CORRIGENDUM: THE BASE CHANGE FUNDAMENTAL LEMMA FOR CENTRAL ELEMENTS IN PARAHORIC HECKE ALGEBRAS

THOMAS J. HAINES

1. Introduction

In section 2.2 of [H09], there is a minor misstatement that this note will correct and clarify. It has no effect on the main results of [H09], but nevertheless this corrigendum seems necessary in order to avoid potential confusion. Also, I take this opportunity to point out a related typographical error in [BT2], section 5.2.4, and to address some matters of a similar nature.

I am very grateful to Brian Smithling and Tasho Kaletha, who informed me that something was amiss in section 2 of [H09].

2. Notation

All notation will be that of [H09], except for the correction in notation discussed below.

3. Correction

In [H09], section 2.2, the "ambient" group scheme $\mathcal{G}_{\mathbf{a}_J}$ was incorrectly identified with the group scheme whose group of $\mathcal{O}_L$-points is the full fixer of the facet $\mathbf{a}_J$. In the notation of Bruhat-Tits [BT2], which I intended to follow in [H09], the group scheme whose group of $\mathcal{O}_L$-points is the full fixer of $\mathbf{a}_J$ is denoted $\widehat{\mathcal{G}}_{\mathbf{a}_J}$. The group scheme $\widehat{\mathcal{G}}_{\mathbf{a}_J}$ is defined and characterized in this way in [BT2], 4.6.26-28.

The group scheme denoted $\mathcal{G}_{\mathbf{a}_J}$ is defined in loc. cit. 4.6.26 (cf. also 4.6.3-6). In general, it can be a bit smaller than $\widehat{\mathcal{G}}_{\mathbf{a}_J}$ (see below). In [H09], the symbol $\mathcal{G}_{\mathbf{a}_J}$ should be interpreted as this potentially proper subgroup of the full fixer $\widehat{\mathcal{G}}_{\mathbf{a}_J}$.

We have, as stated in [H09], (2.3.2) and (2.3.3), the equalities[1]

$$J(L) = \mathcal{G}^\circ_{\mathbf{a}_J}(\mathcal{O}_L) = T(L)_1 \cdot \mathfrak{U}_{\mathbf{a}_J}(\mathcal{O}_L) \tag{3.0.1}$$

$$\mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L) = T(L)_b \cdot \mathfrak{U}_{\mathbf{a}_J}(\mathcal{O}_L). \tag{3.0.2}$$

In general,

$$\mathcal{G}^\circ_{\mathbf{a}_J}(\mathcal{O}_L) = \widehat{\mathcal{G}}^\circ_{\mathbf{a}_J}(\mathcal{O}_L) \subset \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L) \subset \widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L),$$

and both inclusions can be strict.

[1] In light of the typographical error in [BT2], 5.2.4 explained in section 6, the reasoning used in [H09] to justify these equalities is correct.

4. Clarification of subsequent statements in [H09]

1. Theorem 2.3.1 of [H09] remains valid as stated, but can be slightly augmented: equation (2.3.1) can be replaced by

$$J(L) = \mathrm{Fix}(\mathbf{a}_J^{\mathrm{ss}}) \cap G(L)_1 = \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L) \cap G(L)_1 = \widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L) \cap G(L)_1. \tag{4.0.3}$$

Cf. [HRa], Remark 11.

2. Contrary to [H09], line above equation (2.3.2), our $\mathcal{G}_{\mathbf{a}_J}$ should not now be identified with the scheme $\widehat{\mathcal{G}}_{\mathbf{a}_J^{\mathrm{ss}}}$ of [BT2].

3. Corollary 2.3.2 of [H09] remains valid, with the same proof. Indeed, when G_L is split we have $T(L)_b = T(\mathcal{O}_L) = T(L)_1$ and then from (3.0.1) and (3.0.2) above we see that $\mathcal{G}^\circ_{\mathbf{a}_J}(\mathcal{O}_L) = \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L)$.

4. Lemma 2.9.1 of [H09] remains valid as stated, but in the proof (especially in equations (2.9.1) and (2.9.2)) the symbols $\mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L)$ and $\mathcal{G}_{\mathbf{a}_J^M}(\mathcal{O}_L)$ should be replaced by $\widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L)$ and $\widehat{\mathcal{G}}_{\mathbf{a}_J^M}(\mathcal{O}_L)$, respectively.

5. Example

It is sometimes but usually not the case that $\mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L) = \widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L)$. The following is perhaps the simplest example where this equality fails[2]. Take G to be the split group $\mathrm{PSp}(4)$, and let $\mathbf{a}_J$ denote the non-special vertex in a base alcove. Then let τ denote the element in the stabilizer $\Omega \subset \widetilde{W}(L)$ of the base alcove, which interchanges the two special vertices and fixes $\mathbf{a}_J$. The element τ does not belong to the group $\mathcal{G}^\circ_{\mathbf{a}_J}(\mathcal{O}_L) = \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L)$ (cf. **3** above), since τ does not belong to $G(L)_1$. On the other hand $\tau \in \widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L)$ since it fixes $\mathbf{a}_J$ and $G(L)^1 = G(L)$ (cf. [BT2], 4.6.28).

6. Typographical error in [BT2], 5.2.4

Section 5.2.4 of [BT2] contains four displayed equations. In all of these equations, the "hats" should be removed. The fact that the final displayed equation

$$\widehat{\mathfrak{G}}^\natural_\Omega(\mathcal{O}^\natural) = \mathfrak{G}^\circ_\Omega(\mathcal{O}^\natural)\, \mathfrak{Z}(\mathcal{O}^\natural)$$

is incorrect as stated is shown by the Example above (in light of the fact that for a $K^\natural$-split group such as $\mathrm{PSp}(4)$ the group scheme $\mathfrak{Z}$ is connected and the right hand side is simply $\mathfrak{G}^\circ_\Omega(\mathcal{O}^\natural)$).

All of the displayed equations in [BT2], 5.2.4 become correct when the "hats" are removed.

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[2]Brian Smithling and Tasho Kaletha provided me with another example for the split group $\mathrm{SO}(2n)$.

7. When is $\mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L) = \widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L)$?

Let us assume (for simplicity) that G is split over L. Then the following give two cases where the equality $\mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L) = \widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L)$ holds. Since G_L is split, by Corollary 2.3.2 of [H09] we automatically have $\mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L) = \mathcal{G}^\circ_{\mathbf{a}_J}(\mathcal{O}_L)$.

Lemma 7.0.1. *If $G_{\mathrm{der}} = G_{\mathrm{sc}}$, then $\widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L) = \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L)$.*

Proof. Let $\mathcal{I} = \mathrm{Gal}(\overline{L}/L)$ denote the inertia group. Recall that $G(L)_1$ is the kernel of the Kottwitz homomorphism

$$G(L) \to X^*(Z(\widehat{G})^{\mathcal{I}})$$

and $G(L)^1$ is the kernel of the map

$$G(L) \to X^*(Z(\widehat{G})^{\mathcal{I}})/torsion$$

derived from the Kottwitz homomorphism. Our hypotheses imply that $X^*(Z(\widehat{G})^{\mathcal{I}}) = X^*(Z(\widehat{G}))$ is torsion-free, and hence $G(L)^1 = G(L)_1$. But then $\widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L)$, being by [BT2], 4.6.28 the fixer of $\mathbf{a}_J^{\mathrm{ss}}$ in $G(L)^1$, obviously coincides with $\mathcal{G}^\circ_{\mathbf{a}_J}(\mathcal{O}_L)$, the fixer of $\mathbf{a}_J^{\mathrm{ss}}$ in $G(L)_1$ (cf. (4.0.3) above). □

Lemma 7.0.2. *If the closure of $\mathbf{a}_J$ contains a special vertex v, then $\widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L) = \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L)$.*

Proof. By [BT2], 4.6.26, we have $\widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_L) = \widehat{N}^1_{\mathbf{a}_J}\, \mathcal{G}_{\mathbf{a}_J}(\mathcal{O}_L)$, where $\widehat{N}^1_{\mathbf{a}_J}$ denotes the fixer in $N = N_G(T)(L)$ of $\mathbf{a}_J$. Hence, it suffices to show that $\widehat{N}^1_{\mathbf{a}_J} \subset G(L)_1$. Let $K = K_v$ be the special maximal parahoric subgroup of $G(L)$ corresponding to v, and realize the finite Weyl group W at v as $W = (K \cap N_G(T))/T(\mathcal{O}_L)$, cf. [HRa]. As in loc. cit., the choice of the special vertex v gives us a decomposition of the extended affine Weyl group as $X_*(T) \rtimes W$. For $n \in \widehat{N}^1_{\mathbf{a}_J}$ let $t_\lambda w \in X_*(T) \rtimes W$ denote the corresponding element.

We need to show that $t_\lambda w$ belongs to the affine Weyl group, since such an element will automatically belong to $G(L)_1$, and that would be enough to prove that $n \in G(L)_1$. We need to show λ is in the coroot lattice $Q^\vee$. But $t_\lambda w$ fixes v, that is,

$$\lambda + w(v) = v.$$

On the other hand

$$v - w(v) \in Q^\vee,$$

since v is a special vertex. Thus $\lambda \in Q^\vee$ and we are done. □

8. Comparing Iwahori subgroups over F

The "naive" Iwahori subgroup that often appears in the literature (e.g. [C], [Mac]), can be identified with the group

$$\widetilde{I} := G(F) \cap \mathrm{Fix}(\mathbf{a}^\sigma) = G(F)^1 \cap \mathrm{Fix}((\mathbf{a}^{\mathrm{ss}})^\sigma).$$

This contains the group

$$\widehat{\mathcal{G}}_{\mathbf{a}}(\mathcal{O}_F) = G(F)^1 \cap \mathrm{Fix}(\mathbf{a}),$$

(cf. [BT2], 4.6.28). The "true" Iwahori subgroup over F is defined to be

$$I := G(F) \cap (G(L)_1 \cap \mathrm{Fix}(\mathbf{a})) = \mathcal{G}^\circ_{\mathbf{a}}(\mathcal{O}_F)$$

(see [HRa]) which turns out to have the alternative description

$$I = G(F)_1 \cap \mathrm{Fix}(\mathbf{a}^\sigma),$$

see [HRo], Remark 8.0.2. Thus, we always have the inclusions

$$I \subseteq \widehat{\mathcal{G}}_{\mathbf{a}}(\mathcal{O}_F) \subseteq \widetilde{I}.$$

In general, we have $\widetilde{I} \neq I$; for example, in the case of $G = D^\times/F^\times$ we have $\widehat{\mathcal{G}}_{\mathbf{a}}(\mathcal{O}_F) \neq \widetilde{I}$ (see Remark 8.0.2 of [HRo]).

Lemma 8.0.3. *Suppose G is split over L. Then $I = \widehat{\mathcal{G}}_{\mathbf{a}}(\mathcal{O}_F)$.*

Proof. Use Lemma 7.0.2. □

Proposition 8.0.4. *If G is unramified over F, then $I = \widehat{\mathcal{G}}_{\mathbf{a}}(\mathcal{O}_F) = \widetilde{I}$.*

Proof. It is enough to prove $I = \widetilde{I}$. Let v_F denote a hyperspecial vertex in the closure of $(\mathbf{a}^{\mathrm{ss}})^\sigma$, and let $K = K_{v_F}$ denote the corresponding special maximal parahoric subgroup of $G(F)$. Following [HRo], define $\widetilde{K} = G(F)^1 \cap \mathrm{Fix}(v_F)$; recall also that $K = G(F)_1 \cap \mathrm{Fix}(v_F)$. By loc. cit., it is clear that when G is unramified over F we have $\widetilde{K} = K$. On the other hand, the inclusion $\widetilde{I} \subset \widetilde{K}$ clearly induces an injection

$$\widetilde{I}/I \hookrightarrow \widetilde{K}/K.$$

Thus $\widetilde{I}/I$ is trivial. □

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University of Maryland
Department of Mathematics
College Park, MD 20742-4015 U.S.A.
email: tjh@math.umd.edu