\mathbb{Z}$-ind-scheme. For each $w \in W/W_J$, we may consider the Schubert variety $\mathfrak{S}_w \subseteq \mathfrak{F}_J$ obtained as the scheme-theoretic image of the orbit map $\mathfrak{B} \to \mathfrak{F}_J, b \mapsto b \cdot \dot{w} \cdot e$, where $e \in \mathfrak{F}(\mathbb{Z})$ is the base point and $\dot{w} \in \mathfrak{G}(\mathbb{Z})$ some representative of the class w. It is a fundamental theorem of Mathieu [Ma88] and Littelmann [Li98] that the $\mathfrak{S}_w$ are geometrically normal over $\mathbb{Z}$, i.e., the structural map $S_w \to \mathrm{Spec}(\mathbb{Z})$ is a normal morphism in the sense of [StaPro, 038Z].

Next we are going to introduce the negative parabolic subgroups $\mathfrak{P}_J^- = \mathfrak{L}_J \ltimes \mathfrak{U}_J^-$, seemingly a novelty in the literature[4]. As a functor, this is given by the repeller locus of an arbitrary J-regular dominant coweight μ of $\mathfrak{T}$, whereas its unipotent radical $\mathfrak{U}_J^-$ is given by the strict repeller (see [Ri16] or [CGP15, §2.1.] for an exposition of the subject).

Lemma 4.1. *The negative unipotent subgroup $\mathfrak{U}_J^-$ is representable by an ind-affine closed subgroup ind-scheme of $\mathfrak{G}$ of ind-finite presentation, which does not depend on the choice of μ. The multiplication morphism $\mathfrak{U}_J^- \times \mathfrak{P}_J^+ \to \mathfrak{G}$ is a quasi-compact open immersion.*

Proof. We start by observing that $\mathfrak{G}$ can be written as the colimit of μ-invariant closed subschemes $\mathfrak{G}_{w,J}$, for varying w, given as the canonical $\mathfrak{P}_J^+$-bundle over $\mathfrak{S}_{w,J}$. We apply [Ri16, Lem. 1.9] to conclude that the strict repeller $\mathfrak{U}_J^- \cap \mathfrak{G}_{w,J}$ is representable by a closed subscheme. We still need to check that each of the intersections is of finite type over $\mathbb{Z}$. Here we take advantage of the fact

[4] In [Ma89, p. 45], it is even mentioned that $\mathfrak{P}_J^-$ is not defined.

that $\mathfrak{G}_{w,J}$, regarded as a bundle for the congruence subgroup $\mathfrak{P}_J(n)$ with $n >> 0$, becomes $\mathfrak{L}_J$-equivariantly trivial, cf. [Ma89, Lem. 5]. This induces an isomorphism between $\mathfrak{U}_J^- \cap \mathfrak{G}_{w,J}$ and the strict repeller of the finite type affine $\mathbb{Z}$-scheme $\mathfrak{S}_{w,J}^n := \mathfrak{G}_{w,J}/\mathfrak{P}_J(n)$. Independence of μ will follow later, once we have understood the étale formal neighborhood of $\mathfrak{U}_J$.

Next we claim that the multiplication morphism

$$\mathfrak{U}_J^- \times \mathfrak{P}_J^+ \to \mathfrak{G}$$

is representable in schemes by a finitely presented morphism. This can be readily seen by writing it as the composite of the closed immersion $\mathfrak{U}_J^- \times \mathfrak{P}_J^+ \to \mathfrak{G} \times \mathfrak{P}_J^+$ and the multiplication map $\mathfrak{G} \times \mathfrak{P}_J^+ \to \mathfrak{G}$, where $\mathfrak{P}_J^+$ is an affine scheme. Finite presentation results from its analogue in the inductive category, which we have seen for $\mathfrak{U}_J^-$ and $\mathfrak{F}_J$. Finally, notice that the given morphism is a monomorphism, because $\mathfrak{P}_J^+$ lies in the attractor locus of $\mathfrak{G}$ (at the end, the reader should be able to show that these two subfunctors coincide but we will not need it).

By [EGAIV4, Thm. 17.9.1], it remains to show that the multiplication morphism is formally étale. Along the unit section, this follows from compatibility of taking Lie algebras with taking attractors and repellers (see [CGP15, Prop. 2.1.8]), so that we just need to observe that we have a triangular decomposition

$$\widehat{\mathfrak{g}}_R = \widehat{\mathfrak{p}}_{J,R}^+ \oplus \mathfrak{n}_{J,R}^-$$

for every coefficient ring R. The case of a general k-valued point follows from this once we have shown that they lift to Artinian-valued points of $\mathfrak{U}_J^-$ and $\mathfrak{P}_J^+$. This is clear for the formally smooth functor $\mathfrak{P}_J^+$. For $\mathfrak{U}_J^-$, it suffices to show that $\mathfrak{U}_J^-(k)$ equals the subgroup U_J^- generated by k-valued points of root groups $\mathfrak{U}_a \cong \mathbb{G}_a$ attached to negative real roots $a \in \Phi_J^{\mathrm{re},-}$, where $\Phi_J^{\mathrm{re},-}$ consists of those real roots which are strictly negative on any J-regular dominant weight. Since $U_J^- \subseteq \mathfrak{U}_J^-(k)$ is clear, it suffices to prove the opposite inclusion.

Recall that if we set $G = \mathfrak{G}(k)$, $P_J^+ = \mathfrak{P}_J^+(k)$, $L_J = \mathfrak{L}_J(k)$ and $P_J^- = U_J^- \ltimes L_J$, then we have the Birkhoff decomposition for G, see [Rou16, Prop. 3.16]:

$$G = \bigsqcup_{w \in W_J \backslash W / W_J} P_J^- \dot{w} P_J^+. \tag{4.1}$$

It suffices to prove that $\mathfrak{P}_J^-(k) \subseteq P_J^-$. Suppose that an element $\dot{w}p \in \dot{w}P_J^+$ lies in $\mathfrak{P}_J^-(k)$. Then

$$p^{-1}(w^{-1}\mu)(r)p\mu(r^{-1}) = (\dot{w}p)^{-1}\mu(r)\dot{w}p(\mu(r))^{-1}$$

lies in $\mathfrak{U}_J^-(k) \cap \mathfrak{P}_J^+(k) = 1$ because the L_J-components cancel out, as μ centralizes $\mathcal{L}_J$. Now write $p = lu$ with $l \in L_J = \mathfrak{L}_J(k)$ and $u \in U_J^+ = \mathfrak{U}_J^+(k)$ and expand the above product. Coming from the L_J-component, we obtain the identity

$$w^{-1}\mu(r) = \mu(r)$$

for all $r \in k^\times$. This implies triviality of $w \in W_J \backslash W / W_J$, and consequently we get $p \in P_J^+ \cap \mathfrak{P}_J^-(k) = L_J \subseteq P_J^-$. □

For each J-regular dominant weight λ, we may consider the line bundle $\mathfrak{L}(\lambda) := \mathfrak{G} \times_{\mathfrak{P}_J^+} \mathbb{Z}_{-\lambda}$ obtained from the natural $\mathfrak{P}_J^+$-bundle $\mathfrak{G} \to \mathfrak{F}_J$ via the character $-\lambda$ of $\mathfrak{P}_J^+$ (compare with [Ma88] and [Ma89], which use the opposite sign convention). This is a very ample line bundle on $\mathfrak{F}_J$ and we have a natural identification between $\Gamma(\mathfrak{F}_J, \mathfrak{L}(\lambda))^\vee := \mathrm{colim}_{w \in W/W_J} \Gamma(\mathfrak{S}_{w,J}, \mathfrak{L}(\lambda))^\vee$ and the canonical $\mathbb{Z}$-form $V(\lambda)_\mathbb{Z}$ of the highest weight module: indeed, at each finite step, the submodule $\Gamma(\mathfrak{S}_{w,J}, \mathfrak{L}(\lambda))^\vee$ is identified with the integral Demazure module $V_w(\lambda)_\mathbb{Z}$, see [Ma88, Thm. 5]. Finally, we recall that these constructions exhaust after linearization the Picard group of $\mathfrak{F}_J$, in virtue of the isomorphism $P_J/P_I \simeq \mathrm{Pic}(\mathfrak{F}_J)$, where $P_J = \{\lambda \in P \mid \lambda(\alpha_j^\vee) = 0 \, \forall j \in J\}$ (see [Ma88, Prop. 28]).

The following result goes back originally to [Ra87] in the case of classical flag varieties, by using the relatively new at the time method of Frobenius splitting. Credit is also due to Mathieu for showing the existence of an ind-splitting of the diagonal of the flag variety (see [Ma89, Prop. 1]). We are pretty convinced that Littelmann's path model also yields the same type of results, see [Li98].

Theorem 4.2 (Ramanathan, Mathieu)**.** *Given an ample line bundle $\mathfrak{L}$ of $\mathfrak{S}_w$, the corresponding morphism $\mathfrak{S}_w \to \mathbb{P}(\Gamma(\mathfrak{S}_w, \mathfrak{L})^\vee)$ is a closed immersion defined by quadrics. Moreover, the closed immersion $\mathfrak{S}_u \to \mathfrak{S}_w$, $u \le w$ is linearly defined with respect to $\mathfrak{L}$.*

Here we allow $w = \infty$ to get the entire partial affine flag variety $\mathfrak{S}_\infty = \mathfrak{F}_J$.

Proof. The statement refers to the behavior of the graded algebra $\Gamma(\mathfrak{S}_w, \mathfrak{L})$ and its graded module $\Gamma(\mathfrak{S}_u, \mathfrak{L})$ as in Definition A.5. By upper semicontinuity, it suffices to base change to any positive characteristic p field k. Since every ample line bundle on $\mathfrak{S}_w$ extends to $\mathfrak{S}_\infty$, we are reduced to showing, by Proposition A.4 and Proposition A.6, that the compatibly ind-split $\mathfrak{S}_u \subseteq \mathfrak{S}_w \subseteq \mathfrak{S}_\infty$ satisfy: the diagonal $\Delta_{\mathfrak{S}_\infty}$ is compatibly ind-split with $\mathfrak{S}_\infty^2$; the partial mixed diagonals $\Delta_{\mathfrak{S}_\infty} \times \mathfrak{S}_\infty$, $\mathfrak{S}_\infty \times \Delta_{\mathfrak{S}_\infty}$, $\mathfrak{S}_w \times \Delta_{\mathfrak{S}_\infty}$ and $\mathfrak{S}_u \times \Delta_{\mathfrak{S}_\infty}$ are simultaneously compatibly ind-split with $\mathfrak{S}_\infty^3$. We may and do assume $J = \emptyset$ by pushing forward the splitting along the obvious projection.

For this, we need the convoluted flag varieties $\mathfrak{S}_\infty^{\tilde{\times} n} := \mathfrak{S}_\infty \tilde{\times} \dots \tilde{\times} \mathfrak{S}_\infty = \mathfrak{G} \times^{\mathfrak{B}_+} \cdots \times^{\mathfrak{B}_+} \mathfrak{F}$. Note that herein we have convoluted Schubert varieties $\mathfrak{S}_{w_1,\dots,w_n} := \mathfrak{S}_{w_1} \tilde{\times} \dots \tilde{\times} \mathfrak{S}_{w_n}$ which are all compatibly Frobenius split, as one can observe by using appropriate Demazure resolutions (see [Ma89, Lem. 9, Lem. 10]). In particular, under the natural isomorphism $\mathfrak{S}_\infty^{\tilde{\times}} \cong \mathfrak{S}_\infty^n$ given by $(m_1, \dots, m_n)$, where m_i denotes the product of the first i coordinates, the diagonal $\Delta_{\mathfrak{S}_\infty}$ is identified with $\mathfrak{S}_{\infty,1}$ (compare with [Ma89, Prop. 1]), and the partial mixed diagonals $\Delta_{\mathfrak{S}_\infty} \times \mathfrak{S}_\infty$, resp. $\mathfrak{S}_\infty \times \Delta_{\mathfrak{S}_\infty}$, resp. $\mathfrak{S}_w \times \Delta_{\mathfrak{S}_\infty}$, resp. $\mathfrak{S}_u \times \Delta_{\mathfrak{S}_\infty}$ are identified with $\mathfrak{S}_{\infty,1,\infty}$, resp. $\mathfrak{S}_{\infty,\infty,1}$, resp. $\mathfrak{S}_{w,\infty,1}$, resp. $\mathfrak{S}_{u,\infty,1}$. □

The following corollary gives an explicit formula describing the tangent space, which goes back to work of Kumar ([Ku96]) in characteristic 0, and Polo ([Po94]) for classical flag varieties.

Corollary 4.3 (Kumar, Polo)**.** *Let k be a field of arbitrary characteristic. The k-vector space $T_e\mathfrak{S}_{w,J} \otimes k$ consists of all $X \in T_e\mathfrak{F}_J \otimes k$ such that $Xv_\lambda \in V_w(\lambda)_k$, for any fixed J-regular dominant weight λ.*

Proof. We start by noticing that $\mathfrak{U}_J^-$ becomes naturally identified with the distinguished open subset $D_+(v_\lambda^\vee) \subseteq \mathfrak{F}_J$ associated with the dual section $v_\lambda^\vee \in \Gamma(\mathcal{F}\ell_J, \mathcal{L})$ killing all weight spaces different from $V(\lambda)_\lambda = kv_\lambda$. Indeed, this is a consequence of the Birkhoff decomposition, as the respective complements are vertical divisors with equal generic fiber by [Ku96, Lem. 8.3].

By Theorem 4.2, the closed immersion $\mathfrak{U}_{w,J}^- := \mathfrak{S}_{w,J} \cap \mathfrak{U}_J^- \to \mathfrak{U}_J^-$ is defined by the coefficients φ_ξ, where $\xi \in \Gamma(\mathfrak{F}_J, \mathfrak{L}) \otimes k$ runs over all vectors perpendicular to $V_w(\lambda)_k$ and $\varphi_\xi(u) = \xi(uv_\lambda)$ (see also [Po94, Prop. 3.1]). Given a tangent vector $X \in T_e\mathfrak{F}_J \otimes k = T_e\mathfrak{U}_J^- \otimes k$, then it lies in $T_e\mathfrak{S}_{w,J}$ if and only if the associated distribution of $k[\mathfrak{U}_J^-] := \varprojlim k[\mathfrak{U}_{w',J}^-]$ kills all the φ_ξ designated above. Representing X by a $k[\epsilon]$-valued point u of $\mathfrak{U}_J^-$, we get

$$\varphi_\xi(u) = \xi(uv_\lambda) = \epsilon\xi(Xv_\lambda).$$

The right side is obviously zero for all ξ if $Xv_\lambda \in V_w(\lambda)_k$ and the converse follows from the isomorphism $V_w(\lambda)_k^\vee = \Gamma(\mathfrak{F}_J, \mathfrak{L}) \otimes k / V_w(\lambda)_k^\perp$. This proves the formula for the tangent space (compare our argument with Polo's [Po94, Thm. 3.2]). □

Remark 4.4. Polo claims in [Po94, Cor. 4.1] that the dimension of $T_e\mathfrak{S}_{w,J} \otimes k$ does not depend on $p = \operatorname{char} k$. He invokes the fact that $T_e\mathfrak{U}_J^-$ has a natural $\mathbb{Z}$-model given by $\mathfrak{n}_{J,\mathbb{Z}}^-$, whereas the integral model $V_w(\lambda)_\mathbb{Z}$ of $V_w(\lambda)_k$ is a direct summand of the model $V(\lambda)_\mathbb{Z}$ for $V(\lambda)_k$. Now the independence of p of the tangent space dimension is equivalent to the flatness of the cokernel of $\mathfrak{n}_{J,\mathbb{Z}}^- \to V(\lambda)_\mathbb{Z}/V_w(\lambda)_\mathbb{Z}$, $X \mapsto Xv_\lambda$, or equivalently the saturatedness of its image. An argument for this is missing in the proof of [Po94, Cor. 4.1]. It would be interesting to clarify this point. This result has at least been invoked in [Ku96, Rem. 8.10] by Kumar in order to generalize his smoothness criterion to positive characteristic.

Assume $\mathfrak{g}$ is an affine Kac-Moody algebra, that is, the corank of the corresponding generalized Cartan matrix is equal to 1. These are classified by affine Dynkin diagrams and admit very explicit realizations as some mildly modified loop algebras or their fixpoints under order 2 or 3 automorphisms, see [Kac90, §7–8]. More explicitly, the adjoint quotient of $[\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$ can be identified with the graded Lie algebra of the group scheme $L_W \underline{G} \otimes \mathbb{C}$ constructed in (3.4) for a given embedding $K = \mathrm{Frac}\, W \hookrightarrow \mathbb{C}$, where G is the only simply connected absolutely almost simple semisimple group over $k(\!(t)\!)$ having the same Dynkin diagram as $\mathfrak{g}$. Under this correspondence, the Borel subalgebra $\mathfrak{b}$ is linked to the Lie algebra of $\underline{\mathcal{G}}_{\mathbf{a}}$ and every standard parabolic $\mathfrak{p}_J$ to the Lie algebra of $\underline{\mathcal{G}}_{\mathbf{f}_J}$ for some subfacet $\mathbf{f}_J$ of $\mathbf{a}$. We also pick the usual weight lattice P in the Cartan subalgebra $\mathfrak{h}$. We have the following important comparison result, which can be found in [PR08, §9.h.] in a weaker form (see also [Zhu17, §2.5] for an exposition).

Proposition 4.5. *Let k be an algebraically closed field of strictly positive characteristic with ring of Witt vectors $W = W(k)$. There are natural isomorphisms*

$$\mathfrak{P}_J^+ \otimes W \cong \widehat{L^+\underline{\mathcal{G}}_{\mathbf{f}_J}} \rtimes \mathbb{G}_m^{\mathrm{rot}} \quad \textit{and} \quad \mathfrak{G} \otimes W \cong \widehat{L\underline{G}} \rtimes \mathbb{G}_m^{\mathrm{rot}},$$

inducing an equivariant isomorphism $\mathfrak{F}_J \otimes W \cong \underline{\mathcal{F}\ell}_{G,\mathbf{f}_J}$ compatible with $\mathfrak{U}_J^- \otimes W \cdot e \cong L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J} \cdot e$.

Here the hat loop groups are the central extensions of $L\underline{G}$ obtained by trivializing its action against the generator of $\mathrm{Pic}(\underline{\mathcal{F}\ell}_{G,\mathbf{0}})$. In fact, any line bundle of a partial affine flag variety with central charge 1 is enough. The rotation $\mathbb{G}_m$ is induced by automorphisms of the formal disk $R[\![u]\!]$ (as opposed to $R[\![t]\!]$).

Idea of proof. First of all, we construct the isomorphism $\mathfrak{P}_J^+ \otimes W \cong \widehat{L^+\underline{\mathcal{G}}_{\mathbf{f}_J}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$. This essentially amounts to verifing that their algebras of distributions match (see either [PR08, §8.d.] or the more convincing [Loua, §3.8.]).

Next we identify the Demazure varieties $\mathfrak{D}_{\tilde{w}} \cong \underline{D}_{\tilde{w}}$ in the natural way, induced by the twisted product decomposition in terms of parabolics for the left side and jet groups for the right side. By means of a topological argument, we can now get an equivariant identification $\mathfrak{S}_w \cong \underline{S}_w(\mathbf{a}, \mathbf{a})$ (see [Ma88, Lem. 32, Lem. 33] and compare them to the references in the previous paragraph). This yields the desired equivariant identification of full flag varieties $\mathfrak{F} \otimes W \cong \underline{\mathcal{F}\ell}_{G,\mathbf{a}}$, as both are geometrically reduced over W, and then of the partial counterparts by taking quotients.

Now we deduce $\mathfrak{G} \otimes W \cong \widehat{L\underline{G}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$. Consider the $\mathfrak{B}$-bundle $\mathfrak{G}_w$ over $\mathfrak{S}_w$ obtained as the affine hull with respect to $\mathfrak{S}_w$ of the canonical $\mathfrak{B}$-bundle over the Demazure variety $\mathfrak{D}_{\tilde{w}}$, see [Ma88, Ch. XI]. By the universal property of the relative affine hull, we get a $\mathfrak{B} \cong \widehat{L^+\underline{\mathcal{G}}_{\mathbf{a}}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$-equivariant morphism towards the preimage of $\underline{S}_w(\mathbf{a}, \mathbf{a})$ in $\widehat{L\underline{G}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$, which then must be an isomorphism. Taking direct limits now recovers the isomorphism $\mathfrak{G} \otimes W \cong \widehat{L\underline{G}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$.

Finally let us prove $\mathfrak{U}_J^- \otimes W \cong L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J}$ inside $L\underline{G}$. First we note that $\mathfrak{U}_J^- \otimes W$ lies in $\widehat{L\underline{G}}$ by naturality of strict repellers, because any J-regular dominant coweight μ has positive image in $\mathbb{G}_m^{\mathrm{rot}}$. Recall from the proof of Lemma 4.1 that $\mathfrak{U}_J^-(k)$ is generated by the J-negative real root subgroups, and thus $\mathfrak{U}_J^-(k)$ maps into $L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J}(k)$ for any algebraically closed field $W \to k$. Now the composition $\mathfrak{U}_J^- \otimes W \subset \widehat{L\underline{G}} \to L\underline{G}$ is a (representable) quasi-compact monomorphism of reduced ind-schemes. Since $L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J}$ is a closed sub-ind-scheme of $L\underline{G}$, we obtain a map $p\colon \mathfrak{U}_J^- \otimes W \to L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J}$ which is a quasi-compact monomorphism. Now consider the commutative diagram of ind-schemes

$$\begin{tikzcd}
\widehat{L\underline{G}} \arrow[d, "q"', "\text{smooth}"'] & \mathfrak{U}_J^- \otimes W \arrow[l, "\text{closed}"'] \arrow[r, "\text{open}"] \arrow[d, "p"] & \mathfrak{F}_J \otimes W \arrow[d, "\cong"] \\
L\underline{G} & L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J} \arrow[l, "\text{closed}"'] \arrow[r, "\text{open}"] & \underline{\mathcal{F}\ell}_{G,J}
\end{tikzcd}$$

The right square implies that p is an open immersion. Since $q^{-1}(\mathfrak{U}_J^- \otimes W) = \mathfrak{U}_J^- \otimes W \times \mathbb{G}_m^{\mathrm{cent}}$ is closed in $\widehat{L\underline{G}}$, fppf (or smooth) descent for closed immersions implies that p is a closed immersion as well. Since $\mathfrak{U}_J^- \otimes W$ is non-empty and $L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J}$ is connected, the map p is an isomorphism. □

Remark 4.6. The above picture extends to the integers $\mathbb{Z}$ (in other words, to the wildly ramified cases) by work of the second-named author, resting on complicated group-theoretic constructions described in [Loua] for reduced relative root systems and completed in [Louc] for the remaining case of odd unitary groups.

Remark 4.7. We can identify $\mathfrak{U}_J^-(k) \cong L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J}(k)$ for any algebraically closed field $W \to k$ inside $L\underline{G}(k)$ by means of a combinatorial argument. For this, recall that $\mathfrak{U}_J^-(k)$ is generated by the J-negative real root subgroups it contains. The fact that $L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J}(k)$ shares the same generation property is often stated without proof in Kac-Moody theory (compare with [Ti84, App. 2] and [Ti89, §1.3, §4.]), but we will give one for completeness. First of all, assume that $J = \emptyset$, so that $\mathbf{f}_J = \mathbf{a}$ is an alcove. Note that $L^{-}\underline{\mathcal{G}}_{\mathbf{a}}(k) = S \ltimes L^{--}\underline{\mathcal{G}}_{\mathbf{a}}(k)$ fits into an adequate Birkhoff decomposition by [HNY13, Prop. 1.1] which we can compare with the induced Birkhoff decompostion on $L\underline{G}$ coming from (4.1) for the Kac-Moody group in the case $J = \emptyset$ (this is legitimate because the two groups differ by at most $\mathbb{G}_m$-factors in the maximal torus). Write $g \in L^{--}\underline{\mathcal{G}}_{\mathbf{a}}(k)$ as $u\dot{w}b$ with $u \in U_\emptyset^-$ and $b \in L^+\underline{\mathcal{G}}_{\mathbf{a}}(k)$; we see that $w = 1$ and b lies in $L^{--}\underline{\mathcal{G}}_{\mathbf{a}}(k) \cap L^+\underline{\mathcal{G}}_{\mathbf{a}}(k) = 1$ (recall Corollary 3.9). This proves $L^{--}\underline{\mathcal{G}}_{\mathbf{a}}(k) = U_\emptyset^-$, hence the generation result for the left hand side. In general, the subgroup $U_\emptyset^-$ is the semi-direct product of U_J^- and the group generated by all negative real root groups contained in $L_J \subseteq L^+\underline{\mathcal{G}}_{\mathbf{f}_J}(k)$, so the inclusion $L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J}(k) \subseteq U_\emptyset^-$ and the triviality of the intersection $L^+\underline{\mathcal{G}}_{\mathbf{f}_J} \cap L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J}$ forces the desired equality $L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J}(k) = U_J^-$.

Remark 4.8. If we transport the $\mathbb{G}_m$-action on $\mathfrak{G}$ given by a J-regular dominant weight to LG, we can almost immediately deduce that $L^+\underline{\mathcal{G}}_{\mathbf{f}_J}$ is the attractor locus and $L^{--}\underline{\mathcal{G}}_{\mathbf{f}_J}$ the strict repeller locus. Suffice it to say that our proof of Lemma 4.1 (and to some extent of Lemma 3.6) was heavily inspired by the dynamical method of [CGP15, §2.1] and should be regarded as an infinite-dimensional generalization thereof.

5. Tangent spaces at base points

Here we combine the results from §§3–4 to give an effective criterion for the normality of Schubert varieties in §5.4. Somewhat surprisingly the relevant tangent space calculations can essentially be performed in characteristic zero, see §5.3.

5.1. **Preliminaries on tangent spaces.** We start with some general properties of tangent spaces of (ind)-schemes over a general base equipped with a section.

Definition 5.1. Let S be a scheme and X be a sheaf of sets on the category of S-schemes equipped with $x \in X(S)$. The tangent space T_xX of X at x is the sheaf which associates an S-scheme T to the pre-image $p_\epsilon^{-1}(x_T)$ of $x_T \in X(T)$ induced by x along the map $p_\epsilon : X(T[\epsilon]) \to X(T)$. Here by definition $T[\epsilon] = T \times \operatorname{Spec} \mathbb{Z}[\epsilon]$ where $\epsilon^2 = 0$.

If X is representable by a scheme, our tangent space coincides with the (implicit) definition of Demazure-Gabriel (see [DG70, II, §4, Cor. 3.3]):

Proposition 5.2. *Let $X \to S$ be a scheme endowed with a section $x\colon S \to X$. For any S-scheme T, there is a natural bijection of sets $T_xX(T) = \operatorname{Hom}_{\mathcal{O}_T}(x_T^*\Omega_{X_T/T}, \mathcal{O}_T)$. In particular, $T_xX(T)$ has a natural structure of an $\Gamma(T, \mathcal{O}_T)$-module.*

Proof. Since S is an arbitrary scheme, we reduce to the case $S = T$. To give $f \in T_xX(S)$, i.e., a morphism $f\colon S[\epsilon] \to X$ compatible with x, is the same as to give an S-derivation $d\colon \mathcal{O}_X \to x_*\mathcal{O}_S$: since $|S[\epsilon]| = |S|$ on topological spaces, such an f is the same as to a morphism of sheaves of rings $f^\#\colon \mathcal{O}_X \to x_*\mathcal{O}_{S[\epsilon]} = x_*\mathcal{O}_S \oplus \epsilon x_*\mathcal{O}_S$. The compatibility with x implies $f^\# = x^\# + \epsilon d_f$ and it is easily verified that $d_f \in \operatorname{Der}_S(\mathcal{O}_X, x_*\mathcal{O}_S)$ is an S-derivation. We thus get natural bijections

$$T_xX(S) = \operatorname{Der}_S(\mathcal{O}_X, x_*\mathcal{O}_S) = \operatorname{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, x_*\mathcal{O}_S) = \operatorname{Hom}_{\mathcal{O}_S}(x^*\Omega_{X/S}, \mathcal{O}_S),$$

where the second identification is [StaPro, 01UR], and the last identification is the adjunction. □

Hence, if $T = \mathrm{Spec}(R)$ is an affine scheme, then $T_xX(R)$ is an R-module by the preceding proposition.

Corollary 5.3. *Let R be a ring and $i : (X, x) \to (Y, y)$ be a monomorphism of pointed R-schemes. Then the induced homomorphism $i_*\colon T_xX(R) \to T_yY(R)$ of R-modules is injective. If i is an open immersion, then this homomorphism is bijective.*

Proof. This is immediate from the definition, and the fact that monomorphisms are formally unramified, and open immersions are formally étale (see [EGAIV4, Prop. 17.1.3.(i)]), combined with the exact sequence $i^*\Omega_{Y/R} \to \Omega_{X/R} \to \Omega_{X/Y} \to 0$. □

Corollary 5.4. *If $X = \operatorname{colim} X_i$ is a strict pointed ind-scheme over R, then $T_xX = \operatorname{colim} T_xX_i$ (here $T_xX_i := 0$ if $x \notin X_i(R)$) is an R-module independent of the chosen presentation as a strict ind-scheme.*

Proof. This is immediate from Corollary 5.3. □

It is worth noting that the tangent space does not commute with base change in general, whereby we mean the equality $T_xX(R)\otimes_R R' \to T_xX(R')$ for all R-algebras R', but we still have the following:

Lemma 5.5. *Maintain the notation of Proposition 5.2, and suppose moreover that $X \to S$ is of finite type and that $T = \operatorname{Spec} R$ is a Dedekind scheme. Then, for all R-algebras R', the canonical homomorphism $T_xX(R) \otimes_R R' \to T_xX(R')$ is injective. Moreover, it is bijective for all R-algebras R' if and only if $x^*\Omega_{X/R}$ is torsion-free.*

Proof. We may assume $S = T$ and hence that $X \to S$ is of finite presentation. After localizing, we can write $x^*\Omega_{X/R} = R^n \oplus M$ where M is a (finitely generated) torsion module, cf. [StaPro, 01V3]. Let $X' := X_{R'}$ with induced section denoted x'. Since $x'^*\Omega_{X'/R'} = R'^n \oplus (M \otimes_R R')$ by [StaPro, 01UV], we get

$$T_xX(R') = \mathrm{Hom}_{R'}(x'^*\Omega^1_{X'/R'}, R') = R'^n \oplus \mathrm{Hom}_{R'}(M \otimes_R R', R').$$

Using $\mathrm{Hom}_R(M, R) = 0$ (because R is torsion-free), the lemma follows. This also shows that bijectivity is equivalent to $\mathrm{Hom}_R(M, R') = 0$ for all R-algebras R', which in turn amounts to asking $M = 0$ by Nakayama's lemma. □

Lemma 5.6. *Suppose $i\colon (X, x) \to (Y, y)$ is a closed immersion of pointed ind-schemes of ind-finite type over a Dedekind ring R. Then the cokernel of $i_*\colon T_xX(R) \to T_yY(R)$ is a flat R-module.*

Proof. By Corollary 5.4, we may and do assume that X and Y are finite-type schemes. After localization, we may also assume that R is a discrete valuation ring with uniformizer π. Assume there is a $v \in T_yY(R) \setminus T_xX(R)$ such that $\pi v \in T_xX(R)$. By Corollary 5.3 and Lemma 5.5, we have have injections

$$T_xX(R)/\pi T_xX(R) \ \subset\ T_xX(R/\pi R) \ \subset\ T_yY(R/\pi R).$$

Since $\pi v = 0$ in $T_yY(R/\pi R)$, we get $\pi v \in \pi T_xX(R)$, i.e., the existence of some $w \in T_xX(R)$ such that $\pi w = \pi v$. As $T_yY(R)$ is free of finite rank (so in particular R-torsion free) by the proof of Lemma 5.5, we reach a contradiction. This proves the lemma. □

5.2. **Tangent spaces of affine flag varieties.** Let us give a description of the tangent space of the affine flag variety. We proceed with the assumptions and notations of §3.

Lemma 5.7. *For the tangent space at the base point $e \in \underline{\mathcal{F}\ell}_{G,\mathbf{f}}(R)$ with values in a $W(k)$-algebra R, one has as R-modules*

$$T_e\underline{\mathcal{F}\ell}_{G,\mathbf{f}}(R) \;=\; T_eL^{--}\underline{\mathcal{G}}_{\mathbf{f}}(R) \;=\; \Bigg(\bigcap_{w\in W_{H,\mathbf{f}}} {}^{w}\big((L^{--}\mathfrak{h}) \rtimes \mathfrak{u}_H^{\mathrm{op}}\big) \otimes R\Bigg)^{\sigma},$$

where $\mathfrak{h}$ (resp. $\mathfrak{u}_H^{\mathrm{op}}$) denotes the Lie algebra of H (resp. U_H^{op}) over $W(k)$. This is a free R-module and its formation commutes with arbitrary base change.

Proof. By Corollary 3.9, the map $L^{--}\underline{\mathcal{G}}_{\mathbf{f}} \to \underline{\mathcal{F}\ell}_{G,\mathbf{f}}, g \mapsto g \cdot e$ is representable by a quasi-compact open immersion. This immediately implies

$$T_e\underline{\mathcal{F}\ell}_{G,\mathbf{f}}(R) = T_e L^{--}\underline{\mathcal{G}}_{\mathbf{f}}(R) = T_e(L^{--}\underline{\mathcal{H}}_{\mathbf{f}})^\sigma(R).$$

Using that T_e(-) commutes with taking fixed points (-)$^\sigma$ and intersections, the corollary follows from Definition (3.8).

Next we observe that the R-module $T_e\underline{\mathcal{F}\ell}_{G,\mathbf{f}}(R)$ is projective, as the σ-averaging map furnishes a retraction to its inclusion in the free module

$$\bigcap_{w \in W_{H,\mathbf{f}}} {}^w\big((L^{--}\mathfrak{h}) \rtimes \mathfrak{u}_H^{\mathrm{op}}\big) \otimes R.$$

A similar argument shows that the tangent space is compatible with base change, i.e. the natural map $T_e\underline{\mathcal{F}\ell}_{G,\mathbf{f}}(W(k)) \otimes R \to T_e\underline{\mathcal{F}\ell}_{G,\mathbf{f}}(R)$ is an isomorphism for all $W(k)$-algebras R (use the σ-averaging retraction applied to the obvious equality in the split case). Hence it suffices to observe that $T_e\underline{\mathcal{F}\ell}_{G,\mathbf{f}}(W(k))$ is free, which follows from Kaplansky's theorem, because $W(k)$ is local. □

Example 5.8. Assume that $\mathbf{f} = \mathbf{0}$ is the base point of $\mathscr{A}(G, S, F)$ which is an absolutely special vertex. Recall that absolutely special vertices exist for all quasi-split groups by [HRb, Lem. 5.2]. In this case, $L^{--}\mathcal{G}_{\mathbf{f}} = (L^{--}H)^{\sigma,o}$ so that we obtain

$$T_e\underline{\mathcal{F}\ell}_{G,\mathbf{f}}(R) = (L^{--}\mathfrak{h} \otimes R)^\sigma = \bigoplus_{i \geq 1} \big(\mathfrak{h} \otimes R[u^{-i}]\big)^\sigma.$$

5.3. **Tangent spaces of Schubert varieties.** Within this subsection, we additionally assume G to be simply connected. By Proposition 4.5, there is a canonical isomorphism of W-ind-schemes $\underline{\mathcal{F}\ell}_{G,\mathbf{f}} \cong \mathfrak{F}_J \otimes W$ inducing isomorphisms of integral Schubert varieties $\underline{S}_w \cong \mathfrak{S}_w \otimes W$ for all $w \in W/W_{\mathbf{f}}$. Given any ample line bundle $\mathcal{L}$ on $\underline{\mathcal{F}\ell}_{G,\mathbf{f}}$, we note that $\widehat{L\underline{G}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$ naturally acts on $\Gamma(\underline{\mathcal{F}\ell}_{G,\mathbf{f}}, \mathcal{L})^\vee := \mathrm{colim}_w\, \Gamma(\underline{S}_w, \mathcal{L})^\vee$. Restricting this action to $L^{--}\underline{\mathcal{G}}_{\mathbf{f}}$ and taking the tangent spaces at base points, we obtain the action of $T_e\underline{\mathcal{F}\ell}_{G,\mathbf{f}}$ on $\Gamma(\underline{\mathcal{F}\ell}_{G,\mathbf{f}}, \mathcal{L})^\vee$. Under the isomorphisms in Proposition 4.5, this is nothing but the Kac-Moody action used in Corollary 4.3.

Lemma 5.9. *The R-valued tangent space $T_e\underline{S}_w(R)$ identifies with the submodule of $T_e\underline{\mathcal{F}\ell}_{G,\mathbf{f}}(R)$ consisting of those X such that $X\Theta_{\mathcal{L}}^\vee$ lies in $\Gamma(\underline{S}_w, \mathcal{L})^\vee$, where $\Theta_{\mathcal{L}} \in \Gamma(\underline{\mathcal{F}\ell}_{G,\mathbf{f}}, \mathcal{L})$ is the usual theta divisor attached to $\mathcal{L}$ with support given by the complement of $L^{--}\underline{\mathcal{G}}_{\mathbf{f}} \cdot e$.*

Proof. This is nothing but a reformulation of Corollary 4.3 using the isomorphism $\mathfrak{U}_J^- \otimes W \cdot e \cong L^{--}\underline{\mathcal{G}}_{\mathbf{f}} \cdot e$ from Proposition 4.5. □

5.4. **Application to the normality criterion.** Let us now turn to the situation at hand. Let G be a tamely ramified, absolutely almost simple, semisimple F-group which has the same splitting field as its simply connected cover $G_{\mathrm{sc}} \to G$. The set-up of §3 applies to both groups G_{sc}, G and we use it to determine the kernel of the map $\mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}} \to \mathcal{F}\ell_{G,\mathbf{f}}$ on tangent spaces at the base points, cf. Corollary 5.12.

We proceed with the notation of §3. The map $H_{\mathrm{sc}} \to H$ on Chevalley groups extends to a map on parahoric $k[\![u]\!]$-group schemes $\mathcal{H}_{\mathrm{sc},\mathbf{f}} \to \mathcal{H}_{\mathbf{f}}$. This induces a map on strictly negative loop groups $L^{--}\mathcal{H}_{\mathrm{sc},\mathbf{f}} \to L^{--}\mathcal{H}_{\mathbf{f}}$ over k, and hence a map on twisted strictly negative loop groups

$$L^{--}\mathcal{G}_{\mathrm{sc},\mathbf{f}} \to L^{--}\mathcal{G}_{\mathbf{f}}. \tag{5.1}$$

We want to determine the kernel of (5.1). There is a central extension of flat affine $\mathbb{Z}$-group schemes

$$1 \to Z_H \to H_{\mathrm{sc}} \to H \to 1,$$

where Z_H is a suitable σ_0-invariant subgroup of the center of H_{sc}. Then Z_H is a finite flat $\mathbb{Z}$-group scheme of multiplicative type which is étale over $\mathbb{Z}_{(p)}$ if and only if $p \nmid |Z_H| = |\pi_1(G)|$.

Definition 5.10. Let Z be the kernel of $G_{\mathrm{sc}} \to G$. The strictly negative loop group for Z is the subgroup functor of LG_{sc} defined as

$$L^{--}Z \overset{\mathrm{def}}{=} (L^{--}Z_{H,k(\!(u)\!)})^{\sigma,o} \subset (LH_{\mathrm{sc},k(\!(u)\!)})^{\sigma} = LG_{\mathrm{sc}}.$$

Note that $L^{--}Z$ is representable by a closed subgroup ind-scheme of LG_{sc}.

Lemma 5.11. *There is a short exact sequence of group functors*

$$1 \to L^{--}Z \to L^{--}\mathcal{G}_{\mathrm{sc},\mathbf{f}} \to L^{--}\mathcal{G}_{\mathbf{f}}.$$

Proof. Clearly, there is a short exact sequence $1 \to L^{--}Z_{H,k(\!(u)\!)} \to L^{--}H_{\mathrm{sc},k(\!(u)\!)} \to L^{--}H_{k(\!(u)\!)}$. Using that $U^{\mathrm{op}}_{H_{\mathrm{sc}}} = U^{\mathrm{op}}_{H}$ for the opposite unipotent radicals and that $W_{H_{\mathrm{sc}},\mathbf{f}} = W_{H,\mathbf{f}}$ in (3.8), we obtain a short exact sequence

$$1 \to L^{--}Z_{H,k(\!(u)\!)} \to L^{--}\mathcal{H}_{\mathrm{sc},\mathbf{f}} \to L^{--}\mathcal{H}_{\mathbf{f}}.$$

The lemma now follows from Definition (3.8) by passing to σ-invariants (which is left exact) and by taking neutral components. □

By Corollary 5.7, we obtain a k-vector subspace

$$T_eL^{--}Z \;\subset\; T_eL^{--}\mathcal{G}_{\mathrm{sc},\mathbf{f}} \;=\; T_e\mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}},$$

where $e \in \mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}}(k)$ denotes the base point. Recall from (2.1) that there is a map of Schubert varieties $S_{\mathrm{sc},w} = S_{\mathrm{sc},w}(\mathbf{a},\mathbf{f}) \to S_w(\mathbf{a},\mathbf{f}) = S_w$ for each $w \in W_{\mathrm{aff}}/W_{\mathbf{f}}$.

Corollary 5.12. *For each class $w \in W_{\mathrm{aff}}/W_{\mathbf{f}}$, the following are equivalent:*

(1) *The Schubert variety $S_w \subset \mathcal{F}\ell_{G,\mathbf{f}}$ is normal.*
(2) *One has*

$$(T_eL^{--}Z) \cap (T_eS_{\mathrm{sc},w}) \;=\; 0$$

as k-vector subspaces of $T_e\mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}}$.

Proof. By Proposition 2.1, part (1) is equivalent to $\ker(T_eS_{\mathrm{sc},w} \to T_eS_w) = 0$ where e denotes the base point of both $\mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}}$ and $\mathcal{F}\ell_{G,\mathbf{f}}$. Lemma 5.11 implies that there is an exact sequence of k-vector spaces

$$0 \to T_eL^{--}Z \to T_eL^{--}\mathcal{G}_{\mathrm{sc},\mathbf{f}} \to T_eL^{--}\mathcal{G}_{\mathbf{f}},$$

so that $\ker(T_eS_{\mathrm{sc},w} \to T_eS_w) = (T_eL^{--}Z) \cap (T_eS_{\mathrm{sc},w})$. This proves the corollary. □

Remark 5.13. By [PR08, Thm. 0.2], the ind-scheme $\mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}}$ is reduced so that $\mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{f}} = \mathrm{colim}_w\, S_{\mathrm{sc},w}$ is a presentation. Thus, Corollary 5.12 shows that, if $L^{--}Z$ is non-trivial, there are infinitely many $(\mathbf{a},\mathbf{f})$-Schubert varieties inside $\mathcal{F}\ell_{G,\mathbf{f}}$ which are not normal, hence not weakly normal, not Frobenius split and not Cohen-Macaulay.

6. Towards a classification of normal Schubert varieties

Let k be an algebraically closed field of characteristic $p > 0$, and let G be a tamely ramified, absolutely simple group over $F = k(\!(t)\!)$. Examining the tables in [Bou, Ch. VI, Planche IX] and [Ti77, §4], here is the list of all such pairs (G, p) such that $p \mid |\pi_1(G)|$. Split groups:

- type A_n, $n \geq 1$ and $p \,|\, n+1$;
- type B_n, $n \geq 2$ and $p = 2$;
- type C_n, $n \geq 2$ and $p = 2$;
- type D_n, $n \geq 3$ and $p = 2$;
- type E_6 and $p = 3$;
- type E_7 and $p = 2$.

The split groups E_8, F_4 and G_2 have connection index 1, and hence are excluded from the list. Twisted groups:

- type B-C_n, $n \geq 3$ and $p \,|\, 2n$, $p \neq 2$ (even unitary);
- type C-BC_n, $n \geq 1$ and $p \,|\, 2n+1$ (odd unitary);

- type F_4^I and $p = 3$ (ramified E_6);
- type G_2^I and $p = 2$ (ramified triality);

The twisted orthogonal groups C-B_n, $n \geq 2$ are excluded by our tamely ramified hypothesis.

The methods developed in the preceding paragraphs allow us to give a quantitative criterion for the normality of Schubert varieties in general partial affine flag varieties, see Propositions 6.4 and 6.5. The key input is the computation of the tangent spaces of quasi-minuscule Schubert varieties in twisted affine Grassmannians for absolutely special vertices in §6.1. In §6.3 we discuss the example of PGL_2 in characteristic 2 which is much easier. In general the classification of all finitely many normal Schubert varieties in the flag variety for each pair (G, p) in the above list seems to be a challenging problem, see §6.2 for some further discussion.

6.1. **Absolutely special vertices.** We proceed with the assumptions and notations of §5.4. In particular, G is a tamely ramified, absolutely almost simple, semi-simple F-group which has the same splitting field as its simply connected cover $G_{\mathrm{sc}} \to G$.

We further assume $\mathbf{f} = \mathbf{0}$ is the fixed absolutely special vertex in $\mathscr{A}(G, S, F)$. Our aim is to give an effective criterion for the normality of $(\mathbf{a}, \mathbf{0})$-Schubert varieties inside the neutral component of the twisted affine Grassmannian $\mathrm{Gr} := \mathcal{F}\ell_{G,\mathbf{0}}$. For this, we study the tangent spaces of $(\mathbf{a}, \mathbf{0})$-Schubert varieties inside $\mathrm{Gr}_{\mathrm{sc}} := \mathcal{F}\ell_{G_{\mathrm{sc}},\mathbf{0}}$. The $L^+\mathcal{G}_{\mathrm{sc},\mathbf{a}}$-orbits inside $\mathrm{Gr}_{\mathrm{sc}}$ are enumerated by the set $W_{\mathrm{aff}}/W_{\mathbf{0}} = X_*(T_{\mathrm{sc}})_I$, the coinvariants under the Galois group $I := \mathrm{Gal}(F'/F)$ where F'/F is the splitting field. For each $\bar{\mu} \in X_*(T_{\mathrm{sc}})_I$, we denote by $S_{\mathrm{sc},\bar{\mu}} \subset \mathrm{Gr}_{\mathrm{sc}}$ the corresponding $(\mathbf{a}, \mathbf{0})$-Schubert variety. In view of Corollary 5.12 we have to determine exactly those $\bar{\mu} \in X_*(T_{\mathrm{sc}})_I$ such that $(T_e L^{--} Z) \cap (T_e S_{\mathrm{sc},\bar{\mu}}) = 0$ inside

$$T_e \mathrm{Gr}_{\mathrm{sc}} \;=\; \bigoplus_{i \geq 1} \left(\mathfrak{h}_{\mathrm{sc}}[u^{-i}]\right)^{\sigma}, \tag{6.1}$$

cf. Example 5.8. Our normality criterion rests on the following key calculation.

Proposition 6.1. *Let $\bar{\mu} \in X_*(T_{\mathrm{sc}})_I$ be the unique B-dominant, quasi-minuscule element. Then*

$$T_e S_{\mathrm{sc},\bar{\mu}} \;\supset\; \left(\mathfrak{h}_{\mathrm{sc}}[u^{-1}]\right)^{\sigma}$$

as k-vector subspaces of (6.1), and equality holds if $\mathrm{char}(k) = 0$.

Proof. For the proof of this inclusion, we may and do assume $\mathrm{char}(k) = 0$ by Lemmas 5.5 and 5.6 combined with Proposition 3.4, all applied to the normal Schubert variety $\underline{S}_{\mathrm{sc},\bar{\mu}}$. The equality follows then from our work with minimal nilpotent orbits in Appendix C, which extends previous results of [MOV05, §2.9] and [HRb, §8]. For convenience of the reader, let us just note that the inclusion $T_e S_{\mathrm{sc},\bar{\mu}} \supset \left(\mathfrak{h}_{\mathrm{sc}}[u^{-1}]\right)^{\sigma}$ is much simpler - and this is all we will need to prove the important Corollary 6.2 below. Indeed, the intersection of both sides is certainly non-trivial, as we see by looking at the $(\mathbf{a}, 0)$-Schubert variety S_w for the affine simple reflection $w = s_0$. Moreover, both tangent spaces carry an action by the split group $H_{\mathrm{sc}}^{\sigma_0}$. Now we use that the right side is an irreducible $H_{\mathrm{sc}}^{\sigma_0}$-module: this is obvious in the split case, because we get the adjoint representation; in the twisted case, it is proved in Proposition C.1, Proposition C.2 and [HRb, Lem. 8.4]. □

Corollary 6.2. *If $p \mid |\pi_1(G)|$, then the quasi-minuscule Schubert variety inside* Gr *is not normal.*

Proof. Let $\mathfrak{z}_H$ denote the Lie algebra over k of the kernel Z_H of $H_{\mathrm{sc}} \to H$; cf. §5.4. Note $\mathfrak{z}_H$ is nonzero since Z_H is not étale over k by assumption. Combining Proposition 6.1 with Corollary 5.12 (2), it is enough to show that the subspace $(\mathfrak{z}_H[u^{-1}])^{\sigma} \subset (\mathfrak{h}_{\mathrm{sc}}[u^{-1}])^{\sigma}$ is non-trivial. If G is split, so that σ acts trivially, then $\mathfrak{z}_H[u^{-1}]$ is clearly non-trivial. If G is non-split, we go through the possible types for H listed in the beginning of §6. First for simplicity assume G, and hence H, is adjoint, so that Z_H is the center of H_{sc} and $\mathfrak{z}_H$ is the center of $\mathfrak{h}_{\mathrm{sc}}$, that is, the kernel of the adjoint representation, see [Co14, Prop. 3.3.8 ff.]. If H is of type A_n, then $\mathfrak{z}_H$ is spanned by the element

$$\sum_{i=1}^{n} i\alpha_i^{\vee}.$$

Then we notice the congruence $n+1-i \equiv -i$ modulo p and use that $\alpha_i^\vee \mapsto \alpha_{n+1-i}^\vee$ and $u^{-1} \mapsto -u^{-1}$ under σ. If H is of type D_4, then σ_0 permutes the roots as follows: $\alpha_1 \mapsto \alpha_3 \mapsto \alpha_4 \mapsto \alpha_1$, $\alpha_2 \mapsto \alpha_2$. It follows that $\mathfrak{z}_H$ contains in characteristic $p = 2$ the element

$$\alpha_1^\vee + \zeta^{-1}\alpha_3^\vee + \zeta^{-2}\alpha_4^\vee,$$

which becomes σ-invariant after multiplying by u^{-1}. Here ζ is a primitive 3rd root of unity in the notation of §3.1. To check the containment, multiply the Cartan matrix by the column vector $(1, 0, \zeta^{-1}, \zeta^{-2})$ and show that the sum of the entries in each row vanishes modulo 2. Finally in the E_6 type, a similar argument in characteristic $p = 3$ shows that $\mathfrak{z}_H$ contains

$$\alpha_1^\vee + 2\alpha_3^\vee - 2\alpha_5^\vee - \alpha_6^\vee.$$

To show that this element becomes σ-invariant after multiplying by u^{-1} use that $\alpha_1^\vee \leftrightarrow \alpha_6^\vee$, $\alpha_3^\vee \leftrightarrow \alpha_5^\vee$ and $u^{-1} \mapsto -u^{-1}$ under σ. This proves the corollary in the adjoint case.

To handle the general non-split cases where H is of type A_n, note that $\mathfrak{z}_H \neq 0$ must be the entire 1-dimensional center of $\mathfrak{h}_{\mathrm{sc}}$. In type E_6, the group Z_H is a non-trivial subgroup of the center $Z_{H_{\mathrm{sc}}} = \mu_3$ of H_{sc}, hence equal, so that G must be adjoint in this case. If H is of type D_4, then the center of H_{sc} is $Z_{H_{\mathrm{sc}}} = \mu_2 \times \mu_2$, with Lie algebra $\mathfrak{z}_{H_{\mathrm{sc}}} = k \oplus k$. The order 3 automorphism σ_0 preserves $Z_{H_{\mathrm{sc}}}$ and acts on $\mathfrak{z}_{H_{\mathrm{sc}}}$ (up to choice of basis) by $e_1 \mapsto e_1 + e_2$, $e_2 \mapsto e_1$. Now if $1 \subsetneq Z_H \subsetneq Z_{H_{\mathrm{sc}}}$ were σ_0-invariant and non-étale, then $Z_H \cong \mu_2$ (use Cartier duality), and hence σ_0 acts trivially on Z_H (since $\mathrm{Aut}(\mu_2) \cong 1$). In this case σ_0 would fix a vector in $\mathfrak{z}_{H_{\mathrm{sc}}} = k \oplus k$, a contradiction. It follows that only $Z_H = Z_{H_{\mathrm{sc}}}$ occurs so that G must be adjoint in this case as well. This proves the corollary. □

Using the absolutely[5] special vertex $\mathbf{0} \in \bar{\mathbf{a}}$, we identify $\mathscr{A} = \mathscr{A}(G, S, F)$ with $X_*(T)_{I,\mathbb{R}}$, where $I = \mathrm{Gal}(\bar{F}/F)$. Recall that the Iwahori-Weyl group W acts by affine linear transformations on $\mathscr{A}$. We use the Bruhat-Tits convention: $t \in T(F)$ acts by translation by $-\kappa_T(t)$[6], where $\kappa_T : T(F) \twoheadrightarrow X_*(T)_I$ is the Kottwitz homomorphism constructed in [Ko97, §7]. Following [HaRa, Prop. 13, Lem. 14], we get isomorphisms

$$W \xrightarrow{\sim} W_{\mathrm{aff}} \rtimes \Omega_{\mathbf{a}} \cong X_*(T)_I \rtimes W_{\mathbf{0}},$$

where $\Omega_{\mathbf{a}}$ is the subgroup of W preserving $\mathbf{a}$, where the map $\mathrm{Norm}_G T(F) \to X_*(T)_I \rtimes W_{\mathbf{0}}$ extends $\kappa_T : T(F) \twoheadrightarrow X_*(T)_I$. In particular, we have the group embedding $X_*(T)_I \hookrightarrow W$ denoted $\nu \mapsto t^\nu$ where t^ν is characterized by the property $\kappa_T(t^\nu) = \nu$ (if T is split, $t^\nu = \nu(t) \bmod \ker(\kappa_T)$). According to the Bruhat-Tits convention, the element t^ν, and hence ν, acts on $\mathscr{A}$ by translation by the image of $-\nu$ in $X_*(T)_{I,\mathbb{R}}$. We may view W_{aff} as the Coxeter group generated by the reflections through the walls of $\mathbf{a}$. Using the isomorphism, we transport the Bruhat order on $W_{\mathrm{aff}} \rtimes \Omega_{\mathbf{a}}$ to one on W; this induces the Bruhat order on $W/W_{\mathbf{0}}$. Our choice of embedding $X_*(T)_I \hookrightarrow W$ induces a well-defined bijection of sets $X_*(T)_I \xrightarrow{\sim} W/W_{\mathbf{0}}$, and we consider the transported Bruhat order on $X_*(T)_I$. We are going to need the following combinatorial description of the Bruhat order on $X_*(T)_I$, which can be found in [BH, Thm. 2.5] for split groups.

Recall (cf. [HaRa]) that $W_{\mathrm{aff}} = W_{\mathrm{aff}}(\Sigma)$ for the échelonnage roots $\Sigma = \Sigma(G, S, F)$; these have the property that the Coxeter complex determined by the affine roots $\Phi_{\mathrm{af}}(G, S, F)$ of Tits [Ti77, §1.6] is that given by the affine functionals of the form $\beta + n$ for $\beta \in \Sigma$, $n \in \mathbb{Z}$. Let $Q^\vee = \mathbb{Z}[\Sigma^\vee]$ be the échelonnage coroot lattice; it may be identified with $X_*(T_{\mathrm{sc}})_I$. In what follows, all finite and affine roots mentioned will be échelonnage (affine) roots. Let C^+ be the Weyl chamber in $\mathscr{A}$ which contains $\mathbf{a}$ and has apex $\mathbf{0}$. We say a finite root β (resp., affine root $\beta + n$) is positive (and write $\beta > 0$, resp., $\beta + n > 0$) if β takes positive values on C^+ (resp., $\mathbf{a}$). Recall that W_{aff} is the Coxeter group generated by the reflections $s_{\beta+n}$ in the simple positive affine roots $\beta + n$.

Proposition 6.3 (Besson-Hong)**.** *Given two coweights $\lambda, \mu \in X_*(T)_I$, the inequality $\lambda \leq \mu$ holds if and only if $\lambda - \mu \in Q^\vee$ and there is a sequence of coweights $\mu_i \in X_*(T)_I$ such that $\mu_0 = \mu$,*

[5] For this discussion, any special vertex will do.

[6] More precisely, it acts by the image of this element in $X_*(T)_{I,\mathbb{R}}$; recall $X_*(T)_I$ might have torsion.

$\mu_r = \lambda$ and satisfying the following: there is a positive root α_i such that either $\mu_{i+1} = \mu_i - k\alpha_i^\vee$ with $0 \le k \le \langle \alpha_i, \mu_i \rangle$ or $\mu_{i+1} = \mu_i + k\alpha_i^\vee$ with $0 \le k < -\langle \alpha_i, \mu_i \rangle$.

It was already well-known that, if λ and μ lie in a common Weyl chamber, then the Bruhat order described above coincides with the usual dominance partial order with respect to the given Weyl chamber (cf. [Ra05, Lem. 3.8, Prop. 3.5], [BH, Thm. 4.1]).

Proof. By definition $\lambda \le \mu$ if and only if $t^\lambda \le t^\mu$ in the Bruhat order on $W/W_{\mathbf{0}}$. Let $w_\nu \in W$ be the minimal length element in $t^\nu W_{\mathbf{0}}$. The Bruhat order on $W/W_{\mathbf{0}}$ is generated by the following relation between $w_{\nu'}, w_\nu$ for pairs of distinct elements $\nu, \nu' \in X_*(T)_I$: there is an affine reflection $s_{\beta+n}$ with $\beta + n$ positive such that

$$w_{\nu'} > s_{\beta+n} w_{\nu'}$$

in the Bruhat order on W, and $s_{\beta+n} w_{\nu'} W_{\mathbf{0}} = w_\nu W_{\mathbf{0}}$. This is the same as saying that $s_{\beta+n}(-\nu') = -\nu$, and the point $-\nu' + \mathbf{0}$ and the alcove $\mathbf{a}$ are on opposite sides of the affine hyperplane $H_{\beta+n}$, that is, $-\langle \beta, \nu' \rangle + n < 0$.

Therefore, $t^\lambda < t^\mu$ if and only if $\lambda - \mu \in Q^\vee$ and there exists a sequence of reflections $s_i = s_{\beta_i + n_i}$, ($0 \le i \le r-1$, $\beta_i + n_i > 0$), such that as elements in $X_*(T)_{I,\mathbb{R}}$ we have $-\mu_0 = -\mu$, $-\mu_r = -\lambda = s_{r-1} \cdots s_0(-\mu)$, and where, for each $i \ge 0$, if $-\mu_i := s_{i-1} \cdots s_0(-\mu_0)$, then $-\langle \beta_i, \mu_i \rangle + n_i < 0$. Of course, we may assume $\mu_0, \dots, \mu_r$ has no repetitions.

By definition $-\mu_{i+1} = s_i(-\mu_i)$, that is,

$$-\mu_{i+1} = -\mu_i - \big(\langle \beta_i, -\mu_i \rangle + n_i\big)\beta_i^\vee.$$

Because $\beta_i + n_i$ is a positive affine root, we have $n_i \ge 0$ and $n_i = 0 \Rightarrow \beta_i > 0$.

(1) If $\beta_i > 0$ then $n_i \ge 0$ and $\mu_{i+1} = \mu_i - k\beta_i^\vee$ where $k = \langle \beta_i, \mu_i \rangle - n_i$. Note that $0 < k \le \langle \beta_i, \mu_i \rangle$. Set $\alpha_i = \beta_i$.

(2) If $\beta_i < 0$ then $n_i \ge 1$, and $\mu_{i+1} = \mu_i + k(-\beta_i)^\vee$, where $k = \langle \beta_i, \mu_i \rangle - n_i$. Note that $0 < k < -\langle -\beta_i, \mu_i \rangle$. Set $\alpha_i = -\beta_i$.

Conversely, given the positive root α_i and integer k satisfying the given restrictions, we may define the positive affine root $\beta_i + n_i$ using (1) or (2), for which we have $-\langle \beta_i, \mu_i \rangle + n_i < 0$. $\square$

In the following we apply this to uniformly bound the subset of normal Schubert varieties for absolutely almost simple semisimple groups such that $p \mid |\pi_1(G)|$.

Proposition 6.4. *Let G be an absolutely almost simple semisimple group whose simply connected cover is a non-étale isogeny. Then the set of $\lambda \in Q^\vee$ such that S_λ is normal is finite. More precisely, it is contained in the finite complement of all $\lambda \in Q^\vee$ such that $\lambda \ge -2\theta^\vee$, where θ denotes the highest root for the échelonnage root system $\Sigma(G,S,F)$.*

Proof. We start by observing that $-2\theta^\vee$ is bigger than $\theta^\vee$. Indeed, $-\langle \theta, -2\theta^\vee \rangle = 4$ and thus $\theta^\vee = -2\theta^\vee + 3\theta^\vee$ is less than $-2\theta^\vee$ for the partial Bruhat order, see Proposition 6.3. By Corollary 2.2 (2) combined with Corollary 6.2, this gives the proposition as soon as we know that the complement of $\{\lambda \in Q^\vee \mid \lambda \ge -2\theta^\vee\}$ in $Q^\vee$ is finite.

Suppose C_1 and C_2 are two adjacent closed Weyl chambers such that C_1 lies in a minimal gallery connecting the dominant Weyl chamber to C_2. Then there is a positive root α such that the wall of the reflection s_α bounds C_1 and C_2, in such a way that C_1 lies on the nonnegative side with respect to α. In particular, if $\lambda \in C_1$, then $s_\alpha \lambda \in C_2$ and the inequality $s_\alpha \lambda \le \lambda$ holds, again by Proposition 6.3, as $\langle \alpha, \lambda \rangle \ge 0$.

This reduces us to considering only antidominant λ, that is, to showing that the set $\{\lambda \in Q_+^\vee \mid -\lambda \not\ge -\lambda_0\}$ is finite for any fixed $\lambda_0 \in Q_+^\vee$. We will show the equivalent statement that $\{\lambda \in Q_+^\vee \mid \lambda \not\ge \lambda_0\}$ is finite. Dominance ensures we may write $\lambda = \sum n_i \alpha_i^\vee$ and $\lambda_0 = \sum n_{0,i} \alpha_i^\vee$, where $n_i, n_{0,i} \ge 0$ for all i. Writing $\lambda = \sum n_i \alpha_i^\vee$ and $\lambda_0 = \sum n_{i,0} \alpha_i^\vee$, it is enough to prove that for all j, $n_j \le 2^r \max_i\{n_{0,i}\}$ whenever $\lambda \not\ge \lambda_0$, where r is the number of nodes of the Dynkin diagram for $\Sigma(G,S,F)$. In this case, by Proposition 6.3 there is some i such that $n_i < n_{0,i}$. For $j \ne i$, set $r_{ij} = -\langle \alpha_i, \alpha_j^\vee \rangle \in \mathbb{Z}$. Assuming α_j is adjacent to α_i in the Dynkin diagram, $r_{ij} \ge 1$. By the dominance of λ, we see that

$2n_i - r_{ij}n_j \geq \langle \alpha_i, \lambda \rangle \geq 0$, which implies $n_j \leq 2n_i < 2\max_i\{n_{0,i}\}$. Repeating this argument shows $n_j \leq 2^r \max_i\{n_{0,i}\}$ for all j because the Dynkin diagram for $\Sigma(G, S, F)$ is connected. □

6.2. **General facets.** Here we keep virtually all notation introduced in the previous section, in particular we require G is absolutely almost simple, but we no longer assume that $\mathbf{f} = \mathbf{0}$. We rather assume that $\mathbf{f}$ and $\mathbf{0}$ are subfacets of a uniquely determined dominant alcove $\mathbf{a}$. For any $\lambda \in X_*(T)_I$, let w^λ (resp. w_λ) denote the maximal (resp. minimal) length element in $t^\lambda W_0$. Let θ be the highest échelonnage root of G. Then $\bar{\mu} = \theta^\vee$ is the unique quasi-minuscule coweight for the échelonnage root system $\Sigma(G, S, F)$. Fix any regular antidominant element $\delta \in X_*(T)_I$ such that $\delta \geq \theta^\vee$ in the Bruhat order on $X_*(T)_I$.

Proposition 6.5. *Let $\tau \in \Omega_{\mathbf{a}}$. All but finitely many elements of the form $x\tau \in W_{\mathrm{aff}}\tau/W_{\mathbf{f}}$ satisfy $x\tau \geq w_{\delta-\theta^\vee}\tau$ in the Bruhat order on $W/W_{\mathbf{f}}$, and for any such element $S_{x\tau}(\mathbf{a}, \mathbf{f})$ is not normal if $G_{\mathrm{sc}} \to G$ is a non-étale isogeny.*

Note that this proposition proves Theorem 2.5.

Proof. We can immediately reduce to the case $\tau = 1$. Since δ is regular antidominant, we see easily that $w_{\delta-\theta^\vee} > w^\delta \geq w^{\theta^\vee}$ in the Bruhat order on W_{aff}. Indeed, since δ is regular antidominant we have $t^\delta = w^\delta$ (it is known that $l(t^\delta w) = l(t^\delta) - l(w)$, $\forall w \in W_0$, by e.g. [IM65, Prop. 1.23]). The element $t^{-\theta^\vee} s_\theta \in W$ acts on $\mathscr{A}(G, S, F)$ by the simple affine reflection s_0, and so $t^\delta s_0(\mathbf{a}) = t^{\delta-\theta^\vee} s_\theta(\mathbf{a})$ is separated by a wall of type s_0 from $t^\delta(\mathbf{a})$, with $t^\delta(\mathbf{a})$ closer to the base alcove. So $l(t^\delta s_0) = l(t^\delta) + 1$. It follows that $t^\delta s_0 = w_{\delta-\theta^\vee}$, and from this that $w^\delta < w^\delta s_0 = w_{\delta-\theta^\vee}$. Finally, observe that $\delta^\vee \geq \theta^\vee$ is equivalent to $w_\delta \geq w_{\theta^\vee}$, which is equivalent to $w^\delta \geq w^{\theta^\vee}$.

Since $S_{\theta^\vee}(\mathbf{a}, 0)$ is not normal (when $G_{\mathrm{sc}} \to G$ is non-étale), we deduce that $S_{w^{\theta^\vee}}(\mathbf{a}, \mathbf{a})$ is not normal, hence also $S_x(\mathbf{a}, \mathbf{a})$ is not normal whenever $x \geq w_{\delta-\theta^\vee}$ in the Bruhat order on W_{aff}.

Finally we prove that all but finitely many $x \in W_{\mathrm{aff}}$ satisfy $x \geq w_{\delta-\theta^\vee}$. By the proof of Proposition 6.4, all but at most finitely many $\lambda \in Q^\vee$ satisfy $w_\lambda \geq w_{\delta-\theta^\vee}$, equivalently, $w^\lambda \geq w^{\delta-\theta^\vee}$. For any $w \in W_0$ and any such λ, we have $t^\lambda w \geq w_{\delta-\theta^\vee}$. We are done. □

6.3. **The example of** PGL_2**.** Our methods allow us to give a complete classification of normal Schubert varieties for PGL_2 in characteristic 2. In this subsection, let k be a field of characteristic 2.

Lemma 6.6. *The quasi-minuscule Schubert variety inside the affine Grassmannian for* PGL_2 *is not normal. More precisely, an open affine neighborhood of the base point is isomorphic to the spectrum of the k-algebra*

$$k[x, y, v, w]/(vw + x^2y^2, v^2 + x^3y, w^2 + xy^3, xw + yv).$$

Proof. Since 2 divides $|\pi_1(\mathrm{PGL}_2)| = 2$, the non-normality is a special case of Corollary 6.2. It remains to prove the displayed formula for the coordinate ring. By putting $v = xz$, $w = yz$, this k-algebra identifies with the subalgebra of $k[x, y, z]/(z^2 + xy)$ generated by x, y, xz, yz. Now the lemma follows from the calculations in Appendix B, see Corollary B.2. □

Let $\mathcal{F}\ell := \mathcal{F}\ell_{\mathrm{PGL}_2,\mathbf{a}}$ be the affine flag variety. For each w in the Iwahori-Weyl group W, we denote by $S_w \subset \mathcal{F}\ell$ the associated $(\mathbf{a}, \mathbf{a})$-Schubert variety.

Corollary 6.7. *For $w \in W$, the Schubert variety S_w is normal if and only if* $\dim(S_w) \leq 2$ *in which case it is smooth.*

Proof. After possibly multiplying $w \in W$ with an element in the stabilizer of $\mathbf{a}$, we may and do assume that $w \in W_{\mathrm{aff}}$, i.e., S_w lies in the neutral component of $\mathcal{F}\ell$. The affine Weyl group W_{aff} is the free group with generators s_0, s_1 and relations $s_0^2 = s_1^2 = 1$. Here s_0 is the simple affine reflection, and s_1 the simple finite reflection. Consider the projection $\pi\colon \mathcal{F}\ell \to \mathrm{Gr} := \mathrm{Gr}_{\mathrm{PGL}_2}$, a smooth proper morphism of relative dimension 1. Let $S_\mu \subset \mathrm{Gr}$ be the quasi-minuscule Schubert variety, which is not normal by Lemma 6.6. Hence, the Schubert variety $\pi^{-1}(S_\mu) = S_w$, $w = s_1s_0s_1$ is not normal. By Corollary 2.2, all other Schubert varieties S_v with $v \geq w$ are not normal as well. In particular, all Schubert varieties with $\dim(S_w) \geq 4$ are not normal. If $\dim(S_w) \leq 1$, i.e., either $w = 1$, or $w = s_0$, or $w = s_1$, then S_w is clearly smooth, hence normal. In order to treat the remaining cases where $\dim(S_w) = 2$ or $\dim(S_w) = 3$, we observe that the $(\mathbf{a}, \mathbf{a})$-Schubert variety S_w is normal (resp. smooth) if and only if the $(\mathbf{a}, \mathbf{a})$-Schubert variety $S_{\tau w \tau^{-1}}$ is normal (resp. smooth) where $\tau \in W$ is the non-trivial element in the stabilizer of $\mathbf{a}$, see Lemma 2.4. We have $\tau w \tau^{-1} = s_1 s_0$ for $w = s_0 s_1$ and $\tau w \tau^{-1} = s_0 s_1 s_0$ for $w = s_1 s_0 s_1$. Hence, both 3-dimensional Schubert varieties are not normal as argued above. One of the 2-dimensional Schubert varieties is the preimage in $\mathcal{F}\ell$ of the translated to the neutral component minuscule Schubert variety in Gr. Hence, both 2-dimensional Schubert varieties are smooth. This proves the lemma. □

Corollary 6.8. *A Schubert variety in the affine Grassmannian for* PGL_2 *in characteristic* 2 *is normal if and only if it is at most* 1*-dimensional in which case it is already smooth.*

Proof. This is immediate from Corollary 6.7 by considering the smooth projection of relative dimension 1 from the affine flag variety. □

6.4. **Some remarks on the classification.** Our methods from §6.1 do not apply to the case of special, but not absolutely special vertices. This is only an issue in the case of odd unitary groups of type $C\text{-}BC_n$, $n \geq 1$ with $p \,|\, 2n+1$. In this case, there are up to $G_{\mathrm{ad}}(F)$-conjugation two types of special vertices, where exactly one of them is absolutely special, see [HRb, §5]. Here separate methods seem to be required to calculate the tangent space of the quasi-minuscule Schubert variety in the corresponding twisted affine Grassmannian. Furthermore, we note that the normality criterion obtained in Proposition 6.4 is not effective. Indeed, this can be seen already in the case of PGL_2 in characteristic 2 by comparing with the classification in Corollary 6.7. In principle, Corollary 5.12 (2) together with the tangent space formula of Kumar and Polo (Corollary 4.3) gives an effective way of classifying all normal Schubert varieties. Here the main difficulty is the determination of the affine Demazure modules. The case of, say, PGL_3 in characteristic 3 already seems quite involved.

7. Reducedness

In [PR08, Thm. 6.1], the authors show that loop groups (equivalently, their partial affine flag varieties) attached to semisimple groups G over a field k are reduced under the hypothesis $\mathrm{char}(k) \nmid |\pi_1(G)|$. We show in Proposition 7.7 (split case) and Proposition 7.10 (twisted case) that this hypothesis is necessary.

7.1. **The split case.** Throughout this subsection, let k be an arbitrary field and G be a connected split reductive group over k. We are going to use the notion of distributions, which should be regarded as higher order differential operators. For the theory of distributions for (group) schemes we refer to [DG70, II, §4] and [Ja03, §7].

Definition 7.1. Let (X, x), $x \in X(k)$ be a pointed k-ind-scheme. The *space of distributions* $\mathrm{Dist}(X, x)$ is the k-vector space obtained as the filtered colimit of the k-vector space duals of all Artinian closed subschemes of X supported at x.

We record some basic properties.

Lemma 7.2. *Let* (X, x)*,* (Y, y) *be pointed* k*-ind-schemes, and let* $f\colon (Y, y) \to (X, x)$ *be a map of pointed* k*-ind-schemes.*

(1) *If* $(X, x) = \mathrm{colim}(X_i, x)$ *is any presentation, then* $\mathrm{Dist}(X, x) = \mathrm{colim}\,\mathrm{Dist}(X_i, x)$ *with injective transition maps. Further, each* $\mathrm{Dist}(X_i, x)$ *only depends on the formal spectrum* $\mathrm{Spf}(\mathcal{O}_{X_i,x})$ *viewed as an ind-scheme.*
(2) *The map* f *induces a map* $(df)_y\colon \mathrm{Dist}(Y, y) \to \mathrm{Dist}(X, x)$.
(3) *There is a natural map* $\mathrm{Dist}(X, x) \otimes_k \mathrm{Dist}(Y, y) \to \mathrm{Dist}(X \times_k Y, (x, y))$ *which is an isomorphism if both* X*,* Y *are ind-(locally Noetherian) over* k.

Proof. Part (1) is immediate because the transition maps $X_i \to X_j$ are closed immersions. Part (2) and (3) follow from (1) and the case of schemes in [Ja03, I, §7.2 & §7.4]. Note that *loc. cit.* is over more general base rings, and that the assumptions are satisfied for locally Noetherian schemes over fields. □

In particular, for any pointed k-ind-scheme (X, x) which is ind-(locally Noetherian), e.g., of ind-(finite type), the space of distributions $\mathrm{Dist}(X, x)$ is a co-(commutative unital k-algebra) whose co-algebra structure is induced from the diagonal $X \to X \times_k X$ and Lemma 7.2 (3), cf. [Ja03, I, §7.4 (3)] for details. If X is a k-group ind-scheme –possibly of ind-(infinite type)– then we define

$$\mathrm{Dist}(X) \overset{\text{def}}{=} \mathrm{Dist}(X, 1), \tag{7.1}$$

where $1 \in X(k)$ denotes the neutral section. In this case, the action morphism $X \times_k X \to X$ (combined with Lemma 7.2) induces on $\mathrm{Dist}(X)$ the structure of an associative k-algebra under the convolution of distributions, cf. [Ja03, I, §7.7] for details.

For the next lemma recall that a quasi-compact morphism of schemes is called *scheme-theoretically dominant* if its scheme theoretic image [StaPro, 01R5] is equal to its target.

Lemma 7.3. *Let $f\colon (Y, y) \to (X, x)$ be a quasi-compact, scheme-theoretically dominant morphism of locally Noetherian pointed k-schemes. Then the induced homomorphism $(df)_y\colon \mathrm{Dist}(Y, y) \to \mathrm{Dist}(X, x)$ is surjective.*

Proof. Since f is quasi-compact and scheme-theoretically dominant, the induced map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ on local rings is injective, cf. [StaPro, 01R8 (1), (2)]. Also note that the map $(df)_y$ only depends on the induced map on completed local rings $\hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{Y,y}$, which is injective as well. By Krull's intersection theorem, the decreasing sequence of ideals $\{\hat{\mathfrak{m}}_y^n \cap \hat{\mathcal{O}}_{X,x}\}_{n\geq 1}$ has zero intersection, and hence by Chevalley's lemma [Ch43, Lem. 7] defines a cofinal family of Artinian closed subschemes of $\mathrm{Spec}(\hat{\mathcal{O}}_{X,x})$ supported at x. This implies the lemma. □

Remark 7.4. Another interesting example (cf. also [Ja03, I, §7.6]) to which Lemma 7.3 applies is the case of a map $f\colon (Y, y) \to (X, x)$ of locally Noetherian pointed k-schemes which is flat at y. Indeed, then the induced map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is faithfully flat, and hence injective, that is, the map on spectra is scheme-theoretically dense. Also we find it instructive to check Lemma 7.3 "by hand" in the special cases of the normalization of the cusp, and the (relative) Frobenius morphism in strictly positive characteristic, say, of the affine line.

The previous lemma will be used to show that Gr_G for adjoint non-(simply connected) groups is non-reduced in bad characteristics by noticing that the k-vector space of the distributions of its reduction is strictly smaller. The following lemma shows that this space can be easily computed at "infinite level". For later use we formulate this lemma in more generality.

Lemma 7.5. *Let G be a Chevalley group scheme over $\mathbb{Z}$. Let $T \subset G$ be a split, maximal torus over $\mathbb{Z}$, and let $B^{\pm} = T \ltimes U^{\pm}$ be Borel subgroups in G over $\mathbb{Z}$ such that $B^+ \cap B^- = T$. Then the multiplication map on strictly negative loop groups*

$$L^{--}U^- \times_{\mathbb{Z}} L^{--}T \times_{\mathbb{Z}} L^{--}U^+ \to \mathrm{Gr}_G, \quad (u^-, t, u^+) \mapsto u^- \cdot t \cdot u^+ \cdot e \tag{7.2}$$

is formally étale (when viewed as a map of ind-schemes). The source of this map is called the fake open cell. This construction is compatible with arbitrary base change $S \to \mathrm{Spec}(\mathbb{Z})$, e.g., for $S = \mathrm{Spec}(k)$ a field.

Proof. The morphism $U^- \times T \times U^+ \to G$ given by multiplication is an open immersion [Co14, Thm. 5.1.13], and in particular formally étale [StaPro, 04FF]. Passing to negative loop groups (and using that the L^--construction commutes with products), this immediately implies that the top horizontal map

$$\begin{array}{ccc} L^-U^- \times L^-T \times L^-U^+ & \longrightarrow & L^-G \\ \uparrow & & \uparrow \\ L^{--}U^- \times L^{--}T \times L^{--}U^+ & \longrightarrow & L^{--}G, \end{array}$$

is formally étale. Here the vertical maps are the natural inclusions, and one checks that the diagram is Cartesian. Hence, the lower horizontal arrow is formally étale as well. □

Remark 7.6. By the same reasoning, the induced map on loop groups $LU^- \times LT \times LU^+ \to LG$ is formally étale as well.

Lemma 7.5 implies that every Artinian local ring supported at the base point in Gr_G uniquely factors through the fake open cell. We obtain the following proposition which improves on [PR08, Thm. 6.1] in the case of split groups.

Proposition 7.7. *Let G be a split reductive group over a field k. Then the following are equivalent:*

a) *The ind-scheme LG is reduced (and then even geometrically reduced).*
b) *The ind-scheme Gr_G is reduced (and then even geometrically reduced).*
c) *The group G is semisimple, and $\mathrm{char}(k) \nmid |\pi_1(G)|$.*

Proof. We first show the equivalence of a) and b). Recall that the quotient map $LG \to \mathrm{Gr}_G$ is a (right) L^+G-torsor in the étale topology. Thus, the ind-scheme LG is étale locally isomorphic to $\mathrm{Gr}_G \times_k L^+G$. If LG is reduced, then Gr_G is reduced because $L^+G \to \mathrm{Spec}(k)$ is flat [StaPro, 06QM]. Conversely, if Gr_G is reduced, then LG is reduced because L^+G is geometrically reduced [StaPro, 035Z]. This finishes the equivalence of a) and b). Concerning geometrically reducedness, we note that if Gr_G is reduced, then it admits a presentation by Schubert varieties. As Schubert varieties are geometrically reduced, it follows that Gr_G is reduced if and only if Gr_G is geometrically reduced. Since the equivalence of a) and b) is valid for any field, this also implies that LG is reduced if and only if LG is geometrically reduced.

It remains to show the equivalence of b) and c) for which we may (and do) assume that k is algebraically closed. If c) holds, then b) holds by [PR08, Thm. 6.1]. Conversely, if b) holds, i.e., if Gr_G is reduced, then G is semisimple by [PR08, Prop. 6.5]. It remains to show that $p := \mathrm{char}(k)$ does not divide $\pi_1(G)$. We may (and do) assume that $p > 0$ is strictly positive. Let $G_{\mathrm{sc}} \to G$ be the simply connected covering. Fix $T \subset G$, and denote by T_{sc} its preimage in G_{sc}. Let Gr_G^0 denote the neutral connected component of Gr_G. Then the induced map on Schubert varieties $\mathrm{Gr}_{G_{\mathrm{sc}}, \leq\mu} \to \mathrm{Gr}_{G,\leq\mu}$, $\mu \in X_*(T_{\mathrm{sc}})$ is dominant, and hence scheme-theoretically dominant (because the target is reduced by definition). As both ind-schemes $\mathrm{Gr}_{G_{\mathrm{sc}}}$, Gr_G^0 are reduced, they admit presentations by Schubert varieties indexed by dominant $\mu \in X_*(T_{\mathrm{sc}})$. Thus, Lemma 7.3 (combined with Lemma 7.2 (1) for the passage to ind-schemes) implies that the map

$$\mathrm{Dist}(\mathrm{Gr}_{G_{\mathrm{sc}}}, e) \longrightarrow \mathrm{Dist}(\mathrm{Gr}_G, e) \tag{7.3}$$

is surjective where e denotes the base point. This map is calculated using Lemma 7.5 as follows. Let $B^\pm = T \ltimes U^\pm$ Borel subgroups in G such that $B^+ \cap B^- = T$. Then $B^\pm_{\mathrm{sc}} = T_{\mathrm{sc}} \ltimes U^\pm$ are Borel subgroups in G_{sc}. By Lemma 7.5 (combined with Lemma 7.2 (3) for the compatibility with products), the surjectivity of (7.3) implies the surjectivity of

$$\mathrm{Dist}(L^{--}T_{\mathrm{sc}}) \longrightarrow \mathrm{Dist}(L^{--}T). \tag{7.4}$$

Here we use the principle that a tensor product of linear operators on possibly infinite dimensional vector spaces is surjective if and only if each linear operator is surjective.

To make the connection with $n := |\pi_1(G)|$, recall that the kernel Z of $G_{\mathrm{sc}} \to G$ is a finite k-group scheme of order n. Clearly, the subgroup Z is contained in T_{sc} (in fact in any maximal torus) which shows $Z = \ker(T_{\mathrm{sc}} \to T)$. We claim that the surjectivity of (7.4) implies that $p \nmid n$. We need to analyze the map $T_{\mathrm{sc}} \to T$ more carefully. Let $r := \dim(T_{\mathrm{sc}}) = \dim(T)$ denote the rank of the k-tori. Since k is algebraically closed, passing to cocharacter lattices induces an equivalence between k-tori of rank r, and finite free $\mathbb{Z}$-modules of rank r. Hence, the elementary divisor theorem implies that there exist isomorphisms $\mathbb{G}_{m,k}^r \simeq T_{\mathrm{sc}}$ and $T \simeq \mathbb{G}_{m,k}^r$ such that the composite

$$\mathbb{G}_{m,k}^r \simeq T_{\mathrm{sc}} \longrightarrow T \simeq \mathbb{G}_{m,k}^r \tag{7.5}$$

is given by $(\lambda_1, \dots, \lambda_r) \mapsto (\lambda_1^{n_1}, \dots, \lambda_r^{n_r})$ for positive integers $n_1 \geq \dots \geq n_r \geq 1$. We necessarily have $n = n_1 \cdot \ldots \cdot n_r$. Hence, the claim $p \nmid n$ is equivalent to the claim $p \nmid n_i$, $i = 1, \dots, r$. Since (7.5) splits as a product of maps, we can apply Lemma 7.2 (3) to see that the surjectivity of (7.4) implies the surjectivity of each map

$$\mathrm{Dist}(L^{--}\mathbb{G}_{m,k}) \longrightarrow \mathrm{Dist}(L^{--}\mathbb{G}_{m,k}), \tag{7.6}$$

which is induced from $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$, $\lambda \mapsto \lambda^{n_i}$ for $i = 1, \dots, r$. Finally, Lemma 7.8 below implies $p \nmid n_i$ which finishes the proof of the proposition. □

Lemma 7.8. *Let k be a field of characteristic $p > 0$. Let $n \geq 1$ be an integer, and consider the morphism of k-group schemes $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$, $\lambda \mapsto \lambda^n$ given by taking the n-th power. If the induced morphism $\mathrm{Dist}(L^{--}\mathbb{G}_{m,k}) \to \mathrm{Dist}(L^{--}\mathbb{G}_{m,k})$ is surjective, then $p \nmid n$.*

Proof. We immediately reduce to the case that n is a prime number. For a k-algebra R, the n-th power map on $L^{--}\mathbb{G}_{m,k}(R)$ is given by

$$1 + \textstyle\sum_{i\geq 1} a_i u^{-i} \;\mapsto\; \big(1 + \sum_{i\geq 1} a_i u^{-i}\big)^n, \tag{7.7}$$

where all $a_i \in R$ are nilpotent, and almost all a_i are zero. This also shows that there is a presentation $L^{--}\mathbb{G}_{m,k} = \mathrm{colim}_{i\geq 1} \mathrm{Spec}\big(k[a_1, \dots, a_i]/(a_1^i, \dots, a_i^i)\big)$ where a_i are viewed as formal variables. In these coordinates, we have a canonical identification $\mathrm{Dist}(L^{--}\mathbb{G}_{m,k}) = \mathrm{Dist}(\mathbb{A}_k^{\mathbb{N}}, \{0\})$ where $\mathbb{A}_k^{\mathbb{N}} = \mathrm{Spec}(k[\{a_i\}_{i\in\mathbb{N}}])$ is the infinite-dimensional affine space. Hence, a distribution is a k-linear map $\delta \colon k[\{a_i\}_{i\in\mathbb{N}}] \to k$ supported at only finitely many monomials. We see that the space of distribution has a basis given by $\delta_{\underline{r}}$ such that $\underline{r} = (r_i)_{i\in\mathbb{N}}$ is a sequence of positive integers where almost all r_i are zero. Here $\delta_{\underline{r}}$ takes the value 1 on the monomial $\sqcap_{i\in\mathbb{N}} a_i^{r_i}$ and the value 0 on all other monomials (by convention $\delta_{(0,0,\dots)} = 0$). We need to write down the map (7.7) in the basis $\mathrm{Dist}(L^{--}\mathbb{G}_{m,k}) = \mathrm{span}_k\{\delta_{\underline{r}} \mid \underline{r} \in (\mathbb{Z}_{\geq 0})^{\mathbb{N}}\}$. Suppose $n = p$ in which case we have to show that the induced map on the spaces of distributions is not surjective. Since k has characteristic $p > 0$, the formula in (7.7) becomes

$$\big(1 + \textstyle\sum_{i\geq 1} a_i u^{-i}\big)^p = 1 + \sum_{i\geq 1} a_i^p u^{-ip}.$$

This means that the map on spaces of distributions is induced from $\delta_{\underline{r}} \mapsto \delta_{p\star\underline{r}}$ where $p \star \underline{r} \in (\mathbb{Z}_{\geq 0})^{\mathbb{N}}$ is the zero vector (hence $\delta_{p\star\underline{r}} = 0$), unless $p \mid r_i$ for all entries in $\underline{r} = (r_i)_{i\in\mathbb{N}}$ in which case the i-th entry in $p \star \underline{r}$ is given by

$$(p \star \underline{r})_i \;=\; \begin{cases} \frac{r_{i/p}}{p} & \text{if } p \mid i; \\ 0 & \text{else.} \end{cases}$$

Since $p \geq 2$ the distribution $\delta_{(1,0,0,\dots)}$ does not lie in the image of this map. □

Remark 7.9. In fact, the converse to Lemma 7.8 holds as well, i.e., for an integer $n \geq 1$ prime to p the map $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$, $\lambda \mapsto \lambda^n$ induces a surjection on spaces of distributions. Indeed, one reduces to the case where $n \neq p$ is a prime number. Then it follows from an explicit calculation –which we omit– similarly as in the proof of Lemma 7.8, or alternatively using affine Grassmannians as follows. Consider the canonical map $\mathrm{Gr}_{\mathrm{SL}_n} \to \mathrm{Gr}_{\mathrm{PGL}_n}$ on affine Grassmannians, both of which are reduced by Proposition 7.7. Hence, as in (7.3) this induces a surjection on spaces of distributions. Following the proof of Proposition 7.7 further, we see that in (7.5) the elementary divisors $n_1 \geq \dots \geq n_r \geq 1$ (here $r = n$) are necessarily given by $n_1 = n$ and $n_i = 1$, $i \geq 2$ because n is a prime number. Now the surjectivity of the map in (7.6) gives the desired result.

7.2. **Reducedness in the twisted case.** Here we give a different proof of nonreducedness of loop groups of tamely ramified semi-simple groups G such that p divides the order of the fundamental group. The idea consists basically in observing that Weil restriction along purely inseparable extensions preserves loop groups and Grassmannians, but not flat and non-étale isogenies.

Proposition 7.10. *Let k be a perfect field of characteristic $p \geq 0$, and let G be a tamely ramified reductive group over $F = k(\!(t)\!)$. For its loop group LG to be reduced, it is necessary and sufficient that G be semi-simple and the order of $\pi_1(G)$ prime to p*[7].

Proof. By work of Pappas-Rapoport [PR08, Thm. 6.1, Prop. 6.5], we only need to show that LG is non-reduced whenever G is semisimple and the order of the kernel Z of its simply connected cover map $\widetilde{G} := G_{\mathrm{sc}} \to G$ is divisible by $p > 0$. As explained before the statement, we will contemplate the strictly smaller closed subgroup $\overline{G} := \mathrm{Res}_{F/F^p}\widetilde{G}/\mathrm{Res}_{F/F^p}Z$ of $\mathrm{Res}_{F/F^p}G$, as observed in [CGP15, Exam. A.7.9]. Note that Bruhat-Tits theory is available for $\overline{G}$ as well as for $\mathrm{Res}_{F/F^p}G$ by [Loub], their buildings being isomorphic to the building of G over F. We claim that the canonical morphism

$$\overline{\mathcal{G}}_x \to \mathrm{Res}_{\mathcal{O}/\mathcal{O}^p}\mathcal{G}_x$$

between parahoric group schemes is a locally closed immersion and its flat closure defines an invariant smooth subgroup scheme whose quotient is representable by a quasi-affine group scheme, see [Ana73, Thm. 4A].

Let S be a maximal $\breve{F}$-split torus of G, $\widetilde{S}$ be its unique lift to a maximal split torus of $\widetilde{G}$ and $\overline{S}$ be its image in $\overline{G}$. We denote by T (resp. $\widetilde{T}$, resp. $\overline{T}$) the maximal tori of G (resp. $\widetilde{G}$, resp. $\overline{G}$) obtained as centralizers of S (resp. $\widetilde{S}$, resp. $\overline{S}$). Arguing with big cells as in [BT84, §1.2.13, §1.2.14], it suffices for our purposes to show that

$$\overline{\mathcal{T}} \to \mathrm{Res}_{\mathcal{O}/\mathcal{O}^p}\mathcal{T}$$

is a locally closed immersion.

Assume first that T is split. By using the elementary divisor theorem as in (7.5) above, we may assume $T = \mathbb{G}_m = \widetilde{T}$ are 1-dimensional and $Z = \mu_n$. If n is prime to p, then $\overline{T} = \mathrm{Res}_{F/F^p}T$ and the claim is trivial. On the other hand, if n is divisible by p, then $\overline{T} = \mathbb{G}_m \subseteq \mathrm{Res}_{F/F^p}\mathbb{G}_m = \mathrm{Res}_{F/F^p}T$ and the claim is clear as well.

In general, let K/F be a tamely ramified finite Galois extension with group Γ splitting $\widetilde{T}$ and T, and note that K^p/F^p is a (pseudo-)splitting field for $\overline{T}$ with Galois group naturally isomorphic to Γ. Then we have locally closed immersions

$$\begin{aligned}\widetilde{\mathcal{T}} &\hookrightarrow \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}}\widetilde{\mathcal{T}}_{\mathcal{O}_K},\\ \mathcal{T} &\hookrightarrow \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}}\mathcal{T}_{\mathcal{O}_K},\\ \overline{\mathcal{T}} &\hookrightarrow \mathrm{Res}_{\mathcal{O}_K^p/\mathcal{O}^p}\overline{\mathcal{T}}_{\mathcal{O}_K^p},\end{aligned}$$

extending the natural generic homomorphisms. Indeed, the maps exist by the universal property of connected Néron models. Moreover, their scheme-theoretic images are smooth by identifying them with the smooth Γ-invariants of the right hand sides, see [Ed92] and compare also to [PR08, Lem. 6.7], where we use the tameness hypothesis $p \nmid |\Gamma|$. Due to [BLR90, Prop. 10.1.4], the resulting morphisms must be locally closed immersions. Taking restrictions of scalars along $\mathcal{O}_K^p/\mathcal{O}^p$ of the first two maps and applying the split case, we deduce the general claim.

As a consequence of the group-theoretic facts just established, we derive that

$$\mathrm{Gr}^0_{\overline{\mathcal{G}}_x} \to \mathrm{Gr}^0_{\mathrm{Res}_{\mathcal{O}/\mathcal{O}^p}\mathcal{G}_x} \cong \mathrm{Gr}^0_{\mathcal{G}_x}$$

is a closed immersion. Indeed, if we let $\overline{\mathcal{G}}^1_x$ denote the flat closure of $\overline{\mathcal{G}}_x$ inside $\mathrm{Res}_{\mathcal{O}/\mathcal{O}^p}\mathcal{G}_x$, then the morphism

$$\mathrm{Gr}_{\overline{\mathcal{G}}^1_x} \to \mathrm{Gr}_{\mathcal{G}_x}$$

[7] By convention $p = 0$ is prime to every integer.

is a quasi-compact immersion by [Ri, Prop. 3.6], which must be closed, as source and target are ind-projective by [Loub, Thm. 5.2.]. Finally, we have to show that the Galois cover

$$\mathrm{Gr}_{\overline{\mathcal{G}}_x} \to \mathrm{Gr}_{\overline{\mathcal{G}}^1_x}$$

with group $\overline{\mathcal{G}}^1_x(\mathcal{O}^p)/\overline{\mathcal{G}}_x(\mathcal{O}^p)$ induces an isomorphism between neutral components, which can be checked at the level of k-points. Now we notice that $\mathrm{Gr}^0_{\overline{\mathcal{G}}_x}(k) = \overline{G}(F^p)^0/\overline{\mathcal{G}}_x(\mathcal{O}^p)$, where $\overline{G}(F^p)^0$ is the subgroup of $\overline{G}(F^p)$ generated by the parahorics - this was implicitly shown during the proof of [Loub, Thm. 5.2.] by resolving à la Demazure. The claim now follows from [Loub, Prop. 3.9.], which generalizes the main result of [HaRa] for pseudo-reductive groups.

If LG were reduced, then the morphism would have to be an isomorphism, because $\mathrm{Gr}^0_{\tilde{\mathcal{G}}_x} \to \mathrm{Gr}^0_{\mathcal{G}_x}$ is a universal homeomorphism. In particular, that would imply that the F-vector space $\mathrm{Lie}\, G$ is spanned by the strictly smaller subspace $\mathrm{Lie}\, \overline{G}$ and the lattice $\mathrm{Lie}\, \mathcal{G}_x$. This is obviously a contradiction. □

Remark 7.11. One would hope that a similar statement holds beyond the tamely ramified case, but one cannot control the Néron models with the same ease. On the other hand, if one tried to classify reducedness of the loop group for the more general class of pseudo-reductive groups, the above argument suggests this could very well be a nightmare.

8. Ind-flatness

In this section G will denote a Chevalley group scheme over $\mathbb{Z}$. Our aim is to prove in Proposition 8.8 that its affine Grassmannian $\mathrm{Gr}_{G,\mathbb{Z}}$ (equivalently, its loop group) is ind-flat over $\mathbb{Z}$ in the sense of Definition 8.1. In Proposition 8.9 we explain how to generalize our proof to include the case of tamely ramified twisted groups.

8.1. **Preliminaries on ind-flatness.**

Recall our conventions on ind-schemes, see §1.2.

Definition 8.1. Let S be a scheme. An S-ind-scheme X is called ind-flat if there exists a presentation $X = \mathrm{colim}\, X_i$ where X_i are flat S-schemes via the map $X_i \subset X \to S$.

Now let R be a Dedekind ring with fraction field K. For an R-scheme X, the flat closure X^{fl} is the scheme theoretic image of the inclusion $X_K \subset X$. Since $X_K \subset X$ is a quasi-compact map, the scheme theoretic image commutes with localization [StaPro, 01R8], and the closed immersion $X^{\mathrm{fl}} \hookrightarrow X$ is an isomorphism on generic fibers. Then the scheme X is flat over R if and only if the map $X^{\mathrm{fl}} \hookrightarrow X$ is an isomorphism if and only if $\mathcal{O}_X$ is R-torsionfree. If $\varphi\colon X \to Y$ is a map of R-schemes, then there is a map $\varphi^{\mathrm{fl}}\colon X^{\mathrm{fl}} \to Y^{\mathrm{fl}}$ with $\varphi_K = (\varphi^{\mathrm{fl}})_K$.

Lemma 8.2. *Let R be a Dedekind ring with fraction field K. For an R-ind-scheme X the following conditions are equivalent:*

i) X is ind-flat;

ii) for every presentation $X = \mathrm{colim}\, X_i$, the map $\mathrm{colim}\, X_i^{\mathrm{fl}} \hookrightarrow \mathrm{colim}\, X_i$ is an isomorphism;

iii) every ind-(closed immersion) $Y \hookrightarrow X$ which induces $Y_K \cong X_K$ is an isomorphism.

Proof. The implications iii) ⇒ ii) ⇒ i) are immediate, and we prove i) ⇒ iii).

Let $X = \mathrm{colim}\, X_i$ be a flat presentation. Let $Y \hookrightarrow X$ be an ind-(closed immersion) which induces $Y_K \cong X_K$. For each i, the induced map $Y \cap X_i \hookrightarrow X_i$ is an ind-(closed immersion) which induces $(Y \cap X_i)_K \cong (X_i)_K$. We want to show that $Y \cap X_i \cong X_i$. Replacing X by X_i, we may assume that X is a flat R-scheme. Covering X by open affine schemes, we may further assume that X is affine, hence quasi-compact. Now let $Y = \mathrm{colim}\, Y_j$ be any presentation. We will show that $Y_j \cong X$ for $j \gg 0$. As $Y_K \cong X_K$ on generic fibers and X_K is quasi-compact, there is a j with $X_K \hookrightarrow Y_{j,K}$ so that $Y_{j,K} \cong X_K$. As $Y_j \hookrightarrow X$ is a closed immersion and X is R-flat, we must have $Y_j \cong X$. □

Definition 8.3. For an ind-scheme $X = \mathrm{colim}_i\, X_i$, the *flat closure* X^{fl} is the ind-scheme $X^{\mathrm{fl}} = \mathrm{colim}_i\, X_i^{\mathrm{fl}}$.

In view of Lemma 8.2, the ind-(closed immersion) $X^{\mathrm{fl}} \subset X$ is well-defined independently of the choice of a presentation. Also a map of R-ind-schemes $X \to Y$ induces a map $X^{\mathrm{fl}} \to Y^{\mathrm{fl}}$ on the flat closures.

8.2. **Ind-flatness of affine Grassmannians.** The starting point is the following lemma.

Lemma 8.4. *Let H be a smooth, affine group scheme over $\mathbb{Z}$.*

(1) *The positive loop group $L^+H \to \mathrm{Spec}(\mathbb{Z})$ is a flat, affine group scheme.*
(2) *If H is split unipotent or a split torus, then both the loop group LH and the strictly negative loop group $L^{--}H$ are ind-flat over $\mathbb{Z}$.*

Proof. For (1), let $L^+_i H, i \geq 0$ be the smooth, affine $\mathbb{Z}$-group scheme defined by the functor $L^+_i H(R) = H\big(R[u]/(u^{i+1})\big)$ for a ring R. Then $\{L^+_0 H\}_{i\geq 0}$ naturally forms an inverse system, and the canonical map $L^+H \to \lim_{i\geq 0} L^+_i H$ is an isomorphism. This implies (1).

For (2), observe that the map

$$L^{--}H \times_{\mathbb{Z}} L^+H \to LH, \;\; (h^-, h^+) \mapsto h^- \cdot h^+, \tag{8.1}$$

is representable by an open immersion. Since L^+H is faithfully flat by (1), the ind-flatness of $L^{--}H$ follows from the one for LH. Now let H be a split unipotent group scheme. Then $H \simeq \mathbb{A}^n_{\mathbb{Z}}$ as schemes for some $n \geq 0$. Since the formation of loop groups commutes with products, it is enough to show that $L\mathbb{A}^1_{\mathbb{Z}}$ is ind-flat. This is immediate from the identification, which is functorial in the ring R,

$$L\mathbb{A}^1_{\mathbb{Z}}(R) \;=\; R(\!(u)\!) \;=\; \operatorname*{colim}_{i\gg -\infty} \prod_{j\geq i} \mathbb{A}^1_{\mathbb{Z}}(R),$$

given by mapping a Laurent series $\sum a_i u^i$ to the vector (a_i). Next let H be a split torus. Then $H \simeq \mathbb{G}^n_{m,\mathbb{Z}}$ as (group) schemes for some $n \geq 0$, and we reduce to the case $T = \mathbb{G}_{m,\mathbb{Z}}$. In this case, the map (8.1) is surjective and hence an isomorphism. We see that the ind-flatness of $L\mathbb{G}_{m,\mathbb{Z}}$ is equivalent to the one of $L^{--}\mathbb{G}_{m,\mathbb{Z}}$. For the latter we note that for any ring R,

$$L^{--}\mathbb{G}_{m,\mathbb{Z}}(R) \;=\; \big(1 + u^{-1}R[u^{-1}]\big)^{\times} \;=\; \big\{1 + \textstyle\sum_{i\geq 1} a_i u^{-i} \mid a_i \in R \text{ nilpotent}\big\},$$

so that $L^{--}\mathbb{G}_{m,\mathbb{Z}} \simeq \mathrm{colim}_{i\geq 1} \mathrm{Spec}\big(\mathbb{Z}[a_1, \ldots, a_i]/(a_1^i, \ldots, a_i^i)\big)$. This is clearly ind-flat. □

Recall that G denotes a Chevalley group scheme over $\mathbb{Z}$.

Corollary 8.5. *Let $T \subset G$ be a split, maximal torus over $\mathbb{Z}$, and let $B^{\pm} = T \ltimes U^{\pm}$ be Borel subgroups in G such that $B^+ \cap B^- = T$. Then the fake open cell (cf. Lemma 7.5)*

$$L^{--}U^- \times_{\mathbb{Z}} L^{--}T \times_{\mathbb{Z}} L^{--}U^+ \to \mathrm{Spec}(\mathbb{Z})$$

is ind-flat.

Proof. Since $U^{\pm}$ are split unipotent and T is a split torus, this is immediate from Lemma 8.4 (2). □

The ind-flatness of the affine Grassmannian is deduced from Corollary 8.5 using the following observation due to Faltings [Fal03, Proof of Cor. 11].

Lemma 8.6. *Let $Y \hookrightarrow X$ be an ind-(closed immersion) of ind-(locally Noetherian) ind-schemes. If for every local Artinian ring R the induced map*

$$Y(R) \to X(R) \tag{8.2}$$

is bijective, then $Y \hookrightarrow X$ is an isomorphism.

Proof. By Yoneda's lemma, we may view X, Y as set-valued (contravariant) functors on the category of Noetherian, affine schemes. For any such T, the induced map $Y(T) \to X(T)$ is clearly injective, and we need to show the surjectivity. This can be checked after base change $Y \times_X T \to T$ as our assumptions are stable under base change. We reduce to the case where $X = T$ is a Noetherian (affine) scheme. Write $Y = \mathrm{colim}_i Y_i$ as a filtered colimit of closed subschemes of X. We claim that this sequence stabilizes with value X.

Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the ideal sheaf defining Y_i. Since the index set is filtered, it is enough to show the existence of an index i with $\mathcal{I}_i = 0$. For this we note that $\mathcal{I}_i = 0$ if and only if the annihilator ideal sheaf $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{I}_i)$ is equal $\mathcal{O}_X$, or equivalently if the closed subscheme $Z_i \subset X$ defined by $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{I}_i)$ is empty. For $i \leq j$, we have

$$Y_i \subset Y_j \iff I_i \supset I_j \iff \mathrm{Ann}_{\mathcal{O}_X}(\mathcal{I}_i) \subset \mathrm{Ann}_{\mathcal{O}_X}(\mathcal{I}_j) \iff Z_i \supset Z_j.$$

Since X is noetherian and I is filtered, there is an $i_0 \in I$ such that Z_{i_0} is a minimum, i.e. $Z_{i_0} \subseteq Z_i$ for all $i \in I$. Now suppose for the sake of contradiction that $Z_{i_0} \neq \varnothing$.

Now let $\eta \in Z_{i_0}$ be a generic point of an irreducible component. It remains to find an index $i_0 < i_1$ such that $\eta \notin Z_{i_1}$. By construction, this means that the closed subscheme defined by $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{I}_{i_0})_\eta = \mathrm{Ann}_{\mathcal{O}_{X,\eta}}(\mathcal{I}_{i_0,\eta})$ in $\mathrm{Spec}(\mathcal{O}_{X,\eta})$ is supported at the closed point. Since $\mathcal{O}_{X,\eta}$ is Noetherian, this is equivalent to the existence of some integer $N \gg 0$ such that

$$\mathfrak{m}_\eta \supset \mathrm{Ann}_{\mathcal{O}_{X,\eta}}(\mathcal{I}_{i_0,\eta}) \supset \mathfrak{m}_\eta^N,$$

where $\mathfrak{m}_\eta \subset \mathcal{O}_{X,\eta}$ denotes the maximal ideal. Hence, $\mathcal{I}_{i_0,\eta}$ is a finitely generated module over the Artinian ring $\mathcal{O}_{X,\eta}/\mathfrak{m}_\eta^N$, and therefore Artinian itself. Since our index set is filtered, we can choose $i_1 > i_0$ such that $\mathcal{I}_{i_1,\eta}$ is the minimum among the set $\{\mathcal{I}_{j,\eta}\}_{j \geq i_0}$. By Krull's intersection theorem $\cap_{n\geq 0}\mathfrak{m}_\eta^n = 0$, and hence applying (8.2) to the Artinian rings $\mathcal{O}_{X,\eta}/\mathfrak{m}_\eta^n$, $n \geq 1$ shows that we necessarily have $\mathcal{I}_{i_1,\eta} = 0$, i.e., that $\eta \notin Z_{i_1}$. This finishes the proof of the lemma. □

Remark 8.7. The proof of Lemma 8.6 shows that condition (8.2) can be weakened. Namely, it is enough to use local Artinian rings which are strictly Henselian. Indeed, in the last part of the proof it is enough to show $\mathcal{I}_{i_1,\bar\eta} = 0$ where $\bar\eta \to \eta$ is a geometric point and $\mathcal{I}_{i_1,\bar\eta}$ denotes the stalk on the étale site.

Proposition 8.8. *The affine Grassmannian $\mathrm{Gr}_{G,\mathbb{Z}}$ is an ind-flat ind-scheme over $\mathbb{Z}$. In particular, the ind-scheme $\mathrm{Gr}_{G,\mathbb{Z}}$ is reduced if and only if G is semisimple.*

Proof. Let $\mathrm{Gr}^{\mathrm{fl}}_{G,\mathbb{Z}} \subset \mathrm{Gr}_{G,\mathbb{Z}}$ be the flat closure, cf. Definition 8.3. By Lemma 8.6 and Remark 8.7, it is enough to show that every local Artinian, strictly Henselian point $g\colon \mathrm{Spec}(R) \to \mathrm{Gr}_{G,\mathbb{Z}}$ factors through $\mathrm{Gr}^{\mathrm{fl}}_{G,\mathbb{Z}}$. Let k be the residue field of R, and denote by $\bar g\colon \mathrm{Spec}(k) \to \mathrm{Gr}_{G,\mathbb{Z}}$ the reduction of g. Fix a split maximal torus $T \subset G$ over $\mathbb{Z}$, and $B^\pm = T \ltimes U^\pm$ as in Corollary 8.5. By the Cartan decomposition $\mathrm{Gr}_G(k) = \sqcup_{\mu\in X_*(T)} L^+G(k)\cdot u^\mu \cdot e$ (use that k is separably closed)[8], we can write $\bar g$ as a product $\bar h \cdot u^\mu \cdot e$ for some $\bar h \in L^+G(k)$, $\mu \in X_*(T)$. By formal smoothness of $L^+G \to \mathrm{Spec}(\mathbb{Z})$, we can lift $\bar h$ to an R-valued point $h\colon \mathrm{Spec}(R) \to L^+G$. Since $L^+G \to \mathrm{Spec}(\mathbb{Z})$ is flat and $LT \to \mathbb{Z}$ is ind-flat by Lemma 8.4, the inclusion $\mathrm{Gr}^{\mathrm{fl}}_{G,\mathbb{Z}} \subset \mathrm{Gr}_{G,\mathbb{Z}}$ is invariant under the left action of L^+G and LT. Replacing g by $u^{-\mu}\cdot h^{-1}\cdot g$, we may therefore assume that g is supported at the base point. Then g factors through the fake open cell $L^{--}U^- \times L^{--}T \times L^{--}U^+$ by Lemma 7.5. Since this is ind-flat by Corollary 8.5, the map (7.2) factors through $\mathrm{Gr}^{\mathrm{fl}}_{G,\mathbb{Z}}$. This shows $\mathrm{Gr}^{\mathrm{fl}}_{G,\mathbb{Z}} = \mathrm{Gr}_{G,\mathbb{Z}}$.

For the second assertion, note that $\mathrm{Gr}_{G,\mathbb{Q}}$ is reduced if and only if G is semisimple, see [PR08, Thm. 6.1, Prop. 6.5]. Hence, $\mathrm{Gr}_{G,\mathbb{Z}}$ is not reduced whenever G is not semisimple. Conversely, if G is semisimple, then by taking the flat closure of any reduced presentation of $\mathrm{Gr}_{G,\mathbb{Q}}$ gives a reduced presentation of $\mathrm{Gr}^{\mathrm{fl}}_{G,\mathbb{Z}} = \mathrm{Gr}_{G,\mathbb{Z}}$.

This finishes the proof of the proposition. □

We now briefly generalize this to the parahoric group schemes over $W[\![t]\!]$ constructed previously.

Proposition 8.9. *The affine flag variety $\underline{\mathcal{F}\ell}_{G,\mathbf{f}}$ is an ind-flat ind-scheme over W. In particular, the ind-scheme $\underline{\mathcal{F}\ell}_{G,\mathbf{f}}$ is reduced if and only if G is semisimple.*

[8] Alternatively, the reader can note that for tamely ramified groups, with the help of our big cell results and with the rationality results of decompositions in [HaRa], we know the Cartan decomposition holds for any field $W \to k$, so Remark 8.7 is not really needed, even in the proof for the generalization proved in Proposition 8.9.

Proof. Without loss of generality, we may and do assume that $\mathbf{f} = \mathbf{0}$ is an absolutely special vertex. After translating, it suffices to show that $L^{--}\underline{G}$ at e is ind-flat over W. By means of z-extensions (which have smooth kernels, so the transition morphisms of negative loop groups are formally smooth), we may assume that the maximal torus T of G is induced. Now notice that the twisted fake open cell

$$\underline{U}^- \times \underline{T} \times \underline{U}^+ := (\mathrm{Res}_{k[\![u]\!]/k[\![t]\!]} U_H^- \times T_H \times U_H^+)^\sigma$$

gets identified, as a scheme, with a product of restrictions of scalars of open subsets of affine spaces. In particular, its negative loop space is obviously ind-flat. The second assertion follows as in the proof of Proposition 8.8. □

9. Consequences for Pappas-Zhu local models

We discuss some consequences of our findings for the theory of local models in cases where p divides the order of $\pi_1(G_{\mathrm{der}})$.

In this final section, let F be a discretely valued, complete field of characteristic 0 with algebraically closed residue field k of characteristic $p > 0$. We fix a triple $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$ where G is a tamely ramified reductive F-group, $\{\mu\}$ a conjugacy class of geometric cocharacters defined over a finite extension E/F, and $\mathcal{G}_{\mathbf{f}}$ is a parahoric $\mathcal{O}_F$-group scheme with generic fiber G. This notation seems to have first appeared in the survey article of Pappas-Rapoport-Smithling, see [PRS13], and first termed LM triple by He-Pappas-Rapoport in [HPR20, §2.1] (but always under the assumption that $\{\mu\}$ is minuscule). Pappas-Zhu [PZ13] construct from the data $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$ the *Pappas-Zhu local model*

$$\mathbb{M} \;=\; \mathbb{M}(G, \{\mu\}, \mathcal{G}_{\mathbf{f}}),$$

which is a flat, projective $\mathcal{O}_E$-scheme equipped with a left action of a smooth affine group scheme. Recall that the construction of $\mathbb{M}$ requires the construction of a parahoric $\mathcal{O}_F[t]$-group scheme $\underline{\mathcal{G}}_{\mathbf{f}}$ in the sense of [PZ13, Thm. 4.1] which lifts $\mathcal{G}_{\mathbf{f}}$ along the specialization $t \mapsto \varpi$ for some fixed uniformizer $\varpi \in \mathcal{O}_F$. In particular, $\mathbb{M}$ depends a priori on certain auxiliary choices, but it is shown in [HPR20, Thm. 2.7] that $\mathbb{M}$ actually depends, up to equivariant isomorphism, only on the data $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$. The generic fiber $\mathbb{M} \otimes E$ is naturally the Schubert variety in the affine Grassmannian of G/F associated with the class $\{\mu\}$. The special fiber is equidimensional, but not irreducible in general, and is equipped with a closed embedding

$$\mathbb{M} \otimes k \;\hookrightarrow\; \mathcal{F}\ell_{G^\flat, \mathbf{f}^\flat}. \tag{9.1}$$

The pair $(G^\flat, \mathbf{f}^\flat)$ is an equal characteristic analogue over a local function field $F^\flat = k(\!(t)\!)$ of the pair $(G, \mathbf{f})$, see [PZ13] (see also [HRb]).

More precisely, fix a maximal F-split torus S whose apartment $\mathscr{A}(G, S, F)$ contains the facet $\mathbf{f}$. Its centralizer T is a maximal torus since G is quasi-split by Steinberg's theorem. We also have the corresponding data $S^\flat \subset T^\flat$ inside the equal characteristic analogue $G^\flat$. There is an identification of apartments $\mathscr{A} := \mathscr{A}(G, S, F) = \mathscr{A}(G^\flat, S^\flat, F^\flat)$ compatible with the action of the Iwahori Weyl groups $W := W(G, S, F) = W(G^\flat, S^\flat, F^\flat)$ under which $\mathbf{f} = \mathbf{f}^\flat$. Then under (9.1) the reduced locus of $\mathbb{M} \otimes k$ identifies by [HRa, Thm. 6.12] with the admissible locus

$$\mathcal{A}(G, \{\mu\}) \;\overset{\mathrm{def}}{=}\; \bigcup_w S_w(\mathbf{f}, \mathbf{f}),$$

where the union is taken over the finitely many elements w of the admissible set $W_{\mathbf{f}}\mathrm{Adm}(\{\mu\})W_{\mathbf{f}}$ in W, and where $S_w(\mathbf{f}, \mathbf{f})$ denotes the Schubert variety in $\mathcal{F}\ell_{G^\flat, \mathbf{f}}$ using $\mathbf{f} = \mathbf{f}^\flat$. The following result is an application of Proposition 2.3, which is used in [HRc, Rem. 2.2].

Proposition 9.1. *Let $\mathbf{a} \subset \mathscr{A}$ be an alcove whose closure contains $\mathbf{f}$. Suppose $\{\mu\} \subset X_*(T)$ has the property that $S_v(\mathbf{a}, \mathbf{a})$ is normal for all (equivalently, for the maximal elements) $v \in \mathrm{Adm}(\{\mu\})W_{\mathbf{f}}$. Then all $(\mathbf{f}, \mathbf{f})$-Schubert varieties in $\mathcal{A}(G, \{\mu\})$ are normal. In particular, this last conclusion holds when $\bar{\mu} \in X_*(T)_I$ is minuscule for the échelonnage roots and the closure of $\mathbf{f}$ contains a special vertex.*

Proof. Let $v \in W_{\mathbf{f}}\mathrm{Adm}(\{\mu\})W_{\mathbf{f}}$; to prove the first assertion we need to show that $S_v(\mathbf{f},\mathbf{f})$ is normal. By Corollary 2.2, we may assume v is left $\mathbf{f}$-maximal, so that $S_v(\mathbf{f},\mathbf{f}) = S_v(\mathbf{a},\mathbf{f})$ and so we only need to consider $(\mathbf{a},\mathbf{f})$-Schubert varieties. By [HH, Thm. 1.3], we have $W_{\mathbf{f}}\mathrm{Adm}(\{\mu\})W_{\mathbf{f}} = \mathrm{Adm}(\{\mu\})W_{\mathbf{f}}$, so we may reduce ourselves to proving that $S_v(\mathbf{a},\mathbf{f})$ is normal for any $v \in \mathrm{Adm}(\{\mu\})W_{\mathbf{f}}$. By Proposition 2.3, it is enough to prove that $S_{v\eta}(\mathbf{a},\mathbf{a})$ is normal for all $\eta \in W_{\mathbf{f}}$. But these are normal by assumption.

For the second assertion, choose a special vertex $\mathbf{0} \in \bar{\mathbf{f}}$. Since $\bar{\mu}$ is minuscule, the Schubert variety $S_{\bar{\mu}}(\mathbf{a},\mathbf{0})$ is smooth, and hence so is its preimage under $\mathcal{F}\ell_{\mathbf{a}} \to \mathcal{F}\ell_{\mathbf{0}}$. This is itself a Schubert variety $S_{v_0}(\mathbf{a},\mathbf{a})$ indexed by the unique longest element $v_0 \in W_{\mathbf{0}}\mathrm{Adm}(\{\mu\})W_{\mathbf{0}}$, a set which contains $\mathrm{Adm}(\{\mu\})W_{\mathbf{f}}$; thanks again to Proposition 2.3 the latter set indexes normal Schubert varieties, and then we are done by the first assertion. □

As an application we obtain Corollary 1.6 from the introduction:

Corollary 9.2. *Assume p divides the order of $\pi_1(G_{\mathrm{der}})$.*

1. *If every Schubert variety in the admissible locus $\mathcal{A}(G,\{\mu\})$ is normal, then $\mathbb{M}$ is normal and its special fiber is reduced. This is the case when $\bar{\mu}$ is minuscule for the échelonnage roots and $\mathbf{f}$ contains a special vertex in its closure.*
2. *If any Schubert variety inside the admissible locus $\mathcal{A}(G,\{\mu\})$ is not normal, then $\mathbb{M}$ is not normal and its special fiber is not reduced.*

Proof. Part (1) is immediate from Proposition 9.1 and [HRc, Thm. 2.1]. For (2), suppose one of the Schubert varieties inside $\mathcal{A}(G,\{\mu\})$ is not normal. Then the irreducible component containing this Schubert variety is not normal as well by Corollary 2.2 (2). The normalization morphism

$$p\colon \tilde{\mathbb{M}} \longrightarrow \mathbb{M} \tag{9.2}$$

identifies by [HRc, Cor. 2.3, Cor. 2.5] with the map from the Pappas-Zhu local model of some z-extension of G. In particular, (9.2) is a finite, birational, universal homeomorphism and an isomorphism on generic fibers; recall that $\mathbb{M} \otimes E$ is a Schubert variety in characteristic 0. This already shows that $\mathbb{M}$ is not normal, see [HRc, Rem. 2.4]. It remains to show that the special fiber $\mathbb{M} \otimes k$ is not reduced. Arguing by contradiction let us assume that $\mathbb{M} \otimes k$ is reduced. For any line bundle $\mathcal{L}$ on $\mathbb{M}$, this implies the injectivity of the canonical map $H^0(\mathbb{M} \otimes k, \mathcal{L}) \to H^0(\tilde{\mathbb{M}} \otimes k, \mathcal{L})$. Furthermore, if $\mathcal{L}$ is ample and $N > 0$ sufficiently large, then there is an equality

$$\dim_k H^0(\mathbb{M} \otimes k, \mathcal{L}^N) = \dim_k H^0(\tilde{\mathbb{M}} \otimes k, \mathcal{L}^N)$$

by transporting the claim to the generic fiber using flatness. Hence, we get an isomorphism of vector spaces $H^0(\mathbb{M} \otimes k, \mathcal{L}^N) \cong H^0(\tilde{\mathbb{M}} \otimes k, \mathcal{L}^N)$, and thus (9.2) must be an isomorphism on special fibers. We arrive at the desired contradiction since the irreducible components of $\tilde{\mathbb{M}} \otimes k$ are normal by [HRc, Cor. 2.5] for example. □

Let us give two concrete examples of badly behaved PZ local models. The obvious class of examples arises from Weil restrictions of scalars along ramified extensions:

Example 9.3. Let F'/F be a totally ramified extension of 2-adic fields of odd degree $e \geq 1$. Consider the Weil restriction of scalars $G = \mathrm{Res}_{F'/F}(\mathrm{PGL}_2)$, and let $\{\mu\}$ be the unique (nonzero) minuscule conjugacy class defined over F. As parahoric subgroup we take the pointwise fixer of the standard lattice $\mathcal{O}_{F'}^2$, that is, the associated parahoric group scheme is $\mathcal{G} = \mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F}(\mathrm{PGL}_2)$. It corresponds to an absolutely special vertex $\mathbf{0}$, and hence the special fiber of the PZ local model is irreducible. Its underlying reduced subscheme is the unique e-dimensional $(\mathbf{0},\mathbf{0})$-Schubert variety in the affine Grassmannian for PGL_2 in characteristic 2. If $e \geq 2$ this Schubert variety is not normal by our classification in Corollary 6.8. Hence, the special fiber of the PZ local model is not reduced in this case by Corollary 9.2 (2).

We remark that if we take $e = 2$, then F'/F is wildly ramified and we can invoke [Lev16] to define the local models. Again the special fiber of such a local model is not reduced because the corresponding Schubert variety is not normal, as follows immediately from Corollary 6.2. Indeed, the corresponding Schubert variety is the quasi-minuscule one for the group PGL_2 (although it is more natural to think of it as a Schubert variety attached to the standard pseudo-reductive group $\mathrm{Res}_{k(\!(u)\!)/k(\!(t)\!)}\mathrm{PGL}_2$ where $u^2 = t$, compare this to the approach of [Loua] and [Louc]).

A less obvious example is given by ramified unitary groups. In this case, the underlying group is even absolutely simple:

Example 9.4. Let F'/F be a totally ramified quadratic extension of 3-adic fields. Let $G = \mathrm{PU}_3(F'/F)$ be the adjoint, quasi-split unitary group associated with the Hermitian form $x_1\bar{x}_3 + x_2\bar{x}_2 + x_3\bar{x}_1$ on F'^3. Let $\{\mu\}$ be the minuscule conjugacy class corresponding to the coweight $(1, 0, 0)$. As parahoric subgroup we take the pointwise fixer of the standard lattice $\mathcal{O}_{F'}^3$ which corresponds to an absolutely special vertex $\mathbf{0}$ of the building, see [HRb, §7]. In this case, the special fiber is again irreducible and its underlying reduced locus is the unique 2-dimensional Schubert variety in the twisted affine Grassmannian for $\mathrm{PU}_3(F'/F)$ in characteristic 3, that is, the quasi-minuscule one. This Schubert variety is not normal by Corollary 6.2 so that the special fiber of the PZ local model is not reduced by Corollary 9.2 (2).

Next we comment on the behavior of PZ local models relatively to central extensions. The adjoint quotient $(G, \{\mu\}) \to (G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\})$ induces a natural morphism

$$\mathrm{ad}_* : \mathbb{M} \to \mathbb{M}_{\mathrm{ad}} \otimes_{\mathcal{O}_{E_{\mathrm{ad}}}} \mathcal{O}_E,$$

where E_{ad} is the reflex field of $(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\})$ and $\mathbb{M}_{\mathrm{ad}} := \mathbb{M}(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\}, \mathcal{G}_{\mathbf{f},\mathrm{ad}})$. This is a (fiberwise) birational universal homeomorphism, but not always an isomorphism. We are now going to look at the category consisting of all LM triples centrally lifting $(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\}, \mathcal{G}_{\mathbf{f},\mathrm{ad}})$, endowed with the obvious morphisms. It admits fiber products and we use this to study the variation of the PZ local models along central lifts.

Proposition 9.5. *Let $(G_i, \{\mu_i\}, \mathcal{G}_{i,\mathbf{f}})$, $i = 1, 2$, be two LM central lifts of $(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\}, \mathcal{G}_{\mathbf{f},\mathrm{ad}})$ and denote by $(G_3, \{\mu_3\}, \mathcal{G}_{3,\mathbf{f}})$ their fiber product. If $p \nmid \frac{|\pi_1(G_{1,\mathrm{der}})|}{|\pi_1(G_{3,\mathrm{der}})|}$, then the rational map $\mathrm{ad}_{2,*} \circ \mathrm{ad}_{1,*}^{-1}$ extends to an actual morphism of schemes over $\mathcal{O}_{E_3}$*

Proof. Recall that, by construction of $\underline{\mathcal{G}}_{\mathbf{f}}$, we have morphisms

$$\mathbb{M}_3 \to \mathbb{M}_i \otimes \mathcal{O}_{E_3}$$

for $i = 1, 2$. Up to translation by an appropriate element in the loop group of Beilinson-Drinfeld Grassmannian, these morphisms can simultaneously be realized as the scheme-theoretic image of some closed subscheme of $\mathrm{Gr}^{\mathrm{BD}}_{\underline{\mathcal{G}}_{\mathbf{f},3,\mathrm{der}}} \otimes \mathcal{O}_{E_3}$ along the natural map

$$\mathrm{Gr}^{\mathrm{BD}}_{\underline{\mathcal{G}}_{\mathbf{f},3,\mathrm{der}}} \otimes \mathcal{O}_{E_3} \to \mathrm{Gr}^{\mathrm{BD}}_{\underline{\mathcal{G}}_{\mathbf{f},i,\mathrm{der}}} \otimes \mathcal{O}_{E_3}.$$

If the condition $p \nmid \frac{|\pi_1(G_{1,\mathrm{der}})|}{|\pi_1(G_{3,\mathrm{der}})|}$ holds, it means that for $i = 1$ the map between the affine Grassmannians is an étale homeomorphism, hence an isomorphism, giving us the desired extension. □

For defining canonical integral models of Shimura varieties of abelian type, under the assumptions of tame ramification and $p > 2$, Kisin-Pappas need to approach local models in terms of embeddings, see [KP18, §2.3, §3.2].

Definition 9.6. The tuple $(G, \{\mu\})$ is called *of abelian type* if there is a central lift $(G_1, \{\mu_1\})$ of $(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\})$ endowed with a closed embedding $\rho_1 : G_1 \to \mathrm{GL}_n$, $n \geq 1$ such that $\{\rho_1 \circ \mu_1\} = \{\varpi_d^\vee\}$, where $\varpi_d^\vee$ denotes the d-th minuscule coweight of GL_n for some $1 \leq d \leq n-1$. The central lift $(G_1, \{\mu_1\})$ is also called *of Hodge type*.

First note that every Shimura datum of abelian type in the sense of [De79] (see also [Ki10]) gives rise to a tuple of abelian type as above. Further, our definition coincides with the definition given in [Loua, §3.11.], but appears to differ from that of [HPR20, §2.7] in the following way. Let us, for simplicity, assume that G_{ad} contains no F-simple factor over which $\{\mu_{\mathrm{ad}}\}$ becomes trivial. Then the classification of Hodge embeddings due to Deligne, see [De79, 1.3.8., table 1.3.9.] (compare with

[Ki10, Lem. 3.4.13] and [KP18, Lem. 4.6.22.]) implies that ρ_1 is minuscule (as required in [HPR20, §2.7]). The main difference here is not requiring the existence of an isogeny $G_{1,\mathrm{der}} \to G_{\mathrm{der}}$ because we want our class to be stable under central lifts.

Next we note that since G is assumed to be tamely ramified, we can arrange for G_1 to be tamely ramified as well. So the canonical maps $G \to G_{\mathrm{ad}} \leftarrow G_1$ extend to maps of $\mathcal{O}_F[t]$-group schemes $\underline{\mathcal{G}}_{\mathbf{f}} \to \underline{\mathcal{G}}_{\mathbf{f},\mathrm{ad}} \leftarrow \underline{\mathcal{G}}_{\mathbf{f},1}$, and hence to maps of PZ local models $\mathbb{M} \to \mathbb{M}_{\mathrm{ad}} \leftarrow \mathbb{M}_1$ defined over the ring of integers of the compositum $E \cdot E_1$ where E_1 is the reflex field of $\{\mu_1\}$. Also note that adding the center of GL_n to our central lift G_1 changes neither the condition on ρ_1 nor the PZ local model $\mathbb{M}_1$ in view of Proposition 9.5.

The importance of our central lift of Hodge type is that ρ_1 extends by [KP18, Prop. 1.3.3] to a (not necessarily closed) immersion of parahoric group schemes

$$\mathcal{G}_{\mathbf{f},1} \to \mathcal{GL}_n,$$

which is heavily based on work of Landvogt (beware that [KP18] use the notation $\mathcal{G}_x$ for the fixer group scheme of x and reserve $\mathcal{G}_x^0$ for its parahoric neutral component). Then the same authors construct in [KP18, Prop. 2.3.7] a uniquely determined closed embedding:

$$\mathbb{M}_1 \hookrightarrow \mathbb{M}_{\mathrm{lat}}$$

where we set $\mathbb{M}_{\mathrm{lat}} := \mathbb{M}(\mathrm{GL}_n, \{\varpi_d^\vee\}, \mathcal{GL}_n) \otimes \mathcal{O}_{E_1}$. We remark that the symplectic embeddings used in the given reference and the hypothesis $p \nmid |\pi_1(G_{1,\mathrm{der}})|$ are unnecessary, the former pertaining to later applications to Shimura varieties and the latter to ensure normality of $\mathbb{M}_1$. Here we will give a closer look at the possibilities for the geometry of this scheme, analyzing all possible cases.

Proposition 9.7. *Let $(G_1, \{\mu_1\})$ be a central lift of Hodge type as above and let $\mathbb{M}_1$ be the PZ local model attached to the corresponding LM triple. Then the following properties hold:*

(1) *If $p > 2$ or G_{ad} has no D-factors, then $\mathbb{M}_1$ is always normal and only depends on $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$ up to extending scalars.*
(2) *If $p = 2$ and $(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\})$ is $\breve{F}$-simple of type $D_n^{\mathbb{H}}$, $n \geq 5$, then $\mathbb{M}_1$ only depends on $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$ up to base change, but will be non-normal for sufficiently large $\bar{\mu}$.*
(3) *If $p = 2$ and $(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\})$ is $\breve{F}$-simple of type $D_{2m+1}^{\mathbb{R}}$, $m \geq 2$, then $\mathbb{M}_1$ is always normal and only depends on $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$ up to base change.*
(4) *If $p = 2$ and $(G_{\mathrm{ad}}, \{\mu_{\mathrm{ad}}\})$ is $\breve{F}$-simple of type $D_{2m}^{\mathbb{R}}$, $m \geq 2$, then we can always choose $(G_1, \{\mu_1\}, \mathcal{G}_{\mathbf{f},1})$ and ρ_1 such that $\mathbb{M}_1$ is normal. For sufficiently large $\bar{\mu}$, we can simultaneously choose $(G_2, \{\mu_2\}, \mathcal{G}_{\mathbf{f},2})$ and ρ_2, such that $\mathbb{M}_2$ is non-normal.*

Proof. By Corollary 9.2 it suffices to examine the normality of Schubert varieties in the special fiber of $\mathbb{M}_1$ and for this we need to understand when p divides the order of Z, the kernel of $G_{\mathrm{sc}} \to G_{1,\mathrm{der}}$. Inspecting Deligne's table, see [De79, table 1.3.9], we see that Z is always a multiplicative 2-group and trivial if G_{sc} has no D-factors. This gives (1). For simple orthogonal adjoint groups, the pullback of ρ_1 to $G_{\mathrm{sc}} = \mathrm{Res}_{F'/F} \mathrm{Spin}_{2n}$ is the restriction of scalars of one of the three minuscule representations of Spin_{2n}, that is, the two half-spin irreducible factors of the faithful spin representation of Spin_{2n} and the pulled back vector representation of SO_{2n}. Moreover, our inspection of [De79, table 1.3.9] reveals that we can only use the vector representation for $D_n^{\mathbb{H}}$ and the half-spins for $D_n^{\mathbb{R}}$. In (2), the kernel Z is a certain 2-group independent of the choice of a Hodge lift. For (3), Z is always trivial because the half-spin representations are faithful if $n = 2m + 1$ is odd. For (4), we can choose our Hodge lift such that ρ_1 restricts to the faithful spin representation (= sum of the two half-spins), but we can also choose some other Hodge lift with ρ_2 restricting to an half-spin representation, whose kernel is a non-trivial 2-group. □

Let us comment on the relation with the setting of [SW20]. Scholze-Weinstein construct an affine Grassmannian $\mathrm{Gr}^{\mathrm{dR}}_{\mathcal{G}_{\mathbf{f}}}$ using de Rham period rings and yielding an ind-proper v-sheaf, that is, a set-valued pre-sheaf on perfectoid Huber rings which is a sheaf for the v-topology. They conjecture, see [SW20, Conj. 21.4.1] that, for $\{\mu\}$ minuscule, the $\{\mu\}$-bounded portion $\mathrm{Gr}^{\mathrm{dR}}_{\mathcal{G}_{\mathbf{f}}, \leq\{\mu\}}$ is representable by a unique normal projective flat $\mathcal{O}_E$-model with reduced special fiber of the flag variety. This was proved in [HPR20, Thm. 2.14, Cor. 2.15] by He-Pappas-Rapoport in many abelian type tamely ramified cases, and later more generally by the second author in [Loua, Prop. 1.8] (in particular the restriction $p > 2$ for the case $D_n^{\mathbb{H}}$ in [HPR20, Cor. 2.16] is not needed). The argument for the identification in cases of abelian type goes as follows (compare with [Loua, §3.11] which establishes also the wild variant of the conjecture, up to some exceptions if $p = 2$). Observe that the local model $\mathbb{M}_{\mathrm{lat}}$ verifies the conjecture by [SW20, Cor. 21.6.10] (its special fiber is reduced thanks to Görtz [Go01]). On the one hand, we have a closed embedding of v-sheaves $\mathbb{M}_1^{\diamondsuit} \to \mathbb{M}_{\mathrm{lat}}^{\diamondsuit}$ up to extending scalars, and, on the other, we also have an isomorphism $\mathbb{M}_1^{\diamondsuit} \to \mathbb{M}^{\diamondsuit}$, because perfectoid rings forget about universal homeomorphisms (a more precise statement will be given in [Louc]). Putting these two facts together and using the case of GL_n, we obtain a unique equivariant isomorphism

$$\mathrm{Gr}^{\mathrm{dR}}_{\mathcal{G}_{\mathbf{f}}, \leq \{\mu\}} \cong \mathbb{M}^{\diamondsuit}$$

as $\mathcal{O}_E$-models of the generic fiber. The normal local model singled out by the conjecture is the weak normalization $\tilde{\mathbb{M}}$ of $\mathbb{M}$ (simply the local model in the sense of [HPR20, §2.6]) and coincides with the PZ local model of some z-extension, so its special fiber is reduced (see [Zhu14] and Corollary 9.2), as required by [SW20, Conj. 21.4.1]. The closed immersion of v-sheaves $\mathbb{M}^{\diamondsuit} \to \mathbb{M}_{\mathrm{lat}}^{\diamondsuit}$ induces a morphism of $\mathcal{O}_E$-schemes

$$\tilde{\mathbb{M}} \to \mathbb{M}_{\mathrm{lat}}$$

by [SW20, Prop. 18.4.1] and the previous proposition classifies all instances in which this morphism is a closed immersion.

Coming back to integral models of Shimura varieties of abelian type, it turns out that the hypothesis $p \nmid |\pi_1(G_{\mathrm{der}})|$ in [KP18, Thm. 0.4] is unnecessary, as long as one replaces the PZ local model in the statement by its (weak) normalization. For the $p = 2$, $D^{\mathbb{H}}$ case with $\bar{\mu}$ large, the proposition seems to indicate some additional work might be needed in order to construct such integral models, so as to circumvent the fact that the Hodge embedding defines a non-normal orbit closure.

Appendix A. Frobenius ind-splitting

Fix a field k of characteristic $p > 0$. Here we revisit the notion of Frobenius splittings and prove several basic lemmas regarding this technique in the realm of ind-schemes over k.

Definition A.1. A k-scheme X is said to be (Frobenius) split if the morphism of $\mathcal{O}_X$-modules $\mathcal{O}_X \to F_*\mathcal{O}_X$ admits a section s, where F denotes the absolute Frobenius morphism. A closed subscheme Y of X is said to be compatibly split if the splitting of X descends to that of Y. Finally, we say that an ind-scheme X is ind-split (resp. compatibly ind-split with an ind-closed sub-ind-scheme) if it admits a presentation $X = \operatorname{colim} X_i$ (resp. as well as $Y = \operatorname{colim} Y_i$) by simultaneously compatibly split schemes.

Lemma A.2. *Given a collection X_i of simultaneously compatibly ind-split ind-closed sub-ind-schemes of X, finite intersections and finite unions are also simultaneously compatibly split.*

Proof. This is known in the case of schemes ([BK05, Prop. 1.2.1]), and it generalizes to that of ind-schemes by taking appropriate presentations. □

Thanks to [BL94, (3.7)] or [BD99, 7.11.3], we have a good notion of sheaves of modules on ind-schemes $X = \operatorname{colim} X_i$, namely obtained as a family of compatible $\mathcal{O}_{X_i}$-modules in the obvious way. For an ind-proper ind-scheme X over k equipped with a coherent $\mathcal{O}_X$-module $\mathcal{M}$, we define $H^n(X, \mathcal{M}) := \lim H^n(X_i, \mathcal{M}_i)$ for $n \geq 0$. This definition is sensible as the cohomology groups are finite-dimensional and thus $R^n\lim$ vanishes for $n > 0$.

Lemma A.3. *Let $Y \subset X$ be a closed immersion of compatibly ind-split ind-proper ind-schemes over k. If $\mathcal{L}$ is an ample line bundle on X, then $H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L})$ is surjective and $H^{>0}(X, \mathcal{L}) = H^{>0}(Y, \mathcal{L}) = 0$. Additionally, if $Y \subset X$ is a closed immersion, then $H^1(X, \mathcal{I}_Y \otimes \mathcal{L}) = 0$ where $\mathcal{I}_Y$ denotes the ideal sheaf defining Y.*

Proof. At finite level, this is just [BK05, Thm. 1.2.8] and it follows in general by taking projective limits. □

The next results study the implications of certain splittings for the graded algebra $H^0(X, \mathcal{L}^\bullet)$, going back to Ramanathan [Ra87], but we mostly follow the treatment of [BK05, §1.5].

Proposition A.4. *Let $Y \subset X$ be a closed immersion of compatibly ind-split ind-proper ind-schemes over k, and assume that the diagonal Δ_X is compatibly ind-split with $X \times X$. Given an ample line bundle $\mathcal{L}$ on X, the graded k-algebra $H^0(Y, \mathcal{L}^\bullet)$ is generated by its degree one elements, and $\mathcal{L}$ defines an ind-closed immersion of Y into $\mathbb{P}(H^0(Y, \mathcal{L})^\vee)$ where $H^0(Y, \mathcal{L})^\vee := \operatorname{colim} H^0(Y_i, \mathcal{L})^\vee$.*

Proof. Observe that due to the previous lemma and our hypothesis, we get surjectivity of the map $H^0(X, \mathcal{L}^n) \otimes H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}^{n+1})$ which implies the claim as long as $Y = X$, by induction on n. If Y is not necessarily equal to X, we still have an epimorphism $H^0(X, \mathcal{L}^n) \to H^0(Y, \mathcal{L}^n)$. The projective embedding is given by taking the colimit of the resulting closed immersions for a compatibly split presentation. □

Let us recall some terminology regarding commutative graded algebras and modules (compare with [BK05, Def. 1.5.5]).

Definition A.5. A commutative $\mathbb{Z}_{\geq 0}$-graded k-algebra $A_\bullet$ is called quadratic if $A_0 = k$, if it is generated by A_1 and if the kernel $K_\bullet$ of the induced surjection $S^\bullet A_1 \to A_\bullet$ is generated by K_2. An $A_\bullet$-graded module $M_\bullet$ is said to be quadratic if it is generated by M_0 and the kernel $K_\bullet$ of $M_0 \otimes A_\bullet \to M_\bullet$ is generated by K_1.

The next result subsumes [Ra87, Prop. 2.7, Prop. 2.19] and is the ind-scheme version of [BK05, Prop. 1.5.8].

Proposition A.6. *Let $Z \subset Y \subset X$ be closed immersions of simultaneously compatible ind-split ind-proper ind-schemes and $\mathcal{L}$ be an ample line bundle on X. Suppose moreover that $\Delta_{X^2} \times X$, $X \times \Delta_{X^2}$, $Y \times \Delta_{X^2}$ and $Z \times \Delta_{X^2}$ are simultaneously compatibly ind-split in X^3. Then $H^0(Y, \mathcal{L}^\bullet)$ is a quadratic graded algebra and $H^0(Z, \mathcal{L}^\bullet)$ is a quadratic graded module over $H^0(Y, \mathcal{L}^\bullet)$.*

In geometric terms, this tells us that the projective embeddings of Z, Y and X determined by $\mathcal{L}$ are given by quadratic homogeneous polynomials and the transition morphisms are defined by linear ones.

Proof. If we intersect the given ind-schemes with $X \times \Delta_{X^2}$, the conditions of Proposition A.4 are satisfied and hence the graded algebras in the statement are generated by its degree 1 elements. To show that $A_\bullet := H^0(X, \mathcal{L}^\bullet)$ is quadratic, we consider the Mayer-Vietoris short exact sequence $0 \to \mathcal{I}_{\Delta_{X^2} \times X \cup X \times \Delta_{X^2}} \to \mathcal{I}_{\Delta_{X^2} \times X} \oplus \mathcal{I}_{X \times \Delta_{X^2}} \to \mathcal{I}_{\Delta_{X^3}} \to 0$, where the ideals of definition are with respect to X^3, and we tensor it with $\mathcal{L}^{n_1} \boxtimes \mathcal{L}^{n_2} \boxtimes \mathcal{L}^{n_3}$. By taking cohomology, we arrive at surjectivity of $K_{n_1,n_2} \otimes A_{n_3} \oplus A_{n_1} \otimes K_{n_2,n_3} \to K_{n_1,n_2,n_3}$ by the proof of [BK05, Prop. 1.5.8], which implies that $A_\bullet$ is quadratic by [BK05, Lem. 1.5.7]. In order to show that $B_\bullet := H^0(Y, \mathcal{L}^\bullet)$ and $C_\bullet := H^0(Z, \mathcal{L}^\bullet)$ are quadratic algebras, we repeat the same strategy with the couple $(\Delta_{X^2} \times X, Y \times \Delta_{X^2})$ which intersects in Δ_{Y^3} and use surjectivity of the transition maps $A_\bullet \to B_\bullet \to C_\bullet$ to derive the same formulae for $B_\bullet$ and $C_\bullet$. By [BK05, Rmk. 1.5.6 (iii)] every transition map defines a graded quadratic module structure. □

Appendix B. The quasi-minuscule Schubert scheme for PGL_2

Let S_{sc} (resp. S_{ad}) be the quasi-minuscule Schubert variety in $\mathrm{Gr}_{\mathrm{SL}_2}$ (resp. $\mathrm{Gr}_{\mathrm{PGL}_2}$) over $\mathbb{Z}$. The canonical map $\mathrm{Gr}_{\mathrm{SL}_2} \to \mathrm{Gr}_{\mathrm{PGL}_2}$ induces a scheme theoretically surjective morphism $S_{\mathrm{sc}} \to S_{\mathrm{ad}}$, and hence a morphism

$$\operatorname{Spec} A := L^{--}\mathrm{SL}_2 \cap S_{\mathrm{sc}} \to L^{--}\mathrm{PGL}_2 \cap S_{\mathrm{ad}} =: \operatorname{Spec} B,$$

which identifies B with an integral subdomain of A. The aim of this section is to prove the following result.

Proposition B.1. *There is an isomorphism $A \cong \mathbb{Z}[x,y,z]/(z^2+xy)$ under which B is the subring generated by the elements $x, y, 2z, xz, yz$.*

Corollary B.2. *The ring $B\otimes\mathbb{F}_2$ is not reduced. Its reduction identifies with the subring of $A\otimes\mathbb{F}_2 \cong \mathbb{F}_2[x,y,z]/(z^2+xy)$ generated by x, y, xz, yz.*

Proof. The element $u = 2z$ is not 0 in $B \otimes \mathbb{F}_2$ because $z \notin B$. But its square $u^2 = 4z^2 = -4xy$ is 0 in $B\otimes\mathbb{F}_2$ because $x, y \in B$. This shows that $B\otimes\mathbb{F}_2$ is not reduced. Clearly, the image of $B\otimes\mathbb{F}_2 \to A\otimes\mathbb{F}_2 \cong \mathbb{F}_2[x,y,z]/(z^2+xy)$ is the subring generated by x, y, xz, yz. The kernel of this map is nilpotent because the spectra of all rings are irreducible of Krull dimension 2. Hence, the ring $(B\otimes\mathbb{F}_2)_{\mathrm{red}}$ identifies with the desired subring of the integral domain $\mathbb{F}_2[x,y,z]/(z^2+xy)$. $\square$

This corollary shows that the special fibre $S_{\mathrm{ad}}\otimes\mathbb{F}_2$ is not reduced. More precisely, the reduction $(S_{\mathrm{ad}}\otimes\mathbb{F}_2)_{\mathrm{red}}$ is the quasi-minuscule Schubert variety for PGL_2 over $\mathbb{F}_2$, but the inclusion $(S_{\mathrm{ad}}\otimes\mathbb{F}_2)_{\mathrm{red}} \subset S_{\mathrm{ad}}\otimes\mathbb{F}_2$ is strict.

To prove Proposition B.1, we first calculate the ring A. For this, we consider the Lie algebra $\mathfrak{sl}_2$ of SL_2 over $\mathbb{Z}$. The nilpotent cone $\mathfrak{n}$ in $\mathfrak{sl}_2$ is the closed subscheme of matrices whose determinant is zero. We choose the isomorphism $\mathbb{A}^3_{\mathbb{Z}} \cong \mathfrak{sl}_2$ given by the map

$$(x,y,z) \mapsto \begin{pmatrix} z & x \\ y & -z \end{pmatrix},$$

so that $\{z^2+xy=0\} \cong \mathfrak{n}$ as schemes over $\mathbb{Z}$.

Lemma B.3. *Let $e \in \mathrm{Gr}_{\mathrm{SL}_2}(\mathbb{Z})$ denote the base point. The map $\mathfrak{n} \to \mathrm{Gr}_{\mathrm{SL}_2}$, $X \mapsto (1+t^{-1}X)\cdot e$ induces an isomorphism $\mathfrak{n} \cong L^{--}\mathrm{SL}_2 \cap S_{\mathrm{sc}}$, that is, an isomorphism $\mathbb{Z}[x,y,z]/(z^2+xy) \cong A$ on coordinate rings.*

Proof. The map $\mathfrak{n} \to L^{--}\mathrm{SL}_2$, $X \mapsto 1+t^{-1}X$ is well-defined and a closed immersion. It induces an isomorphism onto the closed subscheme $(L^{--}\mathrm{SL}_2)_{[-1,1]}$ of $L^{--}\mathrm{SL}_2$ of all matrices $A = 1 + t^{-1}A_1 + t^{-2}A_2 + \dots$ such that $A_i = 0$ and $(A^{-1})_i = 0$ for $i \geq 2$. We now regard $L^{--}\mathrm{SL}_2$ via the map $g \mapsto g\cdot e$ as an open sub-ind-scheme of $\mathrm{Gr}_{\mathrm{SL}_2}$, see [Fal03, Lem. 2] (cf. Lemma 3.6). It remains to show $L^{--}\mathrm{SL}_2 \cap S_{\mathrm{sc}} = (L^{--}\mathrm{SL}_2)_{[-1,1]}$ as subschemes of $\mathrm{Gr}_{\mathrm{SL}_2}$.

Recall the lattice interpretation of the affine Grassmannian, see [Fal03, p. 42] (cf. [Go01, p. 697]). For any ring R, the R-valued points of $\mathrm{Gr}_{\mathrm{SL}_2}$ are given by $R[\![t]\!]$-lattices $\Lambda \subset R(\!(t)\!)^2$ such that $\det\Lambda = R[\![t]\!]$ in $R(\!(t)\!)$. We denote by $\Lambda_{0,R} = R[\![t]\!]^2$ the standard lattice which corresponds to the base point. Let $\mathrm{Gr}_{\mathrm{SL}_2,[-1,1]}$ denote the closed subscheme of $\mathrm{Gr}_{\mathrm{SL}_2}$ of $R[\![t]\!]$-lattices Λ such that $t\Lambda_{0,R} \subset \Lambda \subset t^{-1}\Lambda_{0,R}$. A direct computation on R-valued points shows $(L^{--}\mathrm{SL}_2)_{[-1,1]} = L^{--}\mathrm{SL}_2 \cap \mathrm{Gr}_{\mathrm{SL}_2,[-1,1]}$. Recall that S_{sc} is defined as the scheme theoretic closure of the orbit map $L^+\mathrm{SL}_2 \to \mathrm{Gr}_{\mathrm{SL}_2}$, $g \mapsto g\cdot\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right)\cdot e$. We see that $S_{\mathrm{sc}} \subset \mathrm{Gr}_{\mathrm{SL}_2,[-1,1]}$, and hence that $L^{--}\mathrm{SL}_2 \cap S_{\mathrm{sc}}$ is a closed subscheme of $(L^{--}\mathrm{SL}_2)_{[-1,1]}$. Since both are integral of Krull dimension $2+\dim\mathbb{Z} = 3$, they must be equal. $\square$

In order to calculate the subring B of $A \cong \mathbb{Z}[x,y,z]/(z^2+xy)$, we consider the adjoint representation of SL_2: The map $g \mapsto (x \mapsto gxg^{-1})$ induces a morphism of $\mathbb{Z}$-group schemes $\mathrm{SL}_2 \to \mathrm{Aut}(\mathfrak{sl}_2) = \mathrm{GL}_3$ given by

$$\mathrm{ad}\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1+2bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2cd & -c^2 & d^2 \end{pmatrix}, \tag{B.1}$$

where we use the ordered basis $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0\end{smallmatrix}\right)$ of $\mathfrak{sl}_2$. This map induces a closed immersion $\mathrm{PGL}_2 \hookrightarrow \mathrm{SL}_3$ of reductive $\mathbb{Z}$-group schemes, and hence a closed immersion $\mathrm{Gr}_{\mathrm{PGL}_2} \hookrightarrow \mathrm{Gr}_{\mathrm{SL}_3}$ of affine Grassmannians over $\mathbb{Z}$. Therefore, the image of S_{ad} in $\mathrm{Gr}_{\mathrm{SL}_3}$ identifies with the scheme theoretic closure of S_{sc} under $\mathrm{ad}\colon \mathrm{Gr}_{\mathrm{SL}_2} \to \mathrm{Gr}_{\mathrm{SL}_3}$.

Proof of Proposition B.1. We identify $A = \mathbb{Z}[x, y, z]/(z^2 + xy)$ under the isomorphism of Lemma B.3. Combining this with (B.1) gives

$$\mathrm{ad}\colon \begin{pmatrix} 1+t^{-1}z & t^{-1}x \\ t^{-1}y & 1-t^{-1}z \end{pmatrix} \mapsto \begin{pmatrix} 1+2t^{-2}xy & -(1+t^{-1}z)t^{-1}y & t^{-1}x(1-t^{-1}z) \\ -2(1+t^{-1}z)t^{-1}x & (1+t^{-1}z)^2 & -t^{-2}x^2 \\ 2t^{-1}y(1-t^{-1}z) & -t^{-2}y^2 & (1-t^{-1}z)^2 \end{pmatrix}.$$

As this formula holds on R-valued points, the ring B is precisely the subring of A generated by the monomials in x, y, z appearing as coefficients of t^i for $i = -1, -2$. An inspection of this matrix using $z^2 + xy = 0$ and $\mathbb{Z}^\times = \pm 1$ shows that $B = \mathbb{Z}[x, y, 2z, xz, yz]$ as a subring of A. □

Appendix C. Minimal nilpotent orbits in twisted affine Grassmannians

Fix an algebraically closed field k of characteristic 0. Let H be a simply connected or adjoint simple k-group of type A_n ($n \geq 2$), D_n ($n \geq 4$), or E_6 endowed with a pinning (H, T_H, B_H, e_H). Let σ_0 be the canonical involution of $\mathrm{Aut}(H, T_H, B_H, e_H)$ induced by the non-trivial involution of the Dynkin diagram of $\Phi(H, T_H)$[9]. Let $M := H^{\sigma_0}$ be the affine k-group deduced from H by taking σ_0-fixed points. It is smooth of finite type, connected, reductive, simple, simply connected or adjoint by [Hai15, Prop. 4.1] (see also [HRb, Prop. A.1]). Moreover, it carries a natural pinning (M, T_M, B_M, e_M) where the middle entries are given by fixed points under the involution and $e_M = e_H$. The root system $\Phi(M, T_M)$ is the set of non-divisible elements in the image of $\Phi(H, T_H)$ under the natural restriction morphism:

$$X^*(T_H) \otimes \mathbb{R} \to X^*(T_M) \otimes \mathbb{R} \cong (X^*(T_H) \otimes \mathbb{R})_{\sigma_0}$$

the latter of which will often be identified with $(X^*(T_H) \otimes \mathbb{R})^{\sigma_0}$ via the obvious averaging map.

Let $k(\!(t)\!)$ be the Laurent series field over k and consider its quadratic Galois extension $k(\!(u)\!)$ with $u = t^{1/2}$. The restriction of scalars $\mathrm{Res}_{k(\!(u)\!)/k(\!(t)\!)} H$ admits the involution $\sigma := \sigma_0 \otimes \iota$ where ι stands for the Galois involution of $k(\!(u)\!)/k(\!(t)\!)$. Its fixed points $G := (\mathrm{Res}_{k(\!(u)\!)/k(\!(t)\!)} H)^\sigma$ form a reductive, quasi-split group equipped with a natural quasi-pinning (G, T_G, B_G, e_G), see [PZ13, §2.] or [Loua, §2.1-2.2]. We have an obvious absolutely special parahoric model of G given by the same formula after replacing $k(\!(u)\!)/k(\!(t)\!)$ by $k[\![u]\!]/k[\![t]\!]$. We will still denote this $k[\![t]\!]$-group by G. It is important to note as well that at a combinatorial level, the groups G and M are not so far from one another, in the sense that $\Phi(M, T_M)$ is the set of non-divisible roots of the relative root system $\Phi(G, S_G)$ via the obvious identification $X^*(S_G) \otimes \mathbb{R} = X^*(T_M) \otimes \mathbb{R}$.

Our goal is to establish a link between certain nilpotent orbits of M (not necessarily for the adjoint representation) and certain Schubert varieties of Gr_G. Note that the classification of simply connected tamely ramified reductive groups would force us to consider the case where σ_0 is either the identity or has order 3 (for the D_4 root system and associated triality). However, the material of this section has already been treated in [MOV05, §2.10] and [HRb, §8] in those additional cases.

C.1. **Minimal nilpotent orbits of the M-module $\mathfrak{g}_{-1}$.** Let $\mathfrak{h} \otimes k[u, u^{-1}]$ be the algebraic loop algebra of $\mathfrak{h}$ with the obvious action of σ by σ_0 on the left and Galois conjugation on the right. We let $\mathfrak{g}$ denote the σ-invariants of this Kac-Moody algebra - this is a graded version of $\mathrm{Lie}\, G$. The action preserves moreover the obvious u-grading and we write $\mathfrak{g}_{-1} := \mathfrak{h}[u^{-1}]^\sigma$. This is acted upon by M in the evident manner and we are going to analyze the structure of this representation as well as some of its nilpotent orbits. In the following, we denote by $\Phi_{M,<}$ the short roots of $\Phi_M := \Phi(M, T_M)$ and by $\theta_{M,<}$ the unique dominant short root of Φ_M.

Proposition C.1. *Suppose that $\Phi_G := \Phi(G, S_G)$ is reduced or, equivalently, that $\Phi_H := \Phi(H, T_H)$ is not of type A_{2m}. The following properties hold:*

(1) *The M-module $\mathfrak{g}_{-1}$ is irreducible and quasi-minuscule, that is, its highest weight equals $\theta_{M,<}$.*

[9] For H of type D_4, there are 3 possible choices of involution, and we pick the one fixing α_1. Note that since these involutions are all conjugate, other choices lead to isomorphic group-theoretic data.

(2) *Let $v \in \mathfrak{g}_{-1}$ be any non-zero weight vector. Then the closed orbit $\mathscr{O}_{\min} := \overline{M \cdot v}$ inside the affine space $\mathfrak{g}_{-1}$ is independent of v and contains the origin. It satisfies the following dimension formula*

$$\dim \mathscr{O}_{\min} = 2 + |\{a \in \Phi_M : a + \theta_{M,<} \in \Phi_{M,<}\}|$$

and its tangent space $T_0\mathscr{O}_{\min}$ at the origin is identified with $\mathfrak{g}_{-1}$.

Proof. The null weight space of our representation equals $\mathfrak{t}_{G,-1} := \mathfrak{t}_H[u^{-1}]^\sigma$ which has dimension equal to the cardinality of $\Delta_{M,<}$, that is, the subset of short positive simple roots. This can be seen by writing down its basis

$$u^{-1}h_\alpha - u^{-1}h_{\sigma_0(\alpha)}, \tag{C.1}$$

where $\alpha \in \Delta_H$ is not σ_0-invariant and $h_\alpha = [e_\alpha, e_{-\alpha}]$ is the canonical coroot element induced by the choice of pinning. Similarly, we see that the only nonzero weights are (short) roots of the form $\frac{\alpha+\sigma_0(\alpha)}{2}$ with $\alpha \in \Phi_H$. Indeed, their weight spaces are 1-dimensional spanned by

$$u^{-1}e_\alpha - u^{-1}e_{\sigma_0(\alpha)},$$

where the e_α are nonzero root vectors belonging to a Chevalley-Steinberg basis of $\mathfrak{h}$ extending the components of e_H, see [BT84, 4.1.3.] and compare with [Loua, §2.1.] for more explanations and references. The reducedness hypothesis is crucial here to ensure that $\sigma_0(e_\alpha) = e_{\sigma_0(\alpha)}$ for all roots $\alpha \in \Phi_H$. Since all the roots in Φ_H have the same length, it follows that the short roots $\Phi_{M,<}$ are those of the form $\frac{\alpha+\sigma_0(\alpha)}{2}$ for non-σ_0-invariant roots $\alpha \in \Phi_H$, and the unique dominant short root $\theta_{M,<}$ is thus the highest weight of the representation $\mathfrak{g}_{-1}$. Since all the weight-spaces with non-zero weight are 1-dimensional, $\mathfrak{g}_{-1}$ is the sum of the quasi-minuscule representation of M, plus possibly a trivial representation with some multiplicity m. But it is known that the weight-zero space in the quasi-minuscule representation of M has dimension $|\Delta_{M,<}|$, and thus it follows that $m = 0$. This completes the proof of (1).

Now we consider the minimal[10] nilpotent orbit $\mathscr{O}_{\min} = \overline{M \cdot v}$. Since all non-zero weight vectors are extremal by (1) and these are conjugate under the M-action, the orbit closure $\mathscr{O}_{\min}$ is independent of the choice of the non-zero weight vector v. Further, it is called nilpotent because v belongs to the nullcone of $\mathfrak{g}_{-1}$. In other words, v is an unstable point in the sense of geometric invariant theory, as one can find a cocharacter λ of M such that

$$\lim_{t\to 0} \lambda(t) \cdot v = 0$$

- this is known as the Hilbert-Mumford criterion. This also proves that $0 \in \mathscr{O}_{\min}$ and the tangent space $T_0\mathscr{O}_{\min}$, being an M-submodule of $\mathfrak{g}_{-1}$, must be the entire space by irreducibility.

As for computing the dimension, we need to subtract from $\dim M$ the dimension of the stabilizer $Z_M(v)$ of v which is preserved under conjugation by T_M. This can be done at the level of Lie algebras and then $\mathfrak{z}_\mathfrak{m}(v)$ actually decomposes into its intersection with weight spaces for the T_M-action. Obviously, $\mathfrak{z}_\mathfrak{m}(v) \cap \mathfrak{t}_M$ is a hyperplane in $\mathfrak{t}_M$ and hence its cokernel contributes once to the dimension of the minimal nilpotent orbit. Now we need to count roots $a \in \Phi_M$ such that e_a annihilates v. Choosing v to be a highest weight vector, it certainly suffices to have $a \notin -\theta_{M,<} + (\Phi_{M,<} \cup \{0\})$.

Suppose, on the other hand, that $a + \theta_{M,<} \in \Phi_{M,<} \cup \{0\}$. If $a = -\theta_{M,<}$ and if we write $\theta_{M,<} = \frac{\psi+\sigma_0(\psi)}{2}$, then $v = u^{-1}e_\psi - u^{-1}e_{\sigma_0(\psi)}$ and $e_{-\theta_{M,<}} \cdot v$ is a non-zero multiple of the averaged coroot element (C.1) for $\alpha = \psi$, using that $\{\psi, \sigma_0(\psi)\}$ form a perpendicular orbit pair. If $a + \theta_{M,<}$ is a short root of M, then we can write $a = \frac{\alpha+\sigma_0(\alpha)}{2}$ without necessarily having $\alpha \neq \sigma_0(\alpha)$ and we claim that we can arrange $\alpha + \psi \in \Phi_H$ up to replacing α by its σ_0-conjugate. Otherwise, the bracket $[e_a, e_{\theta_{M,<}}]$ would have to vanish while simultaneously generating the root space of $a + \theta_{M,<}$. Now if $\alpha \neq \sigma_0(\alpha)$ then $e_a := e_\alpha + e_{\sigma_0(\alpha)}$ and we see that $e_a \cdot v \neq 0$ because after expanding we get a non-zero multiple of $e_{\alpha+\psi} \in \mathbb{C}^\times[e_\alpha, e_\psi]$, which cannot be canceled out since $a + \theta_{M,<}$ is short

[10] To actually know that this is the smallest nilpotent orbit of $\mathfrak{g}_{-1}$ as happens for the adjoint representation, we would need an analogue of the Jacobson-Morozov theorem.

and hence $\sigma_0(\alpha+\psi) \neq \alpha+\psi$ and also since $\alpha \neq \sigma_0(\alpha)$. If $\alpha = \sigma_0(\alpha)$ then $e_a := e_\alpha$ and similarly $[e_a, u^{-1}(e_\psi - e_{\sigma_0(\psi)})] \neq 0$. This yields the dimension formula.[11] □

We treat separately the case when Φ_G is non-reduced, thus of type BC_n, for reasons that will become clear to the reader in a moment. We let θ_G be the highest root of Φ_G (notice that it must always be divisible). Recall that σ_0-invariant roots of Φ_H do not induce roots in Φ_M, but only divisible roots of Φ_G. Under the non-reducedness assumption on Φ_G, the short roots of Φ_M consist of averages of non-orthogonal σ_0-orbit pairs of roots, whereas long roots are the averages of the orthogonal pairs.

Proposition C.2. *Suppose that Φ_G is non-reduced or, equivalently, that Φ_H is of type A_{2n}. The following properties hold:*

(1) *The M-module $\mathfrak{g}_{-1}$ is irreducible of highest weight θ_G.*
(2) *Let $v \in \mathfrak{g}_{-1}$ be any extremal weight vector. Then the closed orbit $\mathscr{O}_{\min} := \overline{M \cdot v}$ inside the affine space $\mathfrak{g}_{-1}$ is independent of v and contains the origin. It satisfies the following dimension formula*

$$\dim \mathscr{O}_{\min} = 1 + |\{a \in \Phi_M : a + \theta_G \in \Phi_M\}|$$

and its tangent space $T_0\mathscr{O}_{\min}$ at the origin is identified with $\mathfrak{g}_{-1}$.

Proof. We start by producing a reasonable basis of $\mathfrak{g}_{-1}$, in the very same spirit of the previous proposition. The null weight space is still $\mathfrak{t}_{G,-1} := \mathfrak{t}_H[u^{-1}]^\sigma$, it has dimension $|\Delta_M|$ spanned by the basis

$$u^{-1}(h_\alpha - h_{\sigma_0(\alpha)})$$

for any orbit pair $\{\alpha, \sigma_0(\alpha)\}$ regardless of their orthogonality behavior. All roots $a = \frac{\alpha+\sigma_0(\alpha)}{2}$ of Φ_M are multiplicity one weights with weight vectors given by

$$v_a = u^{-1}(e_\alpha - e_{\sigma_0(\alpha)}),$$

where we use a Chevalley-Steinberg basis which must necessarily satisfy the property $\sigma_0(e_\alpha) = \epsilon_\alpha e_{\sigma_0(\alpha)}$, with $\epsilon_\alpha \in \{\pm 1\}$ being a fixed sign. Here $\epsilon_\alpha = 1$ if $\alpha \neq \sigma_0(\alpha)$ and $\epsilon_\alpha = -1$ otherwise.

This shows already that $\mathfrak{g}_{-1}$ is not quasi-minuscule, as Φ_M is not simply-laced, but we also have extremal vectors of weight $a = \alpha + \sigma_0(\alpha) \in \Phi_G \backslash \Phi_M$ for all non-orthogonal non-singleton orbit pairs $\{\alpha, \sigma_0(\alpha)\}$, equivalently, all σ_0-invariant roots a of Φ_H. Indeed, these extremal weight spaces are spanned by $v_a = u^{-1}e_a$, which are fixed by σ, because $\sigma_0(e_a) = -e_a$.

Therefore we conclude that $\mathfrak{g}_{-1}$ contains the highest weight module attached to θ_G. Moreover, since every non-zero weight has multiplicity one, belonging to the highest weight module by saturatedness, the only other possible summand would be the trivial representation. However, it is easy to see that for each $a \in \Delta_M$, $[e_{-\alpha} + e_{-\sigma_0(\alpha)}, v_a] = -u^{-1}(h_\alpha - h_{\sigma_0(\alpha)})$, whence irreducibility of $\mathfrak{g}_{-1}$. Indeed, this shows that the entire zero weight space $\mathfrak{t}_H[u^{-1}]^\sigma$ is contained in the module with highest weight θ_G.

As for the remaining assertions on $\mathscr{O}_{\min} := \overline{M \cdot v_{\theta_G}}$, we can argue in the same manner as in the reduced case. Let us take care of the combinatorics. We need to study roots a in Φ_M such that $a + \theta_G \in \Phi_G \cup \{0\}$ and examine whether $e_a v_{\theta_G} \neq 0$. Since $\theta_G \notin \Phi_M$ and $a + \theta_G$ cannot be divisible, we can replace $\Phi_G \cup \{0\}$ by Φ_M. Write $a = \frac{\alpha+\sigma_0(\alpha)}{2}$ and note that we can arrange $\alpha + \theta_G \in \Phi_H$ just as in the proof of Proposition C.1. Then $[e_\alpha + e_{\sigma_0(\alpha)}, u^{-1}e_{\theta_G}]$ is a non-zero multiple of $u^{-1}(e_{\alpha+\theta_G})$ plus a non-zero multiple of $u^{-1}(e_{\sigma_0(\alpha)+\theta_G})$, and cancellation cannot occur since $\alpha \neq \sigma_0(\alpha)$. □

[11] Alternatively, we could have used that the Kac-Moody roots a and $\theta_{M,<} - \delta$ of the Kac-Moody algebra $\mathfrak{g}$, where δ is the minimal positive imaginary root, constitute a prenilpotent pair of real roots in the sense of Tits [Ti87, §3.2.], so their bracket is non-trivial.

C.2. **Quasi-minuscule Schubert variety of** Gr_G**.** Recall that [Hai18, Thm. 6.1] describes the échelonnage root and coroot systems $\Phi_{\breve{\Sigma}}$, resp. $\Phi^\vee_{\breve{\Sigma}}$ for G, in terms of the σ_0-action on Φ_H, resp. on $\Phi^\vee_H$. We obtain

$$\Phi_{\breve{\Sigma}} = N'_{\sigma_0}\big(\Phi_H\big),$$

where the modified norm is defined as in [Hai18, §3]. Dually, we get

$$\Phi^\vee_{\breve{\Sigma}} = \mathrm{res}_{\sigma_0}\big(\Phi^\vee_H\big),$$

which is given by taking σ_0-averages and excluding the resulting divisible coroots. We note that parallel to the above, $\Phi_M = \mathrm{res}_{\sigma_0}(\Phi_H)$.

We are particularly interested in the unique quasi-minuscule $\overline{\psi^\vee}$ coweight of $\Phi^\vee_{\breve{\Sigma}}$. This is obtained from the highest orbit pair $\{\psi^\vee, \sigma_0(\psi)^\vee\}$, the highest orbit pair of $\Phi^\vee_H$ with average $\overline{\psi^\vee}$ of the shortest possible length. In other words, $\{\psi^\vee, \sigma_0(\psi)^\vee\}$ is the set of coroots of $\{\psi, \sigma_0(\psi)\}$, the highest non-singleton orbit pair in Φ_H if Φ_G is reduced and the highest non-orthogonal non-singleton orbit pair otherwise (so ψ carries the same meaning as in the previous section). Indeed, in the reduced case, $\psi + \sigma_0(\psi)$ is the highest long root, hence the highest root, of $N'_{\sigma_0}(\Phi_H) = \Phi_{\breve{\Sigma}}$. So $(\psi + \sigma_0(\psi))^\vee = \frac{\psi^\vee + \sigma_0(\psi)^\vee}{2} = \overline{\psi^\vee}$ is the quasi-minuscule coroot for $\Phi_{\breve{\Sigma}}$ (recall ψ and $\sigma_0(\psi)$ are perpendicular). In the non-reduced case, $2(\psi + \sigma_0(\psi))$ is the highest long root, hence the highest root, of $\Phi_{\breve{\Sigma}}$, and a calculation shows the quasi-minuscule coroot is again expressed as $\overline{\psi^\vee}$, compare with [Hai18, Lem. 3.2].

We have the following important lemma:

Lemma C.3. *The quasi-minuscule Schubert variety $S_{G,\overline{\psi^\vee}}$ and the minimal nilpotent orbit $\mathscr{O}_{\min}$ for $\mathfrak{g}_{-1}$ have the same dimension.*

Proof. This amounts to establishing the combinatorial identity

$$\langle 2\rho_H, \psi^\vee\rangle = \begin{cases} 2 + |\{a \in \Phi_M : a + \theta_{M,<} \in \Phi_{M,<}\}| & \text{if } \Phi_G \text{ is reduced;} \\ 1 + |\{a \in \Phi_M : a + \theta_G \in \Phi_M\}| & \text{else.}\end{cases}$$

Let us first assume Φ_G is reduced. Consider the two types of roots in $\Phi_{H,+}$: β with $\beta \perp \sigma_0(\beta)$ and γ such that $\gamma = \sigma_0(\gamma)$. Write $b := \frac{\beta + \sigma_0(\beta)}{2}$ and $c := \gamma$ for the corresponding positive roots in Φ_M; note that $b \in \Phi_{M,<}$ is a short root and $c \in \Phi_{M,>}$ a long root. Since $\theta^\vee_M := (\theta_{M,<})^\vee = \psi^\vee + \sigma_0(\psi)^\vee$ we have identities

$$\langle b, \theta^\vee_M\rangle = \langle \beta, \psi^\vee\rangle + \langle \sigma_0(\beta), \psi^\vee\rangle \tag{C.2}$$

$$\langle c, \theta^\vee_M\rangle = 2\langle \gamma, \psi^\vee\rangle. \tag{C.3}$$

We claim that (C.2) (resp. (C.3)) takes values in $\{0, 1\}$, if $\beta \notin \{\psi, \sigma_0(\psi)\}$ (resp. $\{0, 2\}$). To see this recall that the root $\beta + \sigma_0(\beta)$ (resp. γ) of $N'_{\sigma_0}(\Phi_H) = \Phi_{\breve{\Sigma}}$ is not proportional to the highest root $\psi + \sigma_0(\psi)$ of $\Phi_{\breve{\Sigma}}$, so by [Bou, VI.1.8, Prop. 25], we obtain $\langle \beta + \sigma_0(\beta), \overline{\psi^\vee}\rangle$ (resp. $\langle \gamma, \overline{\psi^\vee}\rangle$) belongs to $\{0, 1\}$.

Next we observe that $\langle b, \theta^\vee_M\rangle = 1$ if and only $-b + \theta_{M,<} \in \Phi_{M,<}$, when $\beta \notin \{\psi, \sigma_0(\psi)\}$ (resp. $\langle c, \theta^\vee_M\rangle = 2$ if and only if $-c + \theta_{M,<} \in \Phi_{M,<}$). (Note that all roots $a = -b$ (resp. $a = -c$) appearing in the desired formula are necessarily negative.) If $\langle b, \theta^\vee_M\rangle = 1$, then $s_{\theta_{M,<}}(b) = b - \theta_{M,<} \in \Phi_{M,<}$. If $\langle c, \theta^\vee_M\rangle = 2$, then c and $s_{\theta_{M,<}}(c) = c - 2\theta_{M,<}$ are both long roots, so $c - \theta_{M,<}$ is a short root. Conversely, if b (resp. c) and $\theta_{M,<}$ are perpendicular, their difference is longer than $\theta_{M,<}$ so in particular is not a short root.

Finally, note that the contribution $\langle \psi, \psi^\vee\rangle = 2$ to $\langle 2\rho_H, \psi^\vee\rangle$ matches the extra summand 2 found on the right side of the desired identity.

Now consider the case where Φ_G is not reduced of type BC_n so that Φ_H is of type A_{2n}. As much as we could probably give a combinatorial proof, it is quite simple to verify that the right side equals $2n$ by inspecting [Bou, Ch. VI, Planches II-III], whereas a calculation reveals that $\langle 2\rho_H, \psi^\vee\rangle = 2n$ as well. ∎

We have a natural morphism of reduced (ind)-schemes $\exp\colon \mathfrak{n}_{H,-1} \to L^{--}H$ induced by the exponential map of Lie algebras, where $\mathfrak{n}_{H,-1}$ is the set of nilpotent matrices in $\mathfrak{h}_{-1}$. For SL_n, this can be written as the usual exponential $u^{-1}X \mapsto \sum_{i=0}^{\infty} \frac{(u^{-1}X)^i}{i!}$, and it follows that the above morphism is a closed immersion. Moreover, it is σ-equivariant, so we also obtain a closed immersion on fixed points

$$\exp\colon \mathfrak{n}_{G,-1} \longrightarrow L^{--}G,$$

where $\mathfrak{n}_{G,-1} := \mathfrak{n}_{H,-1}^{\sigma}$. We have the following generalization of [HRb, Thm. 8.1, Prop. 8.6]:

Proposition C.4. *The morphism* $\exp\colon \mathfrak{n}_{G,-1} \to L^{--}G$ *restricts to an isomorphism* $\mathscr{O}_{\min} \cong L^{--}G \cap S_{G,\overline{\psi^\vee}}$.

Proof. Once we show that the image of any extremal weight vector lies in the quasi-minuscule Schubert cell, the result follows immediately from Lemma C.3. Indeed, we would have a closed immersion between two varieties of the same dimension, so it has to be an isomorphism.

Now for the factorization claim, we must once again divide our approach depending on the reducedness of Φ_G. Let us first treat the reduced case. We observe that the exponential of $v_{\theta_{M,<}}$ is by definition $x_\psi(u^{-1})x_{\sigma_0(\psi)}(-u^{-1})$. But this product of commuting elements comes from an isogeny $\mathrm{SL}_2 \times \mathrm{SL}_2 \to H$ onto the root group attached to the orbit $\{\psi, \sigma_0(\psi)\}$ of commuting roots. Notice that the element $x_a(\pm u^{-1})$ of $\mathrm{Gr}_{\mathrm{SL}_2}$ belongs to the $a^\vee$-Schubert cell. Hence, by naturality, we get that $\exp(v_{\theta_{M,<}})$ is sent to the Schubert cell of Gr_H associated with $\psi^\vee + \sigma_0(\psi)^\vee$. But this is exactly the image of $t^{\overline{\psi^\vee}} \in T_G(k(\!(t)\!))$ in $T_H(k(\!(u)\!))/T_H([\![u]\!])$ under the Kottwitz map, see [Ko97, (7.3.2)].

Finally, suppose that Φ_G is non-reduced. We have the extremal weight vector v_{θ_G} of $\mathfrak{g}_{-1}$, whose exponential equals $x_{\psi+\sigma_0(\psi)}(u^{-1})$. This element lies again in the Schubert cell of Gr_H attached to $(\psi + \sigma_0(\psi))^\vee = \psi^\vee + \sigma_0(\psi)^\vee$, so we are done again by [Ko97, (7.3.2)]. □

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