

The Jordan-Hölder series for nearby cycles on some Shimura varieties and affine flag varieties

Ulrich Görtz and Thomas J. Haines *

Abstract

We study the Jordan-Hölder series for nearby cycles on certain Shimura varieties and Rapoport-Zink local models, and on finite-dimensional pieces of Beilinson's deformation of the affine Grassmannian to the affine flag variety (and their p -adic analogues). We give a formula for the multiplicities of irreducible constituents in terms of certain cohomology groups, and we also provide an algorithm to compute multiplicities, in terms of the affine Hecke algebra.

MSC-Classification: 14M15, 14G35, 20C08

1 Introduction

This article has several aims. The initial motivation was our desire to understand, as explicitly as possible, the irreducible constituents of the nearby cycles sheaf $R\Psi$ on the reduction modulo p of a Shimura variety with Iwahori-level structure at a fixed prime p . In some sense, the complexity of $R\Psi$ is a measure of the singularities in the reduction modulo p . Moreover, we were motivated by the role $R\Psi$ plays in the computation of the semi-simple local zeta function at p for such a Shimura variety, cf. [RZ1], [HN1].

Via the relationship between the Shimura variety and its Rapoport-Zink local model [RZ2], one can translate the problem into that of understanding the nearby cycles on a local model. The latter can be embedded as a finite-dimensional piece of a \mathbb{Z}_p -ind-scheme M which is a deformation of the affine Grassmannian $\text{Grass}_{\mathbb{Q}_p}$ to the affine flag variety $\mathcal{Fl}_{\mathbb{F}_p}$ for the underlying p -adic group G (M exists at least if G is either GL_n or $GS_{p_{2n}}$; see [HN1] and section 8). A very similar deformation Fl_X over a smooth curve X (due to Beilinson) exists for any group G in the function field setting, and has been extensively studied by Gaitsgory [Ga].

The “maximal parahoric” subgroup $G(\mathcal{O})$ in the loop group of G acts on Grass , the generic fiber of the deformation M or Fl_X . Fixing a dominant coweight μ of

*Research supported in part by NSF grant DMS 0303605

G , one may therefore consider nearby cycles $R\Psi = R\Psi(IC_\mu)$, taken with respect to the deformation, where IC_μ is the intersection complex on the closure $\overline{\mathcal{Q}}_\mu$ of the $G(\mathcal{O})$ -orbit in Grass indexed by μ . Letting \mathcal{B} denote the standard Iwahori subgroup of $G(\mathcal{O})$, one may show that $R\Psi$ is a \mathcal{B} -equivariant perverse sheaf on the affine flag variety, with certain additional properties (e.g., it belongs to the category $P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ of section 4). Thus, quite generally, we were led to the problem of understanding the Jordan-Hölder series for objects \mathcal{F} of the category $P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$. It turns out that the only irreducible objects in this category are Tate-twists of intersection complexes on finite-dimensional Schubert varieties $\mathcal{B}_{\overline{w}}$ in $\mathcal{F}l$ (they are indexed by elements w in the extended affine Weyl group \widetilde{W} for G ; we denote such intersection complexes by IC_w). Thus, we may define non-negative “multiplicities” $m(\mathcal{F}, w, i)$ by the identity in $P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$

$$\mathcal{F}^{ss} = \bigoplus_{w \in \widetilde{W}} \bigoplus_{i \in \mathbb{Z}} IC_w(-i)^{\oplus m(\mathcal{F}, w, i)}.$$

Our main theorem is the following cohomological interpretation for the integers $m(\mathcal{F}, w, i)$ for sheaves \mathcal{F} which satisfy a somewhat technical hypothesis (Property (P) of section 4.3). This property holds for all the nearby cycles we consider, and also for a much larger class of objects (those with a suitable filtration by Wakimoto sheaves; see section 7).

Theorem 1.1 *Suppose $\mathcal{F} \in P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ satisfies property (P) for the integer d . Then for each $w \in \widetilde{W}$,*

$$\sum_i m(\mathcal{F}, w, i)q^i = (-q)^d \text{Tr}(\text{Fr}_q, H_c^\bullet(\mathcal{F}l, D\mathcal{F} \otimes IC(\mathcal{B}^{\overline{w}}))).$$

Here $D\mathcal{F}$ denotes the Verdier dual of \mathcal{F} ; there is also a “dual” formulation (see Theorem 4.7) in which \mathcal{F} appears rather than $D\mathcal{F}$. We introduce and study the intersection complex $IC(\mathcal{B}^{\overline{w}})$ appearing here in section 3. In a certain sense, it is the intersection complex for an orbit $\mathcal{B}^{\overline{w}}$ for the “opposite” Iwahori subgroup \mathcal{B}^- . Such orbits are infinite-dimensional, and so it is necessary to construct $IC(\mathcal{B}^{\overline{w}})$ by pull-back from certain finite-dimensional quotients (introduced by Faltings [F]) of open sets $\Omega \subset \mathcal{F}l$. The basic properties of $IC(\mathcal{B}^{\overline{w}})$ play a key role in the proof. The other main ingredient is the cohomological interpretation of inverse Kazhdan-Lusztig polynomials, announced in [KL2]. In fact, the Kazhdan-Lusztig theorem can be seen as a special case of our main theorem (see section 4.5). Since no proof of the Kazhdan-Lusztig theorem in exactly our setting has appeared in the literature, we provide a proof for this result in section 3. Here again we make essential use of our sheaf $IC(\mathcal{B}^{\overline{w}})$ and of the quotients of Faltings [F] to pass from ind-schemes to ordinary finite-dimensional schemes.

The main theorem was discovered in the attempt to find a conceptual explanation for the following remarkable fact (Cor. 6.3): Let M_μ denote the scheme-theoretic closure in our deformation of $\overline{\mathcal{Q}}_\mu$. Let τ denote the element of \widetilde{W}

indexing the unique dimension zero Iwahori-orbit in the special fiber of M_μ (i.e., the “most singular point”). Then we have

Corollary 1.2 *For $R\Psi = R\Psi(IC_\mu)$, the multiplicity $m(R\Psi, \tau, i)$ is the $2i$ -th intersection Betti number for $\overline{\mathcal{Q}}_\mu$.*

This fact, noticed after several computer-aided calculations using the algorithm described below, gave us the first hint that multiplicity functions have a cohomological interpretation.

It is clear that knowing the multiplicities $m(\mathcal{F}, w, i)$ for every w and every i is equivalent to knowing an explicit formula for the trace function $\mathrm{Tr}(\mathrm{Fr}_q, \mathcal{F})$ as an element in the Iwahori-Hecke algebra (see section 4.2). On the other hand, the Kottwitz conjecture (proved in [Ga] and [HN1], cf. section 2.7) identifies the semi-simple trace function $\mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi(IC_\mu))$ as an explicit sum of functions Θ_λ , each of which can be computed (with a computer program, see sections 7,9). Thus, there is an algorithm to compute the multiplicities $m(R\Psi, w, i)$ in any given case. (Actually, to justify the algorithm, one has to know that $\mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi) = \mathrm{Tr}(\mathrm{Fr}_q, R\Psi)$; see comments below.) Similarly, the algorithm works to compute the numbers $m(\mathcal{F}, w, i)$ whenever \mathcal{F} has a suitable filtration by Wakimoto sheaves, cf. section 7. In section 9, we give the results of several such computer-aided calculations, and we give explanations for some empirical observations we made.

Another consequence of the Kottwitz conjecture is that the functions $\mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi(IC_\mu))$ form a basis for the center $Z(\mathcal{H})$ of the Iwahori-Hecke algebra for G , as μ ranges over dominant coweights. In terms of the standard generators $T_x = \mathrm{char}(\mathcal{B}x\mathcal{B})$ ($x \in \widetilde{W}$) for \mathcal{H} , we have the identity

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi) = \sum_x \left(\sum_{w \geq x} \left(\sum_i m(R\Psi, w, i) q^i \varepsilon_w P_{x,w} \right) \right) T_x,$$

where \leq is the Bruhat order on \widetilde{W} and $P_{x,w}$ are the Kazhdan-Lusztig polynomials [KL1], cf. section 4.2. Our results on $m(R\Psi, w, i)$ therefore give us both a conceptual and an algorithmic way to make this formula explicit, and this works for any group G . Other approaches to computing elements in $Z(\mathcal{H})$ explicitly in terms of the T_x -basis were given previously in [H1],[H3], [HP], and [S].

Other aims of the paper were to provide proofs of several necessary results of a foundational nature, some of which are very general and might be useful in other contexts. Here is a list of the highlights:

- A study of categories of \mathcal{B} -equivariant perverse sheaves on $\mathcal{F}l$; section 4;
- For finite-type schemes X over a trait (S, s, η) , the canonical decomposition of the category $P(X \times_s \eta, \overline{\mathbb{Q}}_\ell)$ of middle perverse sheaves on $X_{\overline{s}}$ endowed with a compatible continuous action of $\mathrm{Gal}(\overline{\eta}/\eta)$, into “unipotent” and “non-unipotent” subcategories; section 5;

- The proof of Gaitsgory’s theorem that our sheaves $R\Psi$ are unipotent (in the above sense), and the consequence that $\mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi) = \mathrm{Tr}(\mathrm{Fr}_q, R\Psi)$, alluded to above; section 5;
- If X is finite-type over S , an unequal characteristic Henselian trait, and X_η is smooth, then the nearby cycles sheaf $R\Psi^X(\overline{\mathbb{Q}}_\ell)$ is *mixed*; section 10;
- If X as above is proper, the existence of the weight spectral sequence; section 8.

Returning now to the situation of Shimura varieties Sh considered in section 8, the Shimura datum (G, X) provides us with a minuscule coweight μ of $G_{\mathbb{Q}_p}$, and the model M_μ above coincides with the Rapoport-Zink local model $\mathbf{M}^{\mathrm{loc}}$, so the above results apply to give the Jordan-Hölder series for $R\Psi_{Sh}$. The irreducible constituents are Tate-twists of intersection complexes $IC_{Sh,w}$ relative to a stratification $\coprod Sh_w$ on the special fiber $Sh_{\mathbb{F}_p}$, induced by the \mathcal{B} -orbit stratification on M_μ (see [GN]), indexed by the subset $\mathrm{Adm}(\mu)$ of section 2.4¹. If Sh is proper over the ring of integers \mathcal{O}_E of the reflex field E , then the weight spectral sequence ${}_W E^{\bullet,\bullet} \Rightarrow H^\bullet(Sh_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell[\ell(\mu)])$ exists (by virtue of the Appendix) and its E_1 -term can be made somewhat explicit. We have the following result (see section 8 for details).

Theorem 1.3 (a) *In the category $P(Sh \times_s \eta, \overline{\mathbb{Q}}_\ell)$, we have*

$$R\Psi_{Sh}^{ss}(\overline{\mathbb{Q}}_\ell[\ell(\mu)]) = \bigoplus_{w \in \mathrm{Adm}(\mu)} \bigoplus_{i=0}^{\ell(\mu) - \ell(w)} IC_{Sh,w}(-i)^{\oplus m(R\Psi_{M_\mu, w, i})}.$$

(b) *Assume Sh is proper over \mathcal{O}_E . Then there is an isomorphism of $\overline{\mathbb{Q}}_\ell$ -spaces*

$${}_W E_1^{pq} = \bigoplus_{\substack{w \in \mathrm{Adm}(\mu) \\ \ell(w) + 2j = -p}} IH^{p+q+\ell(w)}(\overline{Sh}_w, \overline{\mathbb{Q}}_\ell)^{\oplus m(R\Psi_{M_\mu, w, j})}.$$

The multiplicity functions $\sum_i m(R\Psi(IC_\mu), w, i)q^i$ appear to carry fundamental information. In all the cases we computed, they are actually polynomials in q of degree exactly $\ell(\mu) - \ell(w)$ (and they vanish if $w \notin \mathrm{Adm}(\mu)$). It is perhaps striking that we do not know how to prove the polynomial nature of the multiplicity function in general. It can be proved when μ is either minuscule or a coweight for GL_n , by using the existence of minimal expressions for the functions Θ_λ that arise in the Kottwitz conjecture, see section 9. We provide a possible geometrical approach in Part II of the Appendix (section 10).²

¹Recall that Sh is usually geometrically disconnected. However the various connected components carry the “same” stratification, induced in each case by that on M_μ . To be precise, in all statements the symbol $IC_{Sh,w}$ should be interpreted as the direct sum, over all geometric connected components, of the intersection complexes indexed by w on the various components. In particular this applies to the results in section 8.

²After this article was submitted, we proved in the most general case that the multiplicity functions are polynomials in q having degree at most $\ell(\mu) - \ell(w)$, see [GH].

2 Notation and preliminaries

2.1 Galois structures on derived categories

As usual, p will denote a fixed prime number, and q will always denote a power of p . We will work over Henselian traits (S, s, η) of residue characteristic p . These will usually be of the form $S = \text{Spec}(\mathbb{F}_p[[t]])$ (the function field setting), or $S = \text{Spec}(\mathbb{Z}_p)$ (the p -adic setting). We choose a separable closure $\bar{\eta}$ of η and define the Galois group $\Gamma = \text{Gal}(\bar{\eta}/\eta)$ and the inertia subgroup $\Gamma_0 = \ker[\text{Gal}(\bar{\eta}/\eta) \rightarrow \text{Gal}(\bar{s}/s)]$, where \bar{s} is the residue field of the normalization \bar{S} of S in $\bar{\eta}$.

Suppose the residue field $k(s)$ has cardinality q . Any element of Γ which lifts the inverse Frob_q of the topological generator $a \mapsto a^q$ of $\text{Gal}(\bar{s}/s)$ will be called a *geometric Frobenius* and will be denoted by Fr_q . We will often use the same symbol Fr_q to denote the Frobenius automorphism $\text{id}_{X_0} \times \text{Frob}_q : X_0 \times \text{Spec}(\bar{\mathbb{F}}_q) \rightarrow X_0 \times \text{Spec}(\bar{\mathbb{F}}_q)$ for an \mathbb{F}_q -scheme X_0 . For the most part, we leave it to context to dictate what is meant by the symbol Fr_q in each situation.

Let X be a scheme of finite type over a finite (or algebraically closed) field k . (The following also works if we assume that k is the fraction field of a discrete valuation ring R with finite residue field, and that X is finite-type over R , cf. [Ma].) Denote by \bar{k} an algebraic closure of k , and by $X_{\bar{k}}$ the base change.

We denote by $D_c^b(X, \bar{\mathbb{Q}}_\ell)$ the 'derived' category of $\bar{\mathbb{Q}}_\ell$ -sheaves on X . Note that this is not actually the derived category of the category of $\bar{\mathbb{Q}}_\ell$ -sheaves, but is defined via a limit process. See [BBD] 2.2.14 or [Weil2] 1.1.2 for more details; see also section 5. Nevertheless, $D_c^b(X, \bar{\mathbb{Q}}_\ell)$ is a triangulated category which admits the usual functorial formalism, and which can be equipped with a 'natural' t -structure having as its core the category of $\bar{\mathbb{Q}}_\ell$ -sheaves. If $f : X \rightarrow Y$ is a morphism of schemes of finite type over k , we denote by f_* , $f!$ the derived functors $D_c^b(X, \bar{\mathbb{Q}}_\ell) \rightarrow D_c^b(Y, \bar{\mathbb{Q}}_\ell)$.

We will denote by $P(X, \bar{\mathbb{Q}}_\ell)$ the full subcategory of $D_c^b(X, \bar{\mathbb{Q}}_\ell)$ consisting of middle perverse sheaves. See [BBD] or [KW] for a detailed discussion of this notion.

Now let X be a scheme of finite type over s . The absolute Galois group $\text{Gal}(\bar{\eta}/\eta)$ acts on $X_{\bar{s}}$ through its quotient $\text{Gal}(\bar{s}/s)$.

Definition 2.1 (cf. [SGA 7], exp. XIII)

- (1) The category $D_c^b(X \times_s \eta, \bar{\mathbb{Q}}_\ell)$ is the category of sheaves $\mathcal{F} \in D_c^b(X_{\bar{s}}, \bar{\mathbb{Q}}_\ell)$ together with a continuous action of $\text{Gal}(\bar{\eta}/\eta)$ which is compatible with the action on $X_{\bar{s}}$. (Continuity is tested on cohomology sheaves: see section 5.)
- (2) Similarly, $P(X \times_s \eta, \bar{\mathbb{Q}}_\ell)$ denotes the category of perverse sheaves \mathcal{F} in $P(X_{\bar{s}}, \bar{\mathbb{Q}}_\ell)$ together with a continuous action of $\text{Gal}(\bar{\eta}/\eta)$ which is compatible with the action on $X_{\bar{s}}$.

Analogously, assuming $k(s)$ has cardinality q , we let $D_c^{b,\text{Weil}}(X, \overline{\mathbb{Q}}_\ell)$ denote the category of pairs (\mathcal{F}, F_q^*) where \mathcal{F} belongs to $D_c^b(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell)$, and $F_q^* : \text{Fr}_q^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ is an isomorphism in that category. The definition of $P_{\text{Weil}}(X, \overline{\mathbb{Q}}_\ell)$ is similar.

Any choice of geometric Frobenius $\text{Fr}_q \in \Gamma$ gives rise naturally to a functor

$$D_c^b(X \times_s \eta, \overline{\mathbb{Q}}_\ell) \longrightarrow D_c^{b,\text{Weil}}(X, \overline{\mathbb{Q}}_\ell).$$

This induces an analogous functor on the categories of perverse sheaves.

We also have a natural functor

$$P(X_s, \overline{\mathbb{Q}}_\ell) \longrightarrow P_{\text{Weil}}(X, \overline{\mathbb{Q}}_\ell), \quad \mathcal{F}_0 \mapsto (\mathcal{F}, F_q^*),$$

which is a full embedding, and its essential image is a subcategory which is stable under extensions and subquotients; see [BBD], Prop. 5.1.2. In particular, a Weil perverse sheaf which has a filtration such that the graded pieces are all defined over s , lies in the essential image of this functor.

2.2 The affine Grassmannian and flag variety

We briefly recall the definition of the affine Grassmannian Grass , and the affine flag variety $\mathcal{F}l$.

We fix a field k and suppose G is a split connected reductive group over k . Fix a maximal torus T and a Borel subgroup B containing T . Let $\Lambda_+ \subset X_*(T)$ denote the set of B -dominant integral coweights for G . By W we denote the finite Weyl group $N_G(T)/T$, by W_{aff} the affine Weyl group, and by \widetilde{W} the extended affine Weyl group $N_G(T)/T_{\mathcal{O}}$, where $\mathcal{O} = k[[t]]$.

Consider $G(k((t)))$ as an ind-scheme over k . Denote by $\mathcal{B} \subset G(k((t)))$ the standard Iwahori subgroup, i. e. the inverse image of B under the projection $G(k[[t]]) \longrightarrow G(k)$.

The affine Grassmannian Grass (over the field k) is the quotient (of $fpqc$ -sheaves) $G(k((t)))/G(k[[t]])$; it is an ind-scheme. If $G = GL_n$ and R is a k -algebra, $\text{Grass}(R)$ is the set of all $R[[t]]$ -lattices in $R((t))^n$. If $G = GSp_{2n}$, it is the set of lattices in $R((t))^{2n}$ which are self-dual up to an element in $R[[t]]^\times$. In the same way, one can construct the affine Grassmannian over \mathbb{Z} . By the Cartan decomposition we have a stratification into $G(k[[t]])$ -orbits:

$$\text{Grass} = \coprod_{\mu \in \Lambda_+} G(k[[t]])\mu G(k[[t]])/G(k[[t]]).$$

Here we embed $X_*(T)$ into $G(k((t)))$ by the rule $\mu \mapsto \mu(t) \in T(k((t)))$. We will denote the $G(k[[t]])$ -orbit of μ simply by \mathcal{Q}_μ in the sequel. The closure relations are determined by the standard partial order on dominant coweights: $\mathcal{Q}_\lambda \subset \overline{\mathcal{Q}_\mu}$ if and only if $\mu - \lambda$ is a sum of B -positive coroots.

The affine flag variety $\mathcal{F}l$ over k is the quotient $G(k((t)))/\mathcal{B}$; it is an ind-scheme, too. As before, we have a modular interpretation: e.g., for GL_n , $\mathcal{F}l(R)$ is the

space of complete lattice chains in $R((t))^n$. The Iwahori group \mathcal{B} acts on $\mathcal{F}l$, and we get a decomposition

$$\mathcal{F}l = \coprod_{w \in \widetilde{W}} \mathcal{B}w\mathcal{B}/\mathcal{B}.$$

Here we embed $\widetilde{W} = X_*(T) \rtimes W$ into $G(k((t)))$ by using the aforementioned embedding of $X_*(T)$, and by choosing a representative for $w \in W = N_G(T)/T$ in the group $N_G(T) \cap G(k[[t]])$. We write $\mathcal{B}_w = \mathcal{B}w\mathcal{B}/\mathcal{B}$; \mathcal{B}_w is called the Schubert cell associated to w . Its closure, which we denote by $\mathcal{B}_{\overline{w}}$, is a finite-dimensional projective variety, and is called a Schubert variety. We have

$$\mathcal{B}_{\overline{w}} = \coprod_{v \leq w} \mathcal{B}_v.$$

Here \leq denotes the Bruhat order on \widetilde{W} determined by the affine reflections S_{aff} through the walls of the *opposite* base alcove, i.e., the alcove $w_0(A)$, where w_0 is the longest element in W and $A \subset X_*(T) \otimes \mathbb{R}$ is the unique alcove in the B -dominant chamber whose closure contains the origin. With our normalization of the embedding of $X_*(T)$ into $G(k((t)))$, the alcove fixed by the Iwahori \mathcal{B} is $w_0(A)$, see [IM] or [HKP].

The length function $\ell: \widetilde{W} \rightarrow \mathbb{Z}_{\geq 0}$ used throughout this paper is always defined with respect to the Coxeter system $(W_{\text{aff}}, S_{\text{aff}})$. Thus, \mathcal{B}_w is an affine space isomorphic to $\mathbb{A}^{\ell(w)}$. Also, we will often write $\ell(\mu)$ instead of $\ell(t_\mu)$, where t_μ is the translation element in \widetilde{W} corresponding to $\mu \in X_*(T)$.

The analogue of the unipotent radical N^- of the opposite Borel B^- in the finite-dimensional case, is the subgroup $\mathcal{N}^- \subset G(k((t)))$ which is by definition the inverse image of N^- under the projection $G(k[t^{-1}]) \rightarrow G(k)$. We can then decompose $\mathcal{F}l$ into \mathcal{N}^- orbits:

$$\mathcal{F}l = \coprod_{w \in \widetilde{W}} \mathcal{N}^- w\mathcal{B}/\mathcal{B}.$$

We write $\mathcal{B}^w = \mathcal{N}^- w\mathcal{B}/\mathcal{B}$. This is an ind-scheme of 'finite codimension'. More precisely, $\mathcal{B}^w \cap \mathcal{B}^v \neq \emptyset$ if and only if $w \leq v$, and in this case the intersection has dimension $\ell(v) - \ell(w)$; cf. [F]. We denote the closure of \mathcal{B}^w by $\mathcal{B}^{\overline{w}}$; then

$$\mathcal{B}^{\overline{w}} = \coprod_{v \geq w} \mathcal{B}^v.$$

2.3 Intersection complexes

Let X be a finite-type reduced and geometrically irreducible scheme over a finite field k (or, we may assume k is algebraically closed, or that X is finite-type over a discrete valuation ring with finite residue field and with fraction field k). For any open immersion $j: U \rightarrow X$ of a smooth irreducible dense open subset, we define $IC(X) = j_{!*}(\overline{\mathbb{Q}}_\ell)$, where $j_{!*}$ is the Goresky-MacPherson-Deligne extension functor (for the middle perversity), cf. [GM], [BBD]. The

shift $IC(X)[\dim(X)]$ is an irreducible perverse sheaf on X , independent of the choice of U . If X is not reduced, we set $IC(X) := IC(X_{\text{red}})$.

The intersection cohomology groups of X are given by $IH^i(X) = H^i(X, IC(X))$ (similarly for compact supports).

We fix a choice of $\sqrt{q} \in \overline{\mathbb{Q}}_\ell$, needed to define Tate-twists on the categories $D_c^{b, \text{Weil}}(X, \overline{\mathbb{Q}}_\ell)$ and $P_{\text{Weil}}(X, \overline{\mathbb{Q}}_\ell)$. It is known that $IC(X)[\dim(X)](\frac{\dim(X)}{2})$ is a self-dual pure perverse sheaf of weight zero.

For $w \in \widetilde{W}$ we have the locally closed immersion $j_w : \mathcal{B}_w \hookrightarrow \mathcal{F}l$. We define the perverse sheaf $IC_w = IC(\mathcal{B}_w)[\ell(w)]$. For $\mu \in \Lambda_+$, the perverse sheaf IC_μ on Grass has a similar definition. Note that there is no Tate-twist in these definitions. This turns out to be the most suitable normalization for our purposes.

2.4 Deformations of the affine Grassmannian to the flag variety

Suppose $G = GL_n$ or GSp_{2n} , and $S = \text{Spec}(\mathbb{Z}_p)$. There is a deformation from $\text{Grass}_{\mathbb{Q}_p}$ to $\mathcal{F}l_{\mathbb{F}_p}$, i.e. an ind-scheme M over \mathbb{Z}_p with generic fibre $\text{Grass}_{\mathbb{Q}_p}$ and special fibre $\mathcal{F}l_{\mathbb{F}_p}$. The second author and Ngô [HN1] defined such a deformation as a union

$$M = \bigcup_{\mu \in \Lambda_+} M_\mu,$$

where M_μ is a finite-dimensional projective \mathbb{Z}_p -scheme with generic fiber $\overline{\mathcal{Q}}_{\mu, \mathbb{Q}_p}$ and special fiber

$$\prod_{w \in \text{Adm}(\mu)} \mathcal{B}_{w, \mathbb{F}_p}.$$

Here $\text{Adm}(\mu)$ is by definition the following finite subset of \widetilde{W} :

$$\text{Adm}(\mu) = \{x \in \widetilde{W} \mid x \leq t_\lambda, \text{ for some } \lambda \in W\mu\};$$

we refer to [HN2] for further information about this subset. The model M_μ is defined in terms of lattice chains, so this construction works only for $G = GL_n$ or GSp_{2n} . It is conceivable, however, that other groups can be handled by using a Plücker description of Grass and $\mathcal{F}l$, cf. [H2].

In the function field case, suppose G is an arbitrary connected reductive group over $k = \mathbb{F}_p$ and suppose X is a smooth curve over k . Then a deformation Fl_X has been constructed by Beilinson; see [Ga]. For a distinguished point $x_0 \in X$, Fl_X is an ind-scheme over X whose fiber over $x \neq x_0$ is isomorphic to the product $\text{Grass}_k \times G/B$, and whose fiber over x_0 is isomorphic to the affine flag variety $\mathcal{F}l_k$. This deformation also has a modular interpretation, this time in terms of G -bundles on X . We get a deformation over $S = \text{Spec}(k[[t]])$ by base-changing with a formal neighborhood of $x_0 \in X$.

In either the function-field or the p -adic setting, let Grass denote the “constant” deformation of the affine Grassmannian, i.e., we consider the base change

$\text{Grass}_{\mathbb{Z}} \times S$. The modular interpretations of M and Fl_X allow us to define a projective morphism of ind-schemes

$$\pi : M \rightarrow \text{Grass}$$

over S (similarly for Fl_X). In the GL_n case for example, π sends a lattice chain $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \dots$ in M to the lattice \mathcal{L}_0 in Grass . Often, we will use the same letter π to denote the morphism on the special fibers, where it is just the usual projection of ind-schemes $G(k((t)))/\mathcal{B} \rightarrow G(k((t)))/G(k[[t]])$.

In the sequel, when we use the symbol $\mathcal{F}l$ to denote the affine flag variety, we are often thinking of it as the special fiber of one of the deformations above. Occasionally, we even write $\mathcal{F}l$ for the deformation itself.

2.5 Nearby cycles

Let X denote a finite-type scheme over S . For $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$, we define the *nearby cycles sheaf* to be the object in $D_c^b(X \times_s \eta, \overline{\mathbb{Q}}_\ell)$ given by

$$R\Psi^X(\mathcal{F}) = \bar{i}^* R\bar{j}_*(\mathcal{F}_{\bar{\eta}}),$$

where $\bar{i} : X_{\bar{s}} \hookrightarrow X_{\bar{s}}$ and $\bar{j} : X_{\bar{\eta}} \hookrightarrow X_{\bar{s}}$ are the closed and open immersions of the geometric special and generic fibers of X/S , and $\mathcal{F}_{\bar{\eta}}$ is the pull-back of \mathcal{F} to $X_{\bar{\eta}}$. It is known that the functor $R\Psi^X$ preserves perversity, cf. [I].

2.6 Equivariant sheaves

Consider the affine flag variety $\mathcal{F}l$ over \mathbb{F}_p . By definition, a (mixed) perverse sheaf on $\mathcal{F}l$ is a (mixed) perverse sheaf which is supported on some finite-dimensional part.

Definition 2.2 *Denote by $a : \mathcal{B} \times \mathcal{F}l \rightarrow \mathcal{F}l$ the action of the Iwahori group, by $i : \mathcal{F}l \rightarrow \mathcal{B} \times \mathcal{F}l$ the zero section, and by $p_2 : \mathcal{B} \times \mathcal{F}l \rightarrow \mathcal{F}l$ the projection. A perverse sheaf \mathcal{F} is \mathcal{B} -equivariant if there exists an isomorphism $\varphi : a^*\mathcal{F} \rightarrow p_2^*\mathcal{F}$, such that*

i) φ is rigidified along the zero section: $i^\varphi = \text{id}_{\mathcal{F}}$,*

ii) φ satisfies the cocycle condition: $(m \times \text{id}_{\mathcal{F}l})^(\varphi) = p_{23}^*(\varphi) \circ (\text{id}_{\mathcal{B}} \times a)^*(\varphi)$ on $\mathcal{B} \times \mathcal{B} \times \mathcal{F}l$, where $m : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ is the multiplication map (cf. [KW] III.15).*

Note that since by definition all perverse sheaves on $\mathcal{F}l$ are supported on finite-dimensional subschemes of the affine flag variety, and since the Iwahori subgroup acts through a finite quotient on these subschemes (if they are suitably chosen), the fact that we deal with ind-schemes does not pose any problems at this point.

Remark. As Kiehl and Weissauer explain in [KW] III.15, in our situation the conditions i) and ii) in the definition are automatically satisfied; more precisely,

given any isomorphism $a^*\mathcal{F} \rightarrow p_2^*\mathcal{F}$, one can change it into a rigidified isomorphism, and every rigidified isomorphism satisfies the cocycle condition. (Here we use that the Iwahori acts through geometrically connected quotients on suitable finite pieces of the affine flag variety.)

Similarly, on the affine Grassmannian we have an action of the maximal parahoric subgroup $K = G(\mathcal{O})$, where $\mathcal{O} = \mathbb{Z}_p$ in the unequal characteristic case, and $\mathcal{O} = \mathbb{F}_p[[t]]$ in the function field case.

Proposition 2.3 *If \mathcal{F} is a K -equivariant perverse sheaf on Grass, then its sheaf of nearby cycles $R\Psi(\mathcal{F})$ is \mathcal{B} -equivariant.*

Proof. In the function field case, this is [Ga] Prop. 4; for the unequal characteristic case see [HN1] §4. \square

We denote by $P^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ the category of \mathcal{B} -equivariant perverse sheaves on $\mathcal{F}l/\mathbb{F}_p$. Later on (section 4) we will consider \mathcal{B} -equivariant perverse sheaves which have Galois structure compatible with the group action. We postpone the discussion of those sheaves to that section.

We note that the intersection complexes IC_μ on Grass are $G(\mathcal{O})$ -equivariant, and the intersection complexes IC_w on $\mathcal{F}l$ are \mathcal{B} -equivariant.

2.7 The Kottwitz conjecture

For any $\mathcal{F} \in D_c^b(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}}_\ell)$, one can define its semi-simple trace function

$$x \mapsto \mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathcal{F}_x),$$

for any $x \in \mathcal{F}l(\mathbb{F}_q)$; cf. [HN1]. Here $\mathrm{Fr}_q \in \Gamma$ denotes an arbitrary geometric Frobenius (the semi-simple trace is independent of the choice). If \mathcal{F} is a suitably \mathcal{B} -equivariant object (e.g., an object of $P^{\mathcal{B}}(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}}_\ell)$, cf. section 4.1), then this function defines an element in the Iwahori-Hecke algebra

$$\mathcal{H} = C_c(\mathcal{B} \backslash G(\mathbb{F}_q((t))) / \mathcal{B})$$

of \mathcal{B} -bi-invariant compactly-supported $\overline{\mathbb{Q}}_\ell$ -valued functions on $G(\mathbb{F}_q((t)))$. (Remark: in the sequel, we will often conflate the notions of Iwahori-Hecke algebras and affine Hecke algebras without lingering on the difference.)

In the p -adic setting, fix a dominant coweight μ for the group $G = GL_n$ or GS_{p2n} , and the associated model M_μ from [HN1]. For the nearby cycles sheaf $R\Psi := R\Psi^{M_\mu}(IC_\mu)$, the Kottwitz conjecture is the following equality of functions in the Iwahori-Hecke algebra for G :

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi) = \varepsilon_\mu q_\mu^{1/2} \sum_{\lambda \leq \mu} m_\mu(\lambda) z_\lambda.$$

Here, $\varepsilon_\mu = (-1)^{\ell(\mu)}$, $q_\mu = q^{\ell(\mu)}$, λ ranges over dominant coweights preceding μ in the usual partial order, $m_\mu(\lambda)$ is the multiplicity of λ in the character of the

dual group having highest weight μ , and z_λ is the Bernstein function attached to λ . More precisely $z_\lambda = \sum_{\lambda \in W_\mu} \Theta_\lambda$, where Θ_λ is the function $\tilde{T}_{\lambda_1} \tilde{T}_{\lambda_2}^{-1}$; here $\lambda = \lambda_1 - \lambda_2$ and λ_i is dominant, and $\tilde{T}_x := q_x^{-1/2} T_x$ for $x \in \widetilde{W}$. The symbol T_x denotes the generator $\text{char}(\mathcal{B}x\mathcal{B})$ in the Iwahori-Hecke algebra.

This formula was proved in [HN1] in the situation at hand. In the function field case, the analogous formula holds. In that analogue, IC_μ is replaced with the exterior product $IC_\mu \boxtimes \delta$ on $\text{Grass} \times G/B$, where δ is the skyscraper sheaf supported at the base point of G/B . This is a consequence of Gaitsgory's work [Ga], and holds for any connected reductive group G .

We make extensive use of this formula, in particular in the computation of the examples in section 9, and in the proof that $R\Psi$ satisfies the property (P) (cf. section 6).

3 Inverse Kazhdan-Lusztig polynomials

3.1 The sheaf $IC(\mathcal{B}^v)$

Let us recall some of the notation introduced above. Let $k = \overline{\mathbb{F}}_p$. We denote by \mathcal{B} the Iwahori subgroup of $G(k((t)))$, i.e. the inverse image of the fixed Borel group B under the projection $G(k[[t]]) \rightarrow G$. By \mathcal{N}^- we denote the inverse image of the unipotent radical of the opposite Borel under the projection $G(k[t^{-1}]) \rightarrow G$. Furthermore, let $\mathcal{N}^-(n)$ be the inverse image of $T(k[t^{-1}]/t^{-n})$ under the projection $\mathcal{N}^- \rightarrow G(k[t^{-1}]/t^{-n})$ (cf. [F]). Note that $\text{Lie}(\mathcal{N}^-)$ is generated by the negative affine roots (recall we embed $X_*(T) \hookrightarrow T(k((t)))$ by $\lambda \mapsto \lambda(t)$).

For elements w and y in the extended affine Weyl group \widetilde{W} , we denote by \mathcal{B}_y the \mathcal{B} -orbit of y , by \mathcal{B}^w the \mathcal{N}^- -orbit of w , by $\mathcal{B}^{\overline{w}}$ its closure, by \mathcal{B}_y^w the intersection $\mathcal{B}_y \cap \mathcal{B}^w$, and by $\mathcal{B}_y^{\overline{w}}$ the intersection $\mathcal{B}_y \cap \mathcal{B}^{\overline{w}}$. Recall the basic fact (cf. [F]) that $\mathcal{B}_y^w \neq \emptyset \Leftrightarrow w \leq y$ in the Bruhat order on \widetilde{W} .

Further, we define the opposite Iwahori subgroup \mathcal{B}^- to be the inverse image of the opposite Borel subgroup B^- under the projection $G(k[t^{-1}]) \rightarrow G(k)$. Note that $\mathcal{N}^- \triangleleft \mathcal{B}^-$ and $\mathcal{B}^- = \mathcal{N}^- \cdot (\mathcal{B}^- \cap T(k((t))))$. Hence for any $w \in \widetilde{W}$, $\mathcal{N}^- w * = \mathcal{B}^- w *$, where $*$ denotes the base point in $\mathcal{F}l$.

Fix $w \in \widetilde{W}$ such that $\text{Adm}(\mu) \subseteq \{v \mid v \leq w\}$.

Let $\Omega = \bigcup_{v \leq w} v \mathcal{N}^- *$. This is an open subset of $\mathcal{F}l$.

Lemma 3.1 *We have $\Omega = \coprod_{v \leq w} \mathcal{B}^v$. Thus for $y \leq w$ we have $\mathcal{B}_y \subseteq \Omega$.*

Proof. We define

$$\begin{aligned} \mathcal{N}_w^- &= \mathcal{N}^- \cap w \mathcal{N}^- w^{-1} \\ \mathcal{N}_w^+ &= \mathcal{N}^+ \cap w \mathcal{N}^- w^{-1}. \end{aligned}$$

Here \mathcal{N}^+ denotes the preimage of N under the projection $G(k[[t]]) \rightarrow G(k)$; $\text{Lie}(\mathcal{N}^+)$ is generated by the positive affine roots.

We then have

$$\mathcal{N}^- \cong \mathcal{N}_w^- \times \prod_{\substack{\alpha < 0 \\ w^{-1}(\alpha) > 0}} U_\alpha,$$

and

$$\mathcal{N}_w^+ = \prod_{\substack{\alpha > 0 \\ w^{-1}(\alpha) < 0}} U_\alpha = w \left(\prod_{\substack{\alpha < 0 \\ w(\alpha) > 0}} U_\alpha \right) w^{-1}.$$

Furthermore, it is easy to see that $\mathcal{B}_w = \mathcal{N}_w^+ w^*$ and $\mathcal{B}^w = \mathcal{N}_w^- w^*$.

From the above we have

$$v\mathcal{N}^- = v\mathcal{N}_{v^{-1}}^- \cdot (v^{-1}\mathcal{N}_v^+ v) = \mathcal{N}_v^- \mathcal{N}_v^+ v.$$

Thus clearly

$$\mathcal{B}^v = \mathcal{N}_v^- v^* \subseteq v\mathcal{N}^- *$$

so

$$\prod_{v \leq w} \mathcal{B}^v \subseteq \bigcup_{v \leq w} v\mathcal{N}^- *.$$

On the other hand, $\mathcal{N}_v^+ v = \mathcal{B}_v \subseteq \prod_{v' \leq v} \mathcal{B}^{v'}$, hence $\mathcal{N}_v^- \mathcal{N}_v^+ v \subseteq \prod_{v' \leq v} \mathcal{B}^{v'}$ and thus

$$\bigcup_{v \leq w} v\mathcal{N}^- * \subseteq \prod_{v \leq w} \mathcal{B}^v.$$

□

For n sufficiently large, the quotient $\mathcal{N}^-(n) \backslash \Omega$ exists; cf. [F] (choose n large enough so that $\mathcal{N}^-(n) \subset \mathcal{N}_v^-$, for every $v \leq w$). The reason is that we have a product decomposition

$$v\mathcal{N}^- * = \mathcal{N}_v^- \mathcal{N}_v^+ v^* \cong \mathcal{N}^-(n) \times \mathbb{A}^a \times \mathbb{A}^b,$$

where $b = \dim(\mathcal{N}_v^+) = \ell(v)$, and a is the number of affine roots in \mathcal{N}_v^- which are not in $\mathcal{N}^-(n)$; cf. [F]. Thus we see that more precisely, the quotient $\mathcal{N}^-(n) \backslash \Omega$ can be covered by affine spaces, and in particular is smooth.

In the following discussion, we often tacitly identify $\mathcal{B}_y = \mathcal{N}_y^+ y$, $\mathcal{B}^y = \mathcal{N}_y^- y$ and $\mathcal{B}^y \times \mathcal{B}_y = \mathcal{N}_y^- \mathcal{N}_y^+ y = y\mathcal{N}^- *$. We have a commutative diagram ($v \leq y \leq w$)

$$\begin{array}{ccccccc} \mathcal{B}_y^{\bar{v}} \hookrightarrow & \mathcal{B}_y \hookrightarrow & \mathcal{B}^y \times \mathcal{B}_y \hookrightarrow & \Omega \\ \downarrow \cong & \downarrow \cong & \downarrow & \downarrow \pi \\ \pi(\mathcal{B}_y^{\bar{v}}) \hookrightarrow & \pi(\mathcal{B}_y) \hookrightarrow & \mathcal{N}^-(n) \backslash \mathcal{B}^y \times \mathcal{B}_y \hookrightarrow & \mathcal{N}^-(n) \backslash \Omega. \end{array}$$

Note that

$$\pi(\mathcal{B}_y^{\bar{v}} \cap \Omega) = \pi(\mathcal{B}_y) \cap \pi(\mathcal{B}_y^{\bar{v}} \cap \Omega).$$

Furthermore $\pi(\mathcal{B}^v \cap \Omega)$ is isomorphic to an affine space (thus in particular is smooth), and its codimension in $\mathcal{N}^-(n) \setminus \Omega$ is $\ell(v)$. Finally, we have

$$\mathcal{N}^-(n) \setminus (\mathcal{B}^y \times \mathcal{B}_y) = (\mathcal{N}^-(n) \setminus \mathcal{B}^y) \times \mathcal{B}_y =: \mathcal{A}_y$$

and this is an open subset of $\mathcal{N}^-(n) \setminus \Omega$, and $\mathcal{A}_y \cap \pi(\mathcal{B}^{\bar{v}} \cap \Omega) = (\mathcal{N}^-(n) \setminus \mathcal{B}^y) \times \mathcal{B}_y^{\bar{v}}$ is an open subset of $\pi(\mathcal{B}^{\bar{v}} \cap \Omega)$.

By taking the quotient of the morphism $\mathcal{N}^- \times \Omega \longrightarrow \Omega \times \Omega$, $(g, x) \mapsto (x, gx)$ giving the action of \mathcal{N}^- on Ω , we get the groupoid

$$\mathcal{N}^-(n) \setminus \mathcal{N}^- \times^{\mathcal{N}^-(n)} \Omega \longrightarrow \mathcal{N}^-(n) \setminus \Omega \times \mathcal{N}^-(n) \setminus \Omega$$

(cf. [F]).

Proposition 3.2 *This is a smooth groupoid, i.e. the maps $\mathcal{N}^-(n) \setminus \mathcal{N}^- \times^{\mathcal{N}^-(n)} \Omega \longrightarrow \mathcal{N}^-(n) \setminus \Omega$, $(g, x) \mapsto x$, and $\mathcal{N}^-(n) \setminus \mathcal{N}^- \times^{\mathcal{N}^-(n)} \Omega \longrightarrow \mathcal{N}^-(n) \setminus \Omega$, $(g, x) \mapsto gx$, are smooth.*

Proof. Consider the maps

$$\phi_1, \phi_2 : \mathcal{N}^- \times \Omega \longrightarrow \Omega, \quad (g, x) \mapsto x, \text{ resp. } (g, x) \mapsto gx.$$

These are formally smooth maps since \mathcal{N}^- is formally smooth. They induce maps

$$\psi_1, \psi_2 : \mathcal{N}^-(n) \setminus \mathcal{N}^- \times^{\mathcal{N}^-(n)} \Omega \longrightarrow \mathcal{N}^-(n) \setminus \Omega,$$

where $\psi_1(g, x) = x$, $\psi_2(g, x) = gx$.

We must show that ψ_1 and ψ_2 are smooth. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{N}^- \times \Omega & \xrightarrow{\phi, \text{fs}} & \Omega \\ \downarrow \text{fs} & & \downarrow \text{fs} \\ \mathcal{N}^-(n) \setminus \mathcal{N}^- \times^{\mathcal{N}^-(n)} \Omega & \xrightarrow{\psi} & \mathcal{N}^-(n) \setminus \Omega \end{array}$$

where ϕ is either ϕ_1 or ϕ_2 , ψ is the corresponding map and 'fs' means that we already know that this morphism is formally smooth.

Since $\mathcal{N}^-(n) \setminus \mathcal{N}^- \times^{\mathcal{N}^-(n)} \Omega \longrightarrow \mathcal{N}^-(n) \setminus \Omega$ is a morphism of finite type between Noetherian schemes (over $\overline{\mathbb{F}}_p$), in order to prove that it is smooth, it is sufficient to show the infinitesimal lifting criterion for local Artin rings with residue class field $\overline{\mathbb{F}}_p$. (see [SGA 1], Exp. III Thm. 3.1).

So let A be a local Artin ring with residue field $\overline{\mathbb{F}}_p$, let $I \subset A$ be a nilpotent ideal, and look at a diagram

$$\begin{array}{ccc} \text{Spec } A/I & \longrightarrow & \mathcal{N}^-(n) \setminus \mathcal{N}^- \times^{\mathcal{N}^-(n)} \Omega \\ \downarrow & & \downarrow \\ \text{Spec } A & \longrightarrow & \mathcal{N}^-(n) \setminus \Omega. \end{array}$$

We have to find a morphism $\text{Spec } A \longrightarrow \mathcal{N}^-(n) \backslash \mathcal{N}^- \times^{\mathcal{N}^-(n)} \Omega$ which makes the diagram commutative.

Since the morphism $\mathcal{N}^- \times \Omega \longrightarrow \mathcal{N}^-(n) \backslash \mathcal{N}^- \times^{\mathcal{N}^-(n)} \Omega$ is surjective on $\overline{\mathbb{F}}_p$ -valued points, we can extend the previous diagram to a diagram

$$\begin{array}{ccc}
 \text{Spec } \overline{\mathbb{F}}_p & \longrightarrow & \mathcal{N}^- \times \Omega \\
 \downarrow & & \downarrow \xi \\
 \text{Spec } A/I & \longrightarrow & \mathcal{N}^-(n) \backslash \mathcal{N}^- \times^{\mathcal{N}^-(n)} \Omega \\
 \downarrow & & \downarrow \psi \\
 \text{Spec } A & \longrightarrow & \mathcal{N}^-(n) \backslash \Omega.
 \end{array}$$

Now since ξ is formally smooth, we can insert a map $\text{Spec } A/I \longrightarrow \mathcal{N}^- \times \Omega$ into the diagram. Then we get a map $\text{Spec } A \longrightarrow \mathcal{N}^- \times \Omega$, since the composition $\psi\xi$ is formally smooth. Now we can define the map we are looking for as the composition of this map with ξ .

Thus we see that ψ_1 and ψ_2 are indeed smooth. \square

From the Proposition, we immediately get the following corollary which will be used in the proof of the theorem of Kazhdan and Lusztig.

Corollary 3.3 *Let $v \leq z \leq w$. For all $z' \in \mathcal{B}^z(\mathbb{F}_q)$,*

$$IC_{z'}(\pi(\mathcal{B}^{\overline{v}} \cap \Omega)) \cong IC_z(\pi(\mathcal{B}^{\overline{v}} \cap \Omega)).$$

Proof. By pull-back the maps ψ_1 and ψ_2 in the Proposition induce smooth maps

$$\psi_1, \psi_2 : \mathcal{N}^-(n) \backslash \mathcal{N}^- \times^{\mathcal{N}^-(n)} (\mathcal{B}^{\overline{v}} \cap \Omega) \longrightarrow \pi(\mathcal{B}^{\overline{v}} \cap \Omega),$$

where $\psi_1(g, x) = x$, $\psi_2(g, x) = gx$.

Now choose $g \in \mathcal{N}^-$ such that $gz = z'$. The point (g, z) maps to z resp. z' under ψ_1 resp. ψ_2 . Since both these maps are smooth, the corollary follows. \square

However, we can even say more: The sheaf $IC(\pi(\mathcal{B}^{\overline{v}} \cap \Omega))$ on $\pi(\mathcal{B}^{\overline{v}} \cap \Omega) \subseteq \mathcal{N}^-(n) \backslash \Omega$ is equivariant under the above groupoid (our definition of equivariance obviously makes sense in this context). Namely, we have

$$\psi_1^* IC(\pi(\mathcal{B}^{\overline{v}} \cap \Omega)) \cong IC(\mathcal{N}^-(n) \backslash \mathcal{N}^- \times^{\mathcal{N}^-(n)} (\mathcal{B}^{\overline{v}} \cap \Omega)) \cong \psi_2^* IC(\pi(\mathcal{B}^{\overline{v}} \cap \Omega)),$$

since ψ_1 and ψ_2 are both smooth. We can choose an isomorphism which is rigidified along the zero section, and since $\mathcal{N}^-(n) \backslash \mathcal{N}^-$ is connected such an isomorphism will automatically satisfy the cocycle condition.

Thus the pull-back of the sheaf $IC(\pi(\mathcal{B}^{\overline{v}} \cap \Omega))$ to Ω is equivariant with respect to the action of \mathcal{N}^- .

We can now define the intersection complex $IC(\mathcal{B}^{\bar{v}})$ of the anti-cell $\mathcal{B}^{\bar{v}}$. Since $\mathcal{B}^{\bar{v}}$ is just an ind-scheme, it is not a priori clear how to make sense of that. However, by looking at the quotients $\mathcal{N}^-(n)\backslash\Omega$, we can define a sheaf on $\mathcal{F}l$ which merits the name $IC(\mathcal{B}^{\bar{v}})$.

The pull-back of the sheaf $IC(\pi(\mathcal{B}^{\bar{v}} \cap \Omega))$ to Ω is independent of n : in fact, let $n' > n$, and consider the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\pi} & \mathcal{N}^-(n)\backslash\Omega \\ & \searrow \pi' & \swarrow \psi \\ & & \mathcal{N}^-(n')\backslash\Omega. \end{array}$$

The map ψ is an \mathbb{A}^N -bundle (for suitable N), and $\pi(\mathcal{B}^{\bar{v}} \cap \Omega) = \psi^{-1}(\pi'(\mathcal{B}^{\bar{v}} \cap \Omega))$. Thus the pull-back of $IC(\pi'(\mathcal{B}^{\bar{v}} \cap \Omega))$ along ψ is just $IC(\pi(\mathcal{B}^{\bar{v}} \cap \Omega))$. This means that the pull-back of $IC(\pi(\mathcal{B}^{\bar{v}} \cap \Omega))$ to Ω is independent of n . Since it is obviously compatible with enlarging Ω , we indeed get a sheaf on $\mathcal{F}l$, which we denote by $IC(\mathcal{B}^{\bar{v}})$.

We will only ever need to work with the restriction of $IC(\mathcal{B}^{\bar{v}})$ to finite-dimensional subschemes of $\mathcal{F}l$. In particular we do not really need to define this sheaf on $\mathcal{F}l$ — we could always work on a suitable quotient $\mathcal{N}^-(n)\backslash\Omega$.

We will need the following property of $IC(\mathcal{B}^{\bar{v}})$. When restricted to an intersection $\mathcal{B}_y^{\bar{v}}$, we get the intersection complex of that space:

$$IC(\mathcal{B}^{\bar{v}})|_{\mathcal{B}_y^{\bar{v}}} = IC(\mathcal{B}_y^{\bar{v}}).$$

This follows from

Proposition 3.4 *We have*

$$IC(\pi(\mathcal{B}^{\bar{v}} \cap \Omega))|_{\pi(\mathcal{B}_y^{\bar{v}})} = IC(\mathcal{B}_y^{\bar{v}}).$$

Proof. This follows immediately from the fact that $\mathcal{A}_y \cap \pi(\mathcal{B}^{\bar{v}} \cap \Omega) = (\mathcal{N}^-(n)\backslash\mathcal{B}^y) \times \mathcal{B}_y^{\bar{v}}$ is an open subset of $\pi(\mathcal{B}^{\bar{v}} \cap \Omega)$ and that $\mathcal{N}^-(n)\backslash\mathcal{B}^y$ is just an affine space. \square

3.2 The theorem of Kazhdan and Lusztig

We start with two lemmas, and then prove the theorem of Kazhdan and Lusztig.

Lemma 3.5 *We have*

$$\#\mathcal{B}_y^z(\mathbb{F}_q) = R_{z,y}(q).$$

Proof. See [Had], section 2. \square

Lemma 3.6 *On \mathcal{A}_y , which is isomorphic to an affine space \mathbb{A}^N , we have a \mathbb{G}_m -action given by*

$$\lambda \cdot (z_1, \dots, z_N) = (\lambda^{a_1} z_1, \dots, \lambda^{a_N} z_N),$$

for certain $a_i > 0$, which preserves $\pi(\mathcal{B}^{\bar{v}} \cap \Omega) \cap \mathcal{A}_y$ and $\pi(\mathcal{B}^{\bar{v}} \cap \mathcal{B}_y)$.

Proof. We decompose $\mathcal{A}_y = \mathcal{N}^-(n) \backslash \mathcal{B}^y \times \mathcal{B}_y$, and we will define \mathbb{G}_m -actions on the two factors separately. With respect to this decomposition, we have $\pi(\mathcal{B}^{\bar{v}} \cap \Omega) \cap \mathcal{A}_y = \mathcal{N}^-(n) \backslash \mathcal{B}^y \times \mathcal{B}^{\bar{v}} \cap \mathcal{B}_y$ and $\pi(\mathcal{B}^{\bar{v}} \cap \mathcal{B}_y) = \{y\} \times \mathcal{B}^{\bar{v}} \cap \mathcal{B}_y$. Thus the \mathbb{G}_m -action on the first factor must just be a contracting \mathbb{G}_m -action which fixes the origin.

To define the action on the second factor, consider the action of the torus $T(k)$ on \mathcal{B}_y . Obviously it fixes $\mathcal{B}^{\bar{v}} \cap \mathcal{B}_y$, so we only have to find a cocharacter $\phi : \mathbb{G}_m \rightarrow T$ such that the induced \mathbb{G}_m -action is contracting.

This is possible because the set of finite roots α such that for some n , the affine root (α, n) 'occurs' in \mathcal{B}_y is the union of the sets

$$\{\alpha > 0 \mid \langle \alpha, \lambda \rangle \leq 0, \text{ and } \tilde{y}\alpha < 0 \text{ if } \langle \alpha, \lambda \rangle = 0\}$$

and

$$\{\alpha < 0 \mid \langle \alpha, \lambda \rangle \leq -1, \text{ and } \tilde{y}\alpha < 0 \text{ if } \langle \alpha, \lambda \rangle = -1\},$$

where $y^{-1} = \tilde{y}t_\lambda$, $\tilde{y} \in W$, $\lambda \in X_*(T)$. We may put $\phi = -m\lambda - \tilde{y}^{-1}\rho^\vee$ for any sufficiently large integer m . \square

Theorem 3.7 (Interpretation of inverse Kazhdan-Lusztig polynomials. ([KL2], Prop. 5.7))

Let $v \leq y$. Then $IH^i(\mathcal{B}_y^{\bar{v}}) = 0$ for i odd, and

$$Q_{v,y}(q) = \sum_{i \geq 0} \dim IH^{2i}(\mathcal{B}_y^{\bar{v}}) q^i = \text{Tr}(\text{Fr}_q, IH^\bullet(\mathcal{B}_y^{\bar{v}})).$$

Remark. This result is stated in [KL2] as a proposition which 'should not be difficult to prove'.

Kashiwara and Tanisaki give a proof of the analog of this result in the setting of mixed Hodge modules in [KT], Thm. 6.6.4; in particular they work over the complex numbers.

Proof. Given the lemmas above, the theorem is proved much in the same way as Theorem 4.2 in [KL2] which gives an analogous interpretation for the Kazhdan-Lusztig polynomials in the finite dimensional case.

Choose $\Omega = \bigcup_{v \leq w} v\mathcal{N}^-*$ sufficiently large, and denote the projection $\Omega \rightarrow \mathcal{N}^-(n) \backslash \Omega$ by π as before.

We write $B_y^?$ as an abbreviation for $\pi(\mathcal{B}_y^? \cap \Omega)$. Note that $B_y^v \cong \mathcal{B}_y^v$. Denote by $\mathcal{H}^i(B^{\bar{v}})$ the i -th cohomology sheaf of $IC(B^{\bar{v}})$, and by $\mathcal{H}_z^i(B^{\bar{v}})$ its stalk in a point z .

First of all, Lemma 4.5a) in [KL2] together with Lemma 3.6, the fact that $B^y \times B_y^{\bar{v}}$ is open in $B^{\bar{v}}$, and the fact that B^y is just an affine space, yield that $IH^i(B_y^{\bar{v}}) \cong \mathcal{H}_y^i(B^{\bar{v}})$. In the sequel we will thus always work with $\mathcal{H}_y^i(B^{\bar{v}})$.

For $z \geq v$ consider the property

P(z): For any $z' \in B^z(\mathbb{F}_q)$, we have $\mathcal{H}_{z'}^i(B^{\bar{v}}) = 0$ for odd i , and for even i all eigenvalues of Fr_q on $\mathcal{H}_{z'}^i(B^{\bar{v}})$ are equal to $q^{i/2}$.

Obviously **P(v)** is true. Now fix $y > v$ and assume that **P(z)** holds for all $v \leq z < y$. We want to show that this implies **P(y)**.

By assumption $(B^{\bar{v}} \cap \mathcal{A}_y) - B^y$ is very pure. Since

$$(B^{\bar{v}} \cap \mathcal{A}_y) - B^y \cong B^y \times (B_y^{\bar{v}} - \{y\})$$

and B^y is smooth, we see that $B_y^{\bar{v}} - \{y\}$ is very pure. Thus by Lemma 3.6 and Lemma 4.5b) in [KL2], $B_y^{\bar{v}}$ is very pure. This implies that $B^{\bar{v}} \cap \mathcal{A}_y$ is very pure.

Now the Lefschetz trace formula implies (use Lemma 3.5 and Lemma 3.3)

$$\begin{aligned} & \text{Tr}(\text{Fr}_q, IH_c^\bullet(B^{\bar{v}} \cap \mathcal{A}_y)) \\ &= \sum_{v \leq z \leq y} \sum_{z' \in (B^z \cap \mathcal{A}_y)(\mathbb{F}_q)} \text{Tr}(\text{Fr}_q, \mathcal{H}_{z'}^\bullet(B^{\bar{v}})) \\ &= \sum_{v \leq z \leq y} q^{\dim B^y} R_{z,y}(q) \text{Tr}(\text{Fr}_q, \mathcal{H}_z^\bullet(B^{\bar{v}})). \end{aligned}$$

By Poincaré duality we have

$$\text{Tr}(\text{Fr}_q, IH_c^\bullet(B^{\bar{v}} \cap \mathcal{A}_y)) = q^{\dim(B^{\bar{v}} \cap \mathcal{A}_y)} \text{Tr}(\text{Fr}_q^{-1}, IH^\bullet(B^{\bar{v}} \cap \mathcal{A}_y)),$$

and [KL2], Lemma 4.5 gives us

$$\text{Tr}(\text{Fr}_q^{-1}, IH^\bullet(B^{\bar{v}} \cap \mathcal{A}_y)) = \text{Tr}(\text{Fr}_q^{-1}, \mathcal{H}_y^\bullet(B^{\bar{v}})).$$

Altogether, we now get

$$q^{\ell(y) - \ell(v)} \text{Tr}(\text{Fr}_q^{-1}, \mathcal{H}_y^\bullet(B^{\bar{v}})) = \sum_{v \leq z \leq y} R_{z,y} \text{Tr}(\text{Fr}_q, \mathcal{H}_z^\bullet(B^{\bar{v}})). \quad (1)$$

This equation is completely analogous to (4.6.4) in [KL2], and exactly the same arguments as in loc. cit. now yield that, $\forall i$,

$$\text{Tr}(\text{Fr}_q, \mathcal{H}_y^i(B^{\bar{v}})) \in \mathbb{Z}[q],$$

which implies that **P(y)** holds. This proves the second equality in the statement of the theorem.

Furthermore, since the inverse Kazhdan-Lusztig polynomials are characterized by

$$Q_{v,v} = 1, \quad q^{\ell(y) - \ell(v)} Q_{v,y}(q^{-1}) = \sum_{v \leq z \leq y} R_{z,y} Q_{v,z},$$

$$\deg Q_{v,y} \leq (l(y) - l(v) - 1)/2, \quad (v < y)$$

we also get the first equality in the statement of the theorem from equation (1).

□

4 The main theorem

4.1 Decomposition of \mathcal{B} -equivariant sheaves

We begin with a discussion of the categories of perverse sheaves to which our results apply. We will denote by $P(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}}_\ell)$ the full subcategory of $D_c^b(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}}_\ell)$ consisting of middle perverse sheaves endowed with a compatible action of the Galois group $\Gamma = \text{Gal}(\overline{\eta}/\eta)$. (For the precise meaning of “compatible”, see section 5.1). We let $P^\mathcal{B}(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}}_\ell)$ denote the full subcategory of \mathcal{B} -equivariant objects of $P(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}}_\ell)$: these are perverse sheaves on $\mathcal{F}l_{\overline{s}}$ which are equivariant for Γ and \mathcal{B} , in a compatible way.

Similarly, we let $P_{\text{Weil}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ denote the category of pairs (\mathcal{F}, F_q^*) where \mathcal{F} is a perverse sheaf on $\mathcal{F}l_{\overline{s}}$ and $F_q^* : \text{Fr}_q^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ is an isomorphism of perverse sheaves (a Weil structure on \mathcal{F}). Further, let $P_{\text{Weil}}^\mathcal{B}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ denote the full subcategory of \mathcal{B} -equivariant objects of $P_{\text{Weil}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$: the \mathcal{B} -equivariance of \mathcal{F} is assumed compatible with F_q^* .

Lemma 4.1 *The subcategory $P_{\text{Weil}}^\mathcal{B}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ of $P(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ is stable under formation of kernels and cokernels. Objects in this subcategory have finite length. The irreducible objects are intersection complexes of the form $IC(\mathcal{L})[\ell(w)] = j_{!*} \mathcal{L}[\ell(w)]$, where $j : \mathcal{B}_w \rightarrow \mathcal{F}l$ is the locally closed immersion of an Iwahori-orbit \mathcal{B}_w in $\mathcal{F}l$, and \mathcal{L} is a \mathcal{B} -equivariant (thus geometrically constant) irreducible Weil sheaf on \mathcal{B}_w .*

Proof. The statements concerning kernels and cokernels, and finite length, are clear. We prove that all the irreducible subquotients of $\mathcal{F} \in P_{\text{Weil}}^\mathcal{B}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ taken in this category are of the required form $IC(\mathcal{L})[\ell(w)]$, by Noetherian induction on $\text{supp}(\mathcal{F})$. Indeed, we may find an open immersion $j : \mathcal{B}_w \rightarrow \text{supp}(\mathcal{F})$ such that $j^* \mathcal{F}$ is a (necessarily \mathcal{B} -equivariant) Weil-sheaf on \mathcal{B}_w (up to a shift by $\ell(w)$). Now the kernel and cokernel of the adjunction map $j_{!} j^* \mathcal{F} \rightarrow \mathcal{F}$ belong to $P_{\text{Weil}}^\mathcal{B}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ and are supported on proper closed \mathcal{B} -invariant subsets of $\text{supp}(\mathcal{F})$. Further, the kernel of the canonical surjection $j_{!} j^* \mathcal{F} \rightarrow j_{!*} j^* \mathcal{F}$ also has this property. Since for the action of \mathcal{B} on the orbit \mathcal{B}_w , the stabilizer of any point is geometrically connected, any \mathcal{B} -equivariant Weil sheaf on \mathcal{B}_w is geometrically just a constant sheaf.

The fact that the subquotients of \mathcal{F} have the required form is a consequence of these remarks. (Note that $j_!$ etc. denotes the *derived* functor here. We have used the fact that the morphism j is affine to ensure that $j_!$ preserves perversity and that $j_{!*}$ is a quotient of $j_!$.)

It also follows from this that the irreducible objects of $P_{\text{Weil}}^\mathcal{B}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ are of the stated form. \square

Corollary 4.2 *Objects in $P_{\text{Weil}}^\mathcal{B}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ are mixed.*

Now consider a Weil-perverse sheaf $\mathcal{F} \in P_{\text{Weil}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ such that for all $x \in \mathcal{F}l(\mathbb{F}_q)$, we have $\text{Tr}(\text{Fr}_q, \mathcal{F}_x) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$. We will denote the full subcategory of those sheaves by $P_{\sqrt{q}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$. Similarly, let $P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ denote the full subcategory of sheaves such that $\text{Tr}(\text{Fr}_q, \mathcal{F}_x) \in \mathbb{Z}[q, q^{-1}]$ for all $x \in \mathcal{F}l(\mathbb{F}_q)$. The category $P_{\sqrt{q}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ gives rise via the sheaf-function dictionary precisely to elements in the Iwahori-Hecke algebra (for this reason it is sometimes called the Hecke category).

Lemma 4.3 *The subcategory $P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ in $P_{\text{Weil}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ is stable under extensions, quotients, and subquotients, and under Verdier duality. The irreducible objects of $P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ are of the form $IC_w(i)$, for $w \in \widetilde{W}$, $i \in \mathbb{Z}$.*

Proof. The statements concerning extensions, quotients, and Verdier duality are clear. Let $\mathcal{F} \in P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$. We prove all of its irreducible subquotients in the category $P_{\text{Weil}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ are of the form $IC_w(i)$, by Noetherian induction on $\text{supp}(\mathcal{F})$. Replacing \mathcal{F} by its semi-simplification, we may assume \mathcal{F} is semi-simple. Choose an open immersion $j : \mathcal{B}_w \rightarrow \text{supp}(\mathcal{F})$, such that $j^*\mathcal{F}[-\ell(w)]$ is a (geometrically constant) Weil sheaf \mathcal{L} . By hypothesis, $\text{Tr}(\text{Fr}_q, \mathcal{L}_w) \in \mathbb{Z}[q, q^{-1}]$. The linear independence of characters shows that the eigenvalues of Frobenius on \mathcal{L}_w are of form q^i , for various integers i , in other words, \mathcal{L} is a sum of Weil sheaves on \mathcal{B}_w of the form $\overline{\mathbb{Q}}_\ell(i)$, $i \in \mathbb{Z}$. By the previous Lemma, the intersection complex $IC(\mathcal{L})[\ell(w)]$ is a subquotient of \mathcal{F} in the category $P_{\text{Weil}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$. The quotient $\mathcal{F}/IC(\mathcal{L}[\ell(w)])$ belongs to $P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ (because by Kazhdan-Lusztig [KL2], $IC(\mathcal{L}[\ell(w)]) \in P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$), and has support a proper closed subset of $\text{supp}(\mathcal{F})$. Applying the induction hypothesis to this quotient proves that all the irreducible subquotients of \mathcal{F} have the required form.

The fact that the subcategory $P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ of $P_{\text{Weil}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ is closed under formation of subobjects follows. \square

Corollary 4.4 *Let \mathcal{F} be an object of $P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$. Then the constituents of the Jordan-Hölder series of \mathcal{F} have the form $IC_w(-i)$, $w \in \widetilde{W}$, $i \in \mathbb{Z}$. Each of them appears with a certain multiplicity, which we denote by $m(\mathcal{F}, w, i)$.*

4.2 Relation to the Hecke algebra

Denote by $\mathcal{H} = \bigoplus_{w \in \widetilde{W}} \mathbb{Z}[q^{-1/2}, q^{1/2}]T_w$ the Iwahori-Hecke algebra associated to G .

As usual, we write $q_w = q^{\ell(w)}$, $\varepsilon_w = (-1)^{\ell(w)}$.

A perverse sheaf $\mathcal{F} \in P_{\sqrt{q}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ gives rise to a function $\text{Tr}(\text{Fr}_q, \mathcal{F}) = \sum_w \text{Tr}(\text{Fr}_q, \mathcal{F}_w)T_w \in \mathcal{H}$, and we have a sheaf-function dictionary à la Grothendieck.

On the other hand, Kazhdan and Lusztig have defined a different basis C_w of \mathcal{H} . We slightly change the normalization, and work with elements C_w'' instead.

They are related to the C_w resp. to the elements C'_w sometimes appearing in the literature as follows. By definition, C''_w is the function $\text{Tr}(\text{Fr}_q, IC(\mathcal{B}_w)[\ell(w)])$. Also by definition, $C'_w = \text{Tr}(\text{Fr}_q, IC(\mathcal{B}_w)(\ell(w)/2))$, so we have $C''_w = \varepsilon_w q_w^{1/2} C'_w$. But we also have $\sigma(C_w) = \varepsilon_w C'_w$, where σ is the involution of the Hecke algebra given by $q \mapsto q^{-1}$, $T_w \mapsto \varepsilon_w q_w^{-1} T_w$. Thus $C''_w = q_w^{1/2} \sigma(C_w)$. The base change matrix giving the relation between the T_w and the C''_w is given by Kazhdan-Lusztig polynomials resp. inverse Kazhdan-Lusztig polynomials:

$$\begin{aligned} T_w &= \sum_x \varepsilon_w Q_{x,w} C''_x, \\ C''_w &= \sum_x \varepsilon_w P_{x,w} T_x. \end{aligned}$$

Now if $\mathcal{F} \in P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$, the multiplicities $m(\mathcal{F}, w, i)$ defined above are just the coefficients of $\text{Tr}(\text{Fr}_q, \mathcal{F})$ with respect to the basis C''_w :

$$\text{Tr}(\text{Fr}_q, \mathcal{F}) = \sum_w \left(\sum_i m(\mathcal{F}, w, i) q^i \right) C''_w.$$

Thus, knowing the function for \mathcal{F} explicitly is equivalent to knowing all the multiplicities $m(\mathcal{F}, w, i)$ for \mathcal{F} .

For later use note that the identity $\overline{C''_w} = q_w^{-1} C''_w$, together with the fact that Verdier duality descends to the Kazhdan-Lusztig involution on the Iwahori-Hecke algebra, gives the following identity.

Lemma 4.5 *We write $m(\mathcal{F}, w) = \sum_i m(\mathcal{F}, w, i) q^i$ for $\mathcal{F} \in P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$. Then*

$$m(D\mathcal{F}, w) = q_w^{-1} \overline{m(\mathcal{F}, w)}.$$

4.3 The property (P)

We say that a sheaf $\mathcal{F} \in P_{\sqrt{q}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ satisfies the property (P) (for ‘‘palindromic’’), if for all $y \in \mathcal{F}l(\mathbb{F}_q)$ we have

$$\text{Tr}(\text{Fr}_q, D\mathcal{F}_y) = \varepsilon_d \varepsilon_y q^{-d} q_y^{-1} \overline{\text{Tr}(\text{Fr}_q, D\mathcal{F}_y)},$$

for some $d \in \mathbb{Z}$ (setting $\varepsilon_d = (-1)^d$). Here $D\mathcal{F}$ denotes the Verdier dual of \mathcal{F} , and $h \mapsto \bar{h}$ is the involution of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ determined by $q^{1/2} \mapsto q^{-1/2}$.

In other words, we have

$$\text{Tr}(\text{Fr}_q, D\mathcal{F}) = \sum_y \varepsilon_d \varepsilon_y q^{-d} q_y^{-1} \overline{\text{Tr}(\text{Fr}_q, D\mathcal{F}_y)} T_y$$

in the Iwahori-Hecke algebra. By applying Verdier duality (i.e. by applying the Kazhdan-Lusztig involution $\bar{\cdot}$) and using the definition $\overline{T}_y = T_{y^{-1}}^{-1}$, we get

$$\text{Tr}(\text{Fr}, \mathcal{F}) = \sum_x \left(\sum_{y \geq x} \varepsilon_d q^{d_r} \text{Tr}(\text{Fr}_q, D\mathcal{F}_y) \varepsilon_x R_{x,y}(q) \right) T_x.$$

Therefore we see that property (P) implies

$$\mathrm{Tr}(\mathrm{Fr}_q, \mathcal{F}_x) = \varepsilon_d q^d \sum_{y \geq x} \varepsilon_x \mathrm{Tr}(\mathrm{Fr}_q, D\mathcal{F}_y) R_{x,y}(q).$$

Remark. We will see in section 6 that $R\Psi$ satisfies (P). More generally, the notion of Wakimoto sheaf allows us to define a much larger class of sheaves that satisfy (P); cf. section 7.

Arguing as above, it is easy to see from the definitions that property (P) is stable under Verdier duality.

Lemma 4.6 *A sheaf $\mathcal{F} \in P_{\sqrt{q}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ satisfies (P) for an integer d if and only if $D\mathcal{F}$ satisfies (P) for $-d$.*

Proof. Use the identities

$$\begin{aligned} \mathrm{Tr}(\mathrm{Fr}_q, \mathcal{F}_x) &= \sum_{x \leq y} \varepsilon_x \varepsilon_y q_y^{-1} R_{x,y} \overline{\mathrm{Tr}(\mathrm{Fr}_q, D\mathcal{F}_y)} \\ \overline{\mathrm{Tr}(\mathrm{Fr}_q, \mathcal{F}_x)} &= \sum_{x \leq y} q_x R_{x,y} \mathrm{Tr}(\mathrm{Fr}_q, D\mathcal{F}_y). \end{aligned}$$

□

4.4 Cohomological interpretation of the multiplicities

In this section we will give a cohomological interpretation of the multiplicities associated to a sheaf \mathcal{F} which satisfies the property (P). This can be seen as a generalization of the theorem of Kazhdan and Lusztig which gives an interpretation of the inverse Kazhdan-Lusztig polynomials (Theorem 3.7). Their theorem is the key ingredient of the proof.

Theorem 4.7 *Suppose that $\mathcal{F} \in P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ satisfies (P). Then*

$$\begin{aligned} \sum_i m(\mathcal{F}, w, i) q^i &= \varepsilon_d q^d \mathrm{Tr}(\mathrm{Fr}_q, H_c^\bullet(\mathcal{F}l, D\mathcal{F} \otimes IC(\mathcal{B}^{\overline{w}}))) \\ &= \varepsilon_d q^{d-\ell(w)} \mathrm{Tr}(\mathrm{Fr}_q, H_c^\bullet(\mathcal{F}l, \mathcal{F} \otimes IC(\mathcal{B}^{\overline{w}}))^\vee). \end{aligned}$$

Here, $V^\vee = \mathrm{Hom}(V, \overline{\mathbb{Q}}_\ell)$ for a $\overline{\mathbb{Q}}_\ell$ -space V .

Proof. Using the definition of the multiplicities and the fact that \mathcal{F} satisfies (P), we have

$$\begin{aligned} \sum_i m(\mathcal{F}, w, i) q^i &= \sum_{w \leq x} \varepsilon_x \mathrm{Tr}(\mathrm{Fr}_q, \mathcal{F}_x) Q_{w,x} \\ &= \sum_{w \leq x} \sum_{x \leq y} \varepsilon_d q^d \mathrm{Tr}(\mathrm{Fr}_q, (D\mathcal{F})_y) Q_{w,x} R_{x,y}. \end{aligned}$$

Rearranging this and using the formula $\sum_{w \leq x \leq y} Q_{w,x} R_{x,y} = q_y q_w^{-1} Q_{w,y}(q^{-1})$, we get

$$\begin{aligned} \sum_i m(\mathcal{F}, w, i) q^i &= \varepsilon_d q^d \sum_{w \leq y} \text{Tr}(\text{Fr}_q, (D\mathcal{F})_y) \sum_{w \leq x \leq y} Q_{w,x} R_{x,y} \\ &= \varepsilon_d q^d \sum_{w \leq y} (q_y q_w^{-1} Q_{w,y}(q^{-1})) \text{Tr}(\text{Fr}_q, (D\mathcal{F})_y). \end{aligned}$$

By using the Poincaré dual of Theorem 3.7, we see that

$$q_y q_w^{-1} Q_{w,y}(q^{-1}) = \text{Tr}(\text{Fr}_q, IH_c^\bullet(\mathcal{B}_y \cap \mathcal{B}^{\bar{w}})).$$

Thus we get

$$\begin{aligned} &\sum_i m(\mathcal{F}, w, i) q^i \\ &= \varepsilon_d q^d \sum_{w \leq y} \text{Tr}(\text{Fr}_q, IH_c^\bullet(\mathcal{B}_y \cap \mathcal{B}^{\bar{w}})) \text{Tr}(\text{Fr}_q, (D\mathcal{F})_y) \\ &= \varepsilon_d q^d \sum_{w \leq y} \text{Tr}(\text{Fr}_q, H_c^\bullet(\mathcal{B}_y \cap \mathcal{B}^{\bar{w}}, IC(\mathcal{B}_y \cap \mathcal{B}^{\bar{w}})) \otimes D\mathcal{F}|_{\mathcal{B}_y \cap \mathcal{B}^{\bar{w}}}) \\ &= \varepsilon_d q^d \text{Tr}(\text{Fr}_q, H_c^\bullet(\mathcal{F}l, IC(\mathcal{B}^{\bar{w}})) \otimes D\mathcal{F}), \end{aligned}$$

taking Proposition 3.4 into account. This proves the first equality of the theorem.

The second equality can be proved by applying the first equality to the sheaf $D\mathcal{F}$ in place of \mathcal{F} (cf. Lemma 4.6), and by using Lemma 4.5. \square

Remark. It is clear from the proof of the theorem that we could replace the condition that \mathcal{F} satisfies (P) by the following: Let $\tilde{\mathcal{F}}$ be a \mathcal{B} -equivariant sheaf such that for all y

$$\text{Tr}(\text{Fr}_q, D\mathcal{F}_y) = \varepsilon_y q_y^{-1} \overline{\text{Tr}(\text{Fr}_q, \tilde{\mathcal{F}}_y)}.$$

Then

$$\sum_i m(\mathcal{F}, w, i) q^i = \text{Tr}(\text{Fr}_q, H_c^\bullet(\mathcal{F}l, \tilde{\mathcal{F}} \otimes IC(\mathcal{B}^{\bar{w}}))).$$

4.5 The Kazhdan-Lusztig theorem as a consequence of the main theorem

Fix $y \in \widetilde{W}$, and consider the sheaf $\mathcal{F} = j_{y!} \overline{\mathbb{Q}}_\ell[\ell(y)]$. This belongs to $P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$, and satisfies property (P) with $d = \ell(y)$. We may therefore apply the second equality of Theorem 4.7 to this sheaf. Using the projection formula for $j_{y!}$, we have

$$\begin{aligned} \varepsilon_y q_y q_w^{-1} \text{Tr}(\text{Fr}_q, H_c^\bullet(\mathcal{F}l, \mathcal{F} \otimes IC(\mathcal{B}^{\bar{w}}))^\vee) &= q_y q_w^{-1} \text{Tr}(\text{Fr}_q, H_c^\bullet(\mathcal{F}l, j_{y!}(j_y^* IC(\mathcal{B}^{\bar{w}})))^\vee) \\ &= q_y q_w^{-1} \text{Tr}(\text{Fr}_q, H_c^\bullet(\mathcal{B}_y^{\bar{w}}, IC(\mathcal{B}_y^{\bar{w}}))^\vee) \\ &= q_y q_w^{-1} \text{Tr}(\text{Fr}_q^{-1}, H_c^\bullet(\mathcal{B}_y^{\bar{w}}, IC(\mathcal{B}_y^{\bar{w}}))) \\ &= \text{Tr}(\text{Fr}_q, IH_c^\bullet(\mathcal{B}_y^{\bar{w}})), \end{aligned}$$

where the last equality follows from Poincaré duality. Since $\sum m(\mathcal{F}, w, i)q^i = Q_{w,y}(q)$, this is just one version of the theorem of Kazhdan and Lusztig (Theorem 3.7). (Taking into account the very-purity of $\mathcal{B}_y^{\bar{w}}$, one can recover the full statement of that theorem.)

4.6 Consequences

4.6.1 $w \in (\widetilde{W}/W)_{min}$

Denote by $\pi : \mathcal{F}l \rightarrow \text{Grass}$ the projection. The map π is smooth and proper as a morphism of ind-schemes. We denote the projection $\widetilde{W} \rightarrow \widetilde{W}/W$ by π , too.

If $w \in (\widetilde{W}/W)_{min}$, we have $\mathcal{B}^{\bar{w}} = \pi^{-1}(\overline{\mathcal{Q}^{\pi(w)}})$, and so $IC(\mathcal{B}^{\bar{w}}) = \pi^*IC(\overline{\mathcal{Q}^{\pi(w)}})$. Here $\overline{\mathcal{Q}^{\pi(w)}}$ is the closure of the \mathcal{N}^- -orbit of $\pi(w)$ in the affine Grassmannian for G .

Thus in this case we can simplify the formula in the theorem by considering the push-forward to Grass and applying the projection formula. We get

$$\begin{aligned} \sum_i m(\mathcal{F}, w, i)q^i &= \varepsilon_d q^d \text{Tr}(\text{Fr}_q, H_c^\bullet(\mathcal{F}l, D\mathcal{F} \otimes IC(\mathcal{B}^{\bar{w}}))) \\ &= \varepsilon_d q^d \text{Tr}(\text{Fr}_q, H_c^\bullet(\text{Grass}, (\pi_* D\mathcal{F}) \otimes IC(\overline{\mathcal{Q}^{\pi(w)}}))). \end{aligned}$$

We remark that this formula simplifies further in the case $\mathcal{F} = R\Psi$; see section 6.3.

5 The inertia action on $R\Psi$

Elaborating an argument due to Gaitsgory [Ga], in this section we prove that $R\Psi$ carries a purely unipotent action of the inertia group. This has two important consequences. First, it implies that the Jordan-Hölder series for $R\Psi$ in the category $P^{\mathcal{B}}(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}_\ell})$ is determined by that of its image in the category $P_{\text{Weil}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}_\ell})$ (and that the choice of lift of Frobenius is immaterial for the latter). Secondly, it shows that the semi-simple trace of Frobenius and the usual trace of Frobenius agree on these nearby cycles, so that the Kottwitz conjecture can effectively be used to derive information about the Jordan-Hölder series in this latter category. (See sections 6,7 and 9.)

5.1 Unipotent/non-unipotent decompositions

Let (S, η, s) be a Henselian trait, and let $(\bar{S}, \bar{\eta}, \bar{s})$ be the trait constructed by letting $\bar{\eta}$ be a separable closure of η , \bar{S} the normalization of S in $\bar{\eta}$, and \bar{s} the corresponding closed point. We assume the residue field $k(s) = \mathbb{F}_q$, q a power of a prime p . Write $\Gamma = \text{Gal}(\bar{\eta}/\eta)$, $\Gamma^s = \text{Gal}(\bar{s}/s)$, and let Γ_0 denote the inertia

subgroup, i.e., the kernel of the canonical homomorphism $\Gamma \rightarrow \Gamma^s$. For a fixed prime $l \neq p$, there is a canonical surjective homomorphism

$$t_l : \Gamma_0 \rightarrow \mathbb{Z}_l(1).$$

Let $\mathbb{Q}_l\text{-Mod}$ denote the category of finite dimensional \mathbb{Q}_l -vector spaces. We consider the abelian category $\text{Rep}(\Gamma, \mathbb{Q}_l)$ of continuous finite-dimensional l -adic representations (ρ, V)

$$\rho : \Gamma \rightarrow \text{GL}(V).$$

By a theorem of Grothendieck (cf. [ST]), the restricted representation $\rho(\Gamma_0)$ is quasi-unipotent i.e., there exists a finite-index subgroup Γ_1 of Γ_0 which acts unipotently on V . There exists then a unique nilpotent morphism, the *logarithm of the unipotent part of ρ*

$$N : V(1) \rightarrow V$$

characterized by the following property: for all $g \in \Gamma_1$, we have

$$\rho(g) = \exp(Nt_l(g)).$$

We make a choice of geometric Frobenius Φ lifting the automorphism $a \mapsto a^{1/q}$ with respect to the homomorphism $\text{Gal}(\bar{\eta}/\eta) \rightarrow \text{Gal}(\bar{s}/s)$. Then ρ uniquely determines a representation $\tilde{\rho} : \langle \Phi \rangle \times \Gamma_0 \rightarrow \text{GL}(V)$, trivial on some finite-index subgroup Γ_1 of Γ_0 , by the formula

$$\rho(\Phi^n \sigma) = \tilde{\rho}(\Phi^n \sigma) \exp(Nt_l(\sigma))$$

for $n \in \mathbb{Z}$ and $\sigma \in \Gamma_0$ (cf. [Del]); moreover we have the relation $\tilde{\rho}(\Phi^n \sigma) N \tilde{\rho}(\Phi^n \sigma)^{-1} = q^{-n} N$.

We can define an idempotent endomorphism $\gamma = \gamma_V \in \text{End}_{\mathbb{Q}_\ell}(V)$ by

$$\gamma = |\Gamma_0/\Gamma_1|^{-1} \sum_{g \in \Gamma_0/\Gamma_1} \tilde{\rho}(g).$$

Clearly this is independent of the choice of coset representatives for Γ_0/Γ_1 as well as the choice of finite-index subgroup $\Gamma_1 \subset \Gamma_0$ on which $\tilde{\rho}$ is trivial. Using this it is easy to see that γ commutes with $\tilde{\rho}(\langle \Phi \rangle \times \Gamma_0)$ and N , hence also with $\rho(\Gamma)$. We obviously have

$$\gamma^2 = \gamma.$$

We note that if $f : V \rightarrow W$ is a morphism in $\text{Rep}(\Gamma, \mathbb{Q}_l)$, then $f \circ \gamma_V = \gamma_W \circ f$.

We define the *unipotent* and *non-unipotent* subrepresentations of V as follows: V^{un} (resp. V^{non-un}) is the largest Γ -subrepresentation such that, as a representation of Γ_0 , all of its irreducible subquotients are trivial (resp. non-trivial). It is easy to see that these exist, and that $V^{un} \cap V^{non-un} = (0)$.

We denote by $\text{Rep}(\Gamma, \mathbb{Q}_l)^{un}$ (resp. $\text{Rep}(\Gamma, \mathbb{Q}_l)^{non-un}$) the maximal full subcategory of $\text{Rep}(\Gamma, \mathbb{Q}_l)$ whose objects are all unipotent (resp. non-unipotent).

Lemma 5.1 (1) *For every l -adic representation (ρ, V) of Γ , there is a canonical decomposition*

$$V = V^{un} \oplus V^{non-un}.$$

Moreover, any morphism $f : V \rightarrow W$ in $\text{Rep}(\Gamma, \mathbb{Q}_l)$ decomposes canonically as $f = f^{un} \oplus f^{non-un} : V^{un} \oplus V^{non-un} \rightarrow W^{un} \oplus W^{non-un}$. This gives rise to a decomposition of abelian categories

$$\text{Rep}(\Gamma, \mathbb{Q}_l) = \text{Rep}(\Gamma, \mathbb{Q}_l)^{un} \oplus \text{Rep}(\Gamma, \mathbb{Q}_l)^{non-un}.$$

- (2) *The endomorphism γ is $\rho(\Gamma)$ -equivariant. Furthermore, we have $V^{un} = \ker(1 - \gamma) = \text{im}(\gamma)$ and $V^{non-un} = \ker(\gamma) = \text{im}(1 - \gamma)$.*
- (3) *V is unipotent (resp. non-unipotent) if and only if $1 - \gamma$ (resp. γ) acts by zero on V .*

Proof. (1). We only need to show that V has some decomposition $V = V_1 \oplus V_2$, where V_1 is unipotent and V_2 is non-unipotent.

Since the restriction of $\tilde{\rho}$ to Γ_0 is semi-simple, we may consider the decomposition of V according to its isotypical components:

$$V = \bigoplus_{\tau} V_{\tau}.$$

For any fixed σ , the automorphism $\exp(Nt_l(\sigma))$ commutes with every $\tilde{\rho}(\sigma')$, $\sigma' \in \Gamma_0$, hence preserves each component V_{τ} . Thus $\rho(\sigma)$ preserves each component V_{τ} , for $\sigma \in \Gamma_0$. On the other hand, it is easily seen that $\rho(\Phi)$ preserves the isotypical component $V_{\mathbb{I}}$ corresponding to the trivial representation, while permuting the components V_{τ} with $\tau \neq \mathbb{I}$. Since the Weil group $\langle \Phi \rangle \rtimes \Gamma_0$ is dense in Γ , it follows that $V_1 = V_{\mathbb{I}}$ and $V_2 = \bigoplus_{\tau \neq \mathbb{I}} V_{\tau}$ have the required properties.

- (2). We have already noted that γ is $\rho(\Gamma)$ -equivariant. The exact sequence

$$0 \longrightarrow V^{\tilde{\rho}(\Gamma_0)} \xrightarrow{i} V \xrightarrow{p} V/V^{\tilde{\rho}(\Gamma_0)} \longrightarrow 0$$

splits in the category of $\rho(\Gamma)$ -modules; in fact we have $\gamma \circ i = \text{id}$ and $p \circ (1 - \gamma) = \text{id}$. This shows that

$$\begin{aligned} V^{un} &= V^{\tilde{\rho}(\Gamma_0)} = \text{im}(\gamma) \\ V^{non-un} &\cong V/V^{\tilde{\rho}(\Gamma_0)} \cong \ker(\gamma). \end{aligned}$$

- (3). This follows from (2). □

Let π_1 denote a pro-finite group (e.g. the algebraic fundamental group of a locally Noetherian scheme at a geometric point). Let $\text{Rep}(\Gamma, \pi_1, \mathbb{Q}_l)$ denote the category whose objects V are \mathbb{Q}_l -modules equipped with continuous commuting actions of π_1 and Γ and whose morphisms are compatible with the actions. Then Lemma 5.1 implies that this category decomposes into unipotent and non-unipotent subcategories.

Lemma 5.2 *There is a canonical decomposition of abelian categories*

$$\mathrm{Rep}(\Gamma, \pi_1, \mathbb{Q}_l) = \mathrm{Rep}(\Gamma, \pi_1, \mathbb{Q}_l)^{un} \oplus \mathrm{Rep}(\Gamma, \pi_1, \mathbb{Q}_l)^{non-un}.$$

Similarly, there is a decomposition

$$\mathrm{Rep}(\Gamma_0, \pi_1, \mathbb{Q}_l) = \mathrm{Rep}(\Gamma_0, \pi_1, \mathbb{Q}_l)^{un} \oplus \mathrm{Rep}(\Gamma_0, \pi_1, \mathbb{Q}_l)^{non-un},$$

and the restriction functor $\mathrm{Rep}(\Gamma, \pi_1, \mathbb{Q}_l) \rightarrow \mathrm{Rep}(\Gamma_0, \pi_1, \mathbb{Q}_l)$ is compatible with the decompositions. \square

Let X denote a locally Noetherian scheme. Let $LS(X, \mathbb{Q}_l)$ denote the category of locally constant and constructible \mathbb{Q}_l -sheaves on X , i.e., the *l -adic local systems on X* (see [SGA 5], Ex. VI §1.4). An object in this category is a surjective projective system $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{N}}$ where \mathcal{F}_n is a finite and locally constant $\mathbb{Z}/l^n\mathbb{Z}$ -sheaf on $X_{\acute{e}t}$. If X is connected, then by loc.cit. §1.4.2, there is an equivalence of categories

$$LS(X, \mathbb{Q}_l) \cong \mathrm{Rep}(\pi_1(X, a), \mathbb{Q}_l),$$

given by the functor $F_a : (\mathcal{F}_n)_n \mapsto \varprojlim (\mathcal{F}_n)_a \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, where \mathcal{G}_a is the fiber of a sheaf \mathcal{G} at a geometric point a of X . (The choice of a is essentially irrelevant, since X is connected.)

Now suppose X is a scheme of finite-type over $s = \mathrm{Spec}(\mathbb{F}_q)$, and denote the geometric special fiber by $X_{\bar{s}} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. If $X_{\bar{s}}$ is connected, we denote its fundamental group by $\bar{\pi}_1 = \pi_1(X_{\bar{s}})$.

We will define several categories attached to X/s .

- $LS(X \times_s \eta, \mathbb{Q}_l)$: objects are objects in $LS(X_{\bar{s}}, \mathbb{Q}_l)$ equipped with a continuous action of Γ compatible with the action of Γ on $X_{\bar{s}}$ (the quotient Γ^s of Γ acts on $X_{\bar{s}}$ by transport of structure). In other words, $(\mathcal{F}_n)_n$ is an object of this category if for each $g \in \Gamma$ there is an isomorphism of projective systems

$$\phi_g : \bar{g}_* \mathcal{F}_n \cong \mathcal{F}_n$$

such that $\phi_{gh} = \phi_g \circ \bar{g}_*(\phi_h)$. Here \bar{g} denotes the image of g under the projection $\Gamma \rightarrow \Gamma^s$. (Note: The ‘‘continuity’’ condition is that for each quasi-compact étale morphism $U \rightarrow X_s$ and corresponding base change $\bar{U} \rightarrow X_{\bar{s}}$, the action of Γ on the (finite) set $\mathcal{F}_n(\bar{U})$ is continuous for each n (cf. [SGA 7], Ex. XIII).)

Let $LS(X \times_s \eta^{nr}, \mathbb{Q}_l)$ denote the category consisting of local systems with an action of Γ_0 rather than a compatible action of the group Γ . If $X_{\bar{s}}$ is connected, then taking stalks $\mathcal{G} \mapsto \mathcal{G}_a$ gives an equivalence of categories

$$LS(X \times_s \eta^{nr}, \mathbb{Q}_l) \cong \mathrm{Rep}(\Gamma_0, \bar{\pi}_1, \mathbb{Q}_l).$$

(Remark: This means that to define the notions of unipotent and non-unipotent for objects on the left hand side, we should consider their stalks, where such notions are already defined; cf. Lemma 5.2.)

- $Sh_c(X \times_s \eta, \mathbb{Q}_l)$: Let $Sh_c(X_{\bar{s}}, \mathbb{Q}_l)$ denote the abelian category consisting of \mathbb{Q}_l -sheaves $(\mathcal{F}_n)_n$ on $X_{\bar{s}}$ where each \mathcal{F}_n is finite and constructible.

We define $Sh_c(X \times_s \eta, \mathbb{Q}_l)$: the objects are objects of $Sh_c(X_{\bar{s}}, \mathbb{Q}_l)$ equipped with a continuous action of Γ , compatible with the action of Γ on $X_{\bar{s}}$, in the sense defined above. The morphisms are compatible with the action of Γ . Continuity has the same meaning as in $LS(X \times_s \eta, \mathbb{Q}_l)$ above.

The category $Sh_c(X \times_s \eta^{nr}, \mathbb{Q}_l)$ is defined analogously to $LS(X \times_s \eta^{nr}, \mathbb{Q}_l)$.

- $P(X \times_s \eta, \mathbb{Q}_l)$: the objects are middle perverse sheaves on $X_{\bar{s}}$ equipped with a continuous action of Γ compatible with its action on $X_{\bar{s}}$. (Here continuity is tested on cohomology sheaves.) Morphisms are compatible with the action of Γ . This category is naturally a full subcategory of the category $D_c^b(X \times_s \eta, \mathbb{Q}_l)$ defined below.

The category $P(X \times_s \eta^{nr}, \mathbb{Q}_l)$ is defined analogously to $LS(X \times_s \eta^{nr}, \mathbb{Q}_l)$.

- $D_c^b(X \times_s \eta, \mathbb{Q}_l)$: the objects are objects of the “derived” category $D_c^b(X_{\bar{s}}, \mathbb{Q}_l)$ equipped with a continuous action of Γ compatible with its action on $X_{\bar{s}}$. (Continuity is tested on the cohomology sheaves.) The morphisms are compatible with the action of Γ .

The category $D_c^b(X \times_s \eta^{nr}, \mathbb{Q}_l)$ is analogous to $LS(X \times_s \eta^{nr}, \mathbb{Q}_l)$.

The following lemma extends certain properties of $\text{Rep}(\Gamma_0, \mathbb{Q}_l)$ to each of the analogous categories above.

Lemma 5.3 *Let M be an object of one of the categories $LS(X \times_s \eta^{nr}, \mathbb{Q}_l)$, $Sh_c(X \times_s \eta^{nr}, \mathbb{Q}_l)$, or $D_c^b(X \times_s \eta^{nr}, \mathbb{Q}_l)$. Then*

- (1) *The action of $\rho(\Gamma_0)$ on M is quasi-unipotent: there exists a finite-index subgroup $\Gamma_1 \subset \Gamma_0$ such that $(\rho(g) - 1)$ acts nilpotently on M , for every $g \in \Gamma_1$.*
- (2) *There is a unique nilpotent morphism $N : M(1) \rightarrow M$ characterized by the equality*

$$\rho(g) = \exp(Nt_l(g))$$

for every $g \in \Gamma_1$.

- (3) *There is a unique homomorphism $\tilde{\rho} : \Gamma_0 \rightarrow \text{Aut}(M)$ which satisfies*
 - (a) $\rho(\sigma) = \tilde{\rho}(\sigma)\exp(Nt_l(\sigma))$, for $\sigma \in \Gamma_0$,
 - (b) $\tilde{\rho}$ is trivial on a finite-index subgroup of Γ_0 ,
 - (c) $\tilde{\rho}(\sigma)N\tilde{\rho}(\sigma)^{-1} = N$, for $\sigma \in \Gamma_0$.

This is closely related to the next result, whose proof works along the same lines. To state it we need a suitable notion of “unipotent” and “non-unipotent” object for the categories above.

Definition 5.4 (1) Let M be an object of $LS(X \times_s \eta, \mathbb{Q}_l)$ or $Sh_c(X \times_s \eta, \mathbb{Q}_l)$.

We say M is unipotent (resp. non-unipotent) if for every geometric point $\bar{x} \in X_s(\bar{\mathbb{F}}_q)$, the stalk $M_{\bar{x}}$ is a unipotent (resp. non-unipotent) object of $\text{Rep}(\Gamma_0, \mathbb{Q}_l)$.

(2) Let M be an object of $P(X \times_s \eta, \mathbb{Q}_l)$ or $D_c^b(X \times_s \eta, \mathbb{Q}_l)$. We say M is unipotent (resp. non-unipotent) if every cohomology sheaf $\mathcal{H}^i M$ is a unipotent (resp. non-unipotent) object of $Sh_c(X \times_s \eta, \mathbb{Q}_l)$.

Remark: In the same way we can define the notions of unipotent and non-unipotent for objects of $LS(X \times_s \eta^{nr}, \mathbb{Q}_l)$, $Sh_c(X \times_s \eta^{nr}, \mathbb{Q}_l)$, etc. It is clear that $M \in LS(X \times_s \eta, \mathbb{Q}_l)$ etc. is unipotent (resp. non-unipotent) if and only if it has unipotent (resp. non-unipotent) image under the forgetful functor $LS(X \times_s \eta, \mathbb{Q}_l) \rightarrow LS(X \times_s \eta^{nr}, \mathbb{Q}_l)$ etc.

Lemma 5.5 (1) Let \mathcal{C} be one of the categories $LS(X \times_s \eta^{nr}, \mathbb{Q}_l)$, $Sh_c(X \times_s \eta^{nr}, \mathbb{Q}_l)$, or $D_c^b(X \times_s \eta^{nr}, \mathbb{Q}_l)$. Let M be an object of \mathcal{C} . Define the idempotent $\gamma = \gamma_M \in \text{End}_{\mathcal{C}}(M)$ by

$$\gamma = |\Gamma_0/\Gamma_1|^{-1} \sum_{g \in \Gamma_0/\Gamma_1} \tilde{\rho}(g)$$

where $\tilde{\rho}$ is as above and $\Gamma_1 \subset \Gamma_0$ is any finite-index subgroup of Γ_0 on which $\tilde{\rho}$ is trivial. Then M is unipotent (resp. non-unipotent) if and only if $1 - \gamma_M$ (resp. γ_M) acts by zero on M .

(2) If M is an object of one of the abelian categories $LS(X \times_s \eta^{nr}, \mathbb{Q}_l)$, $Sh_c(X \times_s \eta^{nr}, \mathbb{Q}_l)$, or $P(X \times_s \eta^{nr}, \mathbb{Q}_l)$ then the canonical decomposition

$$M = \ker(1 - \gamma_M) \oplus \ker(\gamma_M)$$

is a canonical decomposition of M into unipotent and non-unipotent direct factors. The same holds for the categories $LS(X \times_s \eta, \mathbb{Q}_l)$, $Sh_c(X \times_s \eta, \mathbb{Q}_l)$, and $P(X \times_s \eta, \mathbb{Q}_l)$.

Proof of Lemmas 5.3 and 5.5.

To begin, we assume $M \in D_c^b(X \times_s \eta^{nr}, \mathbb{Q}_l)$, the most general category in Lemma 5.3.

Once it is known that $\rho(\Gamma_0)$ acts quasi-unipotently on M , the existence and uniqueness of N follow in general from the same argument used in the category $\text{Rep}(\Gamma_0, \mathbb{Q}_l)$. Indeed, let $P_l = \ker(t_l)$ and let $\Gamma_1 \subset \Gamma_0$ denote a finite-index subgroup such that $\rho(\Gamma_1)$ acts unipotently on M . One shows that $\rho(\Gamma_1 \cap P_l) = \text{id}_M$ (this can be tested on cohomology stalks, effectively reducing to the case $M \in \text{Rep}(\Gamma_0, \mathbb{Q}_l)$ where it is already known), and thence that $\rho|_{\Gamma_1}$ factors through t_l ; then define $N : M(1) \rightarrow M$ by $Nt_l(\sigma) = \log(\rho(\sigma))$ for all $\sigma \in \Gamma_1$.

Moreover, given N we then define $\tilde{\rho}$ by the equality

$$\tilde{\rho}(\sigma) = \rho(\sigma) \exp(-Nt_l(\sigma))$$

for $\sigma \in \Gamma_0$. The identities proving that $\tilde{\rho}$ is a homomorphism and the equality in Lemma 5.3 (3(c)) reduce to the case $M \in \text{Rep}(\Gamma_0, \mathbb{Q}_l)$. Indeed, passing to cohomology sheaves, we may assume $M \in \text{Sh}_c(X \times_s \eta^{nr}, \mathbb{Q}_l)$. Then the constructibility of M allows us to view M as the result of “gluing” a local system on an open subscheme with a constructible sheaf supported on a proper closed subscheme (for the precise statement of gluing, see for example [Mi], II §3). By Noetherian induction therefore, we may assume $M \in \text{LS}(X \times_s \eta^{nr}, \mathbb{Q}_l)$. Passing to the finitely many connected components of X , we are reduced to $M \in \text{Rep}(\Gamma_0, \bar{\pi}_1, \mathbb{Q}_l)$, where the desired identities are already known.

To prove that $\tilde{\rho}$ is trivial on a finite-index subgroup $\Gamma_1 \subset \Gamma_0$, one uses the same reduction steps, keeping in mind that for $M \in \text{Sh}_c(X \times_s \eta^{nr}, \mathbb{Q}_l)$, there is a finite stratification X_i ($i \in I$) of X by locally closed subschemes X_i such that $M|_{X_i} \in \text{LS}(X_i \times_s \eta^{nr}, \mathbb{Q}_l)$, and each X_i has only finitely many connected components.

To show that $\rho(\Gamma_0)$ acts quasi-unipotently on M is similar. This completes the proof of Lemma 5.3.

Now again suppose M is an object of $D_c^b(X \times_s \eta^{nr}, \mathbb{Q}_l)$. We wish to prove M is unipotent (resp. non-unipotent) if and only if $1 - \gamma_M$ (resp. γ_M) acts by zero on M . Passing to cohomology sheaves and using the obvious identity

$$\gamma_{\mathcal{H}^i M} = \mathcal{H}^i(\gamma_M)$$

we can assume $M \in \text{Sh}_c(X \times_s \eta^{nr}, \mathbb{Q}_l)$. Taking stalks and using the identity

$$\gamma_{i_{\bar{x}}^* M} = i_{\bar{x}}^*(\gamma_M),$$

where $i_{\bar{x}} : \bar{x} \rightarrow X_s$ is the embedding of a geometric point in $X_s(\bar{\mathbb{F}}_q)$, we can assume that $M \in \text{Rep}(\Gamma_0, \mathbb{Q}_l)$, where the result follows from Lemma 5.1. This proves Lemma 5.5 part (1).

Concerning part (2), if M belongs to one of the abelian categories $\text{LS}(X \times_s \eta^{nr}, \mathbb{Q}_l)$, $\text{Sh}_c(X \times_s \eta^{nr}, \mathbb{Q}_l)$, or $P(X \times_s \eta^{nr}, \mathbb{Q}_l)$, then the idempotent endomorphism γ_M gives rise to the required decomposition. This proves part (2), and thus Lemma 5.5. \square

Remark: If \mathcal{C} is one of the categories in Lemma 5.5 (2), there is a decomposition of abelian categories

$$\mathcal{C} = \mathcal{C}^{un} \oplus \mathcal{C}^{non-un},$$

and we denote the corresponding components of an object M by M^{un} and M^{non-un} .

This argument does not prove that the category $D_c^b(X \times_s \eta^{nr}, \mathbb{Q}_l)$ decomposes in a like manner: since it is not abelian, we cannot even define the kernel and image of γ_M for a general object M .

5.2 Permanence properties of unipotency and non-unipotency

Suppose $f : X \rightarrow Y$ is a finite-type morphism of finite-type S -schemes. This gives rise to the “derived” functors $f_! : D_c^b(X \times_s \eta^{nr}, \mathbb{Q}_l) \rightarrow D_c^b(Y \times_s \eta^{nr}, \mathbb{Q}_l)$ and $f^* : D_c^b(Y \times_s \eta^{nr}, \mathbb{Q}_l) \rightarrow D_c^b(X \times_s \eta^{nr}, \mathbb{Q}_l)$.

Lemma 5.6 (1) *Let $M \in D_c^b(X \times_s \eta^{nr}, \mathbb{Q}_l)$ be unipotent (resp. non-unipotent). Then $f_!M \in D_c^b(Y \times_s \eta^{nr}, \mathbb{Q}_l)$ is unipotent (resp. non-unipotent).*

(2) *Let $M' \in D_c^b(Y \times_s \eta^{nr}, \mathbb{Q}_l)$, and suppose $f_{\bar{s}} : X_{\bar{s}} \rightarrow Y_{\bar{s}}$ is surjective. Then M' is unipotent (resp. non-unipotent) if and only if f^*M' is unipotent (resp. non-unipotent).*

Proof. (1). By the local-global spectral sequence

$$\mathcal{H}^p(f_! \mathcal{H}^q M) \Rightarrow \mathcal{H}^{p+q} f_! M$$

we are reduced to the case $M \in Sh_c(X \times_s \eta^{nr}, \mathbb{Q}_l)$. Taking cohomology stalks we know that $f_!M$ is unipotent if and only if for every $\bar{y} \in Y_s(\bar{\mathbb{F}}_q)$, and every i , $1 - \gamma$ acts by zero on $H_c^i(f^{-1}(\bar{y}), M)$. But as M is unipotent, $1 - \gamma$ acts by zero on M , hence on the cohomology group with coefficients in M as well. The non-unipotent case is similar.

(2). This is straightforward. \square

5.3 Unipotency/non-unipotency of products and convolutions

The following facts follow easily from Lemmas 5.5 and 5.6.

Let \mathcal{F} and \mathcal{G} be objects of $D_c^b(X \times_s \eta, \mathbb{Q}_l)$ and suppose that \mathcal{F} is unipotent. Then $(\mathcal{F} \boxtimes \mathcal{G})^{non-un} = \mathcal{F} \boxtimes \mathcal{G}^{non-un}$.

Now let \mathcal{B} denote the Iwahori-group scheme acting on the affine flag variety $\mathcal{Fl} = G(\bar{\mathbb{F}}_p((t)))/\mathcal{B}$. If the objects \mathcal{F} and \mathcal{G} above are \mathcal{B} -equivariant perverse sheaves on \mathcal{Fl} , then we may define their twisted product $\mathcal{F} \tilde{\boxtimes} \mathcal{G}$ and their convolution product $\mathcal{F} \star_s \mathcal{G}$ (cf. [HN1]). Still under the assumption $\mathcal{F} = \mathcal{F}^{un}$, one can prove first $(\mathcal{F} \tilde{\boxtimes} \mathcal{G})^{non-un} = \mathcal{F} \tilde{\boxtimes} \mathcal{G}^{non-un}$, and then $(\mathcal{F} \star_s \mathcal{G})^{non-un} = \mathcal{F} \star_s \mathcal{G}^{non-un}$.

5.4 $R\Psi = R\Psi^{un}$

First, we remark that in the above discussion in 5.1-5.3, we can without any problems replace \mathbb{Q}_ℓ by a finite extension, and can then also pass to the 2-inductive limit of the categories $D_c^b(X, E)$, E/\mathbb{Q}_ℓ finite. Thus all the statements above hold for $\bar{\mathbb{Q}}_\ell$ -adic sheaves as well.

We now come back to the affine flag variety. Fix a dominant coweight μ and let $R\Psi = R\Psi(IC(\mathcal{Q}_\mu)[\ell(\mu)]) \in P^{\mathcal{B}}(\mathcal{Fl} \times_s \eta, \bar{\mathbb{Q}}_\ell)$.

Theorem 5.7 (Gaitsgory) *The action of Γ_0 on the sheaf $R\Psi$ of nearby cycles is purely unipotent.*

Proof. Because $R\Psi = R\Psi^{un} \oplus R\Psi^{non-un}$ it suffices to prove that $R\Psi^{non-un} = 0$.

Note that $R\Psi^{non-un}$ is a \mathcal{B} -equivariant object of $P(\mathcal{F}l \times_s \eta, \mathbb{Q}_\ell)$ (for equivariance, use Lemma 5.6). By using 5.3 we see that, as for $R\Psi$ itself, $R\Psi^{non-un}$ is central with respect to convolution of Iwahori-equivariant perverse sheaves on the affine flag variety. To show it is zero, it suffices to show that its semi-simple trace function $\tau_{R\Psi^{non-un}} : x \mapsto \mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi_x^{non-un})$, an element in $Z(\mathcal{H}_I)$, vanishes.

Now consider the projection $\pi : \mathcal{F}l \rightarrow \mathrm{Grass}$. Because the semi-simple trace provides a good sheaf-function dictionary in the sense of Grothendieck (cf. [HN1], Proposition 10), the semi-simple trace function $\mathrm{Tr}^{ss}(\mathrm{Fr}_q, \pi_*(R\Psi^{non-un}))$ corresponds to summing $\mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi^{non-un})$ along the fibers of π , i. e., we have the following equality of functions in the spherical Hecke algebra \mathcal{H}_K :

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, \pi_*(R\Psi^{non-un})) = \mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi^{non-un}) \star \mathbb{I}_K.$$

By a theorem of Bernstein (cf. [L2]), $-\star \mathbb{I}_K : Z(\mathcal{H}_I) \xrightarrow{\sim} \mathcal{H}_K$ is an isomorphism, so it suffices to prove $\pi_*(R\Psi^{non-un}) = 0$. But by Lemma 5.6 we have

$$\pi_*(R\Psi^{non-un}) = (\pi_* R\Psi)^{non-un} = (\mathcal{A}_{\mu,s})^{non-un} = 0.$$

Here \mathcal{A}_μ is the “constant” intersection complex $IC(\mathcal{Q}_\mu)[\ell(\mu)]$ on the “constant family” Grass, hence it has trivial inertia action. We are done.

5.5 Semi-simple trace vs. usual trace

Proposition 5.8 *The trace and the semi-simple trace on $R\Psi$ coincide:*

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi) = \mathrm{Tr}(\mathrm{Fr}_q, R\Psi^{ss}).$$

Here, on the left hand side $R\Psi$ is regarded as an object of the category $P^{\mathcal{B}}(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}}_\ell)$, and the semi-simple trace is defined as in [HN1]. On the right hand side, Fr_q represents a choice of lift of geometric Frobenius, and $R\Psi^{ss}$ denotes the semi-simplification of $R\Psi$ when regarded as an object of $P_{\mathrm{Weil}}^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$ (via that choice of lift).

Proof. It suffices to find a Galois-invariant filtration on $R\Psi$ such that Γ_0 acts trivially on the subquotients. Since Γ_0 acts unipotently on $R\Psi$ by 5.4, such a filtration is given by the kernels of powers of the operator N :

$$F^i R\Psi = \ker N^i.$$

5.6 A different proof

Here is a sketch of a different argument for the result above: after fixing a Weil sheaf structure on $R\Psi$, the usual trace of Frobenius satisfies the same sheaf-function dictionary as the semi-simple trace: the same proof as for semi-simple trace shows that the usual trace yields a central function in the Iwahori-Hecke algebra, characterized by

$$\mathrm{Tr}(\mathrm{Fr}_q, R\Psi) \star \mathbb{1}_K = \mathrm{Tr}(\mathrm{Fr}_q, IC(\mathcal{Q}_\mu)[\ell(\mu)]).$$

It follows from this that it coincides with the semi-simple trace function. This in turn implies that the action of inertia on $R\Psi$ must be purely unipotent. To prove the last statement, it is enough to show that $R\Psi$ admits a filtration such that the inertia group Γ_0 acts trivially on the graded pieces. Consider the filtration

$$F^i R\Psi = \ker N^i,$$

where N is the logarithm of the unipotent part of the local monodromy, as before. The inertia group acts on the F^i/F^{i-1} through a finite quotient, so the definition of semi-simple trace and the above remarks yields

$$\mathrm{Tr}(\mathrm{Fr}_q, (\bigoplus F^i/F^{i-1})^{\Gamma_0}) = \mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi) = \mathrm{Tr}(\mathrm{Fr}_q, R\Psi) = \mathrm{Tr}(\mathrm{Fr}_q, \bigoplus F^i/F^{i-1}).$$

Thus the function of the (necessarily perverse) quotient $(\bigoplus F^i/F^{i-1})/(\bigoplus F^i/F^{i-1})^{\Gamma_0}$ is zero, so we have $(\bigoplus F^i/F^{i-1})^{\Gamma_0} = \bigoplus F^i/F^{i-1}$. (We have used the fact that if $\mathcal{F} \in P(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}}_\ell)$ carries a *finite* action of inertia, then $\mathcal{F}^{\Gamma_0} = \mathrm{im}(\gamma_{\mathcal{F}})$, hence is also perverse.)

6 Applying the theorem to $R\Psi$

Fix a dominant coweight μ and write $R\Psi = R\Psi(IC(\mathcal{Q}_\mu)[\ell(\mu)]) \in P^{\mathcal{B}}(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}}_\ell)$.

6.1 $R\Psi$ as a Weil sheaf

Fixing a lift $\mathrm{Fr}_q \in \Gamma$ of the inverse of the Frobenius morphism enables us to view $R\Psi$ as a Weil-perverse sheaf. Since it is also a \mathcal{B} -equivariant perverse sheaf, it is a *mixed* Weil-perverse sheaf (this also follows from the Appendix, Theorem 10.6).

The Kottwitz conjecture holds for $R\Psi$ (see section 2.7), so

$$\mathrm{Tr}(\mathrm{Fr}_q, R\Psi) = \varepsilon_\mu q_\mu^{1/2} \sum_{\lambda \leq \mu} m_\mu(\lambda) z_\lambda.$$

In particular, $\mathrm{Tr}(\mathrm{Fr}_q, R\Psi_x) \in \mathbb{Z}[q, q^{-1}]$, so we can consider $R\Psi$ as an object of $P_q^{\mathcal{B}}(\mathcal{F}l, \overline{\mathbb{Q}}_\ell)$.

We will show that the nearby cycles sheaf satisfies the property (P), and hence we can apply Theorem 4.7, and get some information on the Jordan-Hölder series of $R\Psi$ as a Weil sheaf.

But furthermore, we know that the inertia group Γ_0 acts purely unipotently on $R\Psi$, so in this case, we get a stronger independence statement with respect to the choice of the lift Fr_q . Namely, we can regard the sheaves $IC_w(-i)$, with trivial Γ_0 -action, as objects of $P(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}}_\ell)$, and then the Jordan-Hölder series of $R\Psi \in P(\mathcal{F}l \times_s \eta, \overline{\mathbb{Q}}_\ell)$ is 'the same' as that of $R\Psi$ as a Weil-perverse sheaf.

6.2 $R\Psi$ and property (P)

To show that $R\Psi$ satisfies the property (P), we need to prove

$$\overline{\text{Tr}(\text{Fr}_q, R\Psi_x)} = \varepsilon_x \varepsilon_\mu q_x q_\mu^{-1} \text{Tr}(\text{Fr}_q, R\Psi_x),$$

because $DR\Psi = R\Psi(\ell(t_\mu))$, i.e. $\text{Tr}(\text{Fr}_q, DR\Psi) = q_\mu^{-1} \text{Tr}(\text{Fr}_q, R\Psi)$.

Note the following general lemma. We write $Q = q^{-1/2} - q^{1/2}$.

Lemma 6.1 *Let $f \in \mathbb{Z}[q, q^{-1}]$, $\alpha \in \frac{1}{2}\mathbb{Z}$ such that $f = q^\alpha R(Q)$ for some $R \in \mathbb{Z}[Q]$. Then all Q -powers occurring in R have the parity of 2α , and $f(q^{-1}) = (-1)^{2\alpha} q^{-2\alpha} f(q)$. \square*

Recalling that $z_\lambda = \sum_{\nu \in W(\lambda)} \Theta_\nu$ and the fact that $\Theta_\nu = \tilde{T}_{\nu_1} \tilde{T}_{\nu_2}^{-1}$, where $\nu_1 - \nu_2 = \nu$ and ν_i is dominant, we see (using the Kottwitz conjecture) that

$$R\Psi(x) = \varepsilon_\mu q_\mu^{1/2} q_x^{-1/2} R(Q),$$

where $R \in \mathbb{N}[Q]$. Thus we can apply the lemma with $\alpha = (\ell(\mu) - \ell(x))/2$, and get that $R\Psi$ satisfies (P).

6.3 Consequences of the theorem for $R\Psi$

Since $R\Psi$ is self-dual up to a Tate twist, from Theorem 4.7 we get

Corollary 6.2 *We have*

$$\sum_i m(R\Psi, w, i) q^i = \varepsilon_\mu \text{Tr}(\text{Fr}_q, H_c^\bullet(\mathcal{F}l, R\Psi \otimes IC(\mathcal{B}^w))).$$

In particular, by setting $w = \tau$, the unique element of length 0 in the support of $R\Psi$, we get by using the projection formula for $\pi : \mathcal{F}l \rightarrow \text{Grass}$ that

$$\begin{aligned} \sum_i m(R\Psi, \tau, i) q^i &= \varepsilon_\mu \text{Tr}(\text{Fr}_q, H_c^\bullet(\mathcal{F}l, R\Psi)) \\ &= \varepsilon_\mu \text{Tr}(\text{Fr}_q, H_c^\bullet(\text{Grass}, \pi_* R\Psi)) \\ &= \varepsilon_\mu \text{Tr}(\text{Fr}_q, H_c^\bullet(\text{Grass}, IC(\mathcal{Q}_\mu)[\ell(\mu)])) \\ &= \sum_i \dim IH^{2i}(\mathcal{Q}_\mu, \overline{\mathbb{Q}}_\ell) q^i. \end{aligned}$$

Corollary 6.3 *The multiplicity $m(R\Psi, \tau, i)$ is the $2i$ -th intersection Betti number of $\bar{\mathcal{Q}}_\mu$.*

Remark. This corollary was initially noticed as a purely empirical fact with the aid of a computer program, and it was the starting point of this work: the effort to explain it conceptually lead to the discovery of the more general Theorem 4.7. The corollary can also be proved, taking the Kottwitz conjecture into account, by purely combinatorial considerations in the Iwahori-Hecke algebra \mathcal{H} (recall that the traces of Frobenius and the multiplicities are related by a base change in \mathcal{H}). But it seems impossible to prove anything about the multiplicities for other w 's in that way.

More generally, by applying section 4.6.1 to $\mathcal{F} = R\Psi$, we get

Corollary 6.4 *Suppose $w \in \text{Adm}(\mu)$ belongs to $(\tilde{W}/W)_{\min}$. Then the multiplicity function is given by*

$$\sum_i m(R\Psi, w, i)q^i = \text{Tr}(\text{Fr}_q, H_c^\bullet(\bar{\mathcal{Q}}_\mu, IC(\bar{\mathcal{Q}}_\mu) \otimes IC(\overline{\mathcal{Q}^{\pi(w)}}))).$$

7 Wakimoto sheaves

7.1 Definition and properties

We have a convolution product

$$\star: P_{\text{Weil}}^{\mathcal{B}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell) \times P_{\text{Weil}}^{\mathcal{B}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell) \longrightarrow D_{c, \text{Weil}}^b(\mathcal{F}l, \bar{\mathbb{Q}}_\ell),$$

see [L3], [HN1]. Using the usual sheaf-function dictionary, it is easy to see that convolution of sheaves corresponds to multiplication of their functions in the Hecke algebra.

Definition 7.1 *The Wakimoto sheaf associated to $v, w \in \tilde{W}$ is the sheaf $M_{v, w} := j_{v!}\bar{\mathbb{Q}}_\ell[\ell(v)] \star j_{w*}\bar{\mathbb{Q}}_\ell[\ell(w)]$.*

Note that, as usual, $j_{v!}$ and j_{w*} denote the *derived* functors. The function in the Iwahori-Hecke algebra \mathcal{H} corresponding to $M_{v, w}$ is $\varepsilon_v \varepsilon_w q_w T_v T_{w^{-1}}^{-1} = \varepsilon_v \varepsilon_w q_v^{1/2} q_w^{1/2} \tilde{T}_v \tilde{T}_{w^{-1}}^{-1}$. As usual, we write $\tilde{T}_x = q_x^{-1/2} T_x$.

Proposition 7.2 Properties of Wakimoto sheaves

- (1) $M_{v, w}$ is perverse.
- (2) $M_{v, w}$ satisfies (P), with $d = \ell(v) + \ell(w)$.

Proof. (1) This result is due to Mirkovic. See [HP] Prop. 6.2, or [AB] Thm. 5.

(2) The function corresponding to the dual of $M_{v,w}$ is $\varepsilon_v \varepsilon_w q_w^{-1} T_w T_{v^{-1}}^{-1} = \varepsilon_v \varepsilon_w q_w^{-1/2} q_v^{-1/2} \tilde{T}_w \tilde{T}_{v^{-1}}^{-1}$. We have

$$DM_{v,w}(y) = \varepsilon_v \varepsilon_w q_w^{-1/2} q_v^{-1/2} q_y^{-1/2} R_{y,v}^w(Q)$$

where $R_{y,v}^w \in \mathbb{N}[Q]$ is such that Q^i occurs only for $i \equiv \ell(w) + \ell(v) + \ell(y) \pmod{2}$. (Use, for example, Lemma 6.1.) Thus,

$$\overline{DM_{v,w}(y)} = \varepsilon_v \varepsilon_w \varepsilon_y q_w q_v q_y DM_{v,w}(y),$$

which proves property (P) with $d = \ell(v) + \ell(w)$. \square

The proposition shows that there is a large class of sheaves which satisfy the property (P).

Proposition 7.3 *Let $\mathcal{F} \in P_{\sqrt{q}}^{\mathcal{B}}(\mathcal{Fl}, \overline{\mathbb{Q}}_\ell)$ be a sheaf which has a filtration whose graded pieces are of the form $M_{v_i, w_i}(-\alpha_i)$, where $\alpha_i \in \mathbb{Z}$ is such that $2\alpha_i + \ell(v_i) + \ell(w_i)$ is independent of i . Then \mathcal{F} satisfies (P), with $d = \ell(v_i) + \ell(w_i) + 2\alpha_i$.*

Proof. The function corresponding to \mathcal{F} is

$$\sum_i q^{\alpha_i} q_{w_i} \varepsilon_{v_i} \varepsilon_{w_i} T_{v_i} T_{w_i^{-1}}^{-1},$$

and the function of the dual of \mathcal{F} is obtained by applying $\bar{\cdot}$. A computation similar to the one above yields the result. \square

Remark. In their recent paper [AB], Arkhipov and Bezrukavnikov show that the sheaf of nearby cycles $R\Psi$ has a filtration whose graded pieces are Wakimoto sheaves.

Their theorem relies on the sheaf of nearby cycles being central, so that we do not get a proof of (P) for $R\Psi$ which does not use the Kottwitz conjecture.

7.2 The elements in the Hecke algebra corresponding to Wakimoto sheaves

Fix $v, w \in W_{\text{aff}}$, and suppose $w = s_1 \cdots s_r$ is reduced. There are unique polynomials $R_{x,w}^v(Q)$ in the parameter $Q := q^{-1/2} - q^{1/2}$ and with non-negative integral coefficients, such that

$$\tilde{T}_v \tilde{T}_{w^{-1}}^{-1} = \sum_{x \leq v|w} R_{x,w}^v(Q) \tilde{T}_x.$$

Here the notation $x \leq v|w$ means by definition that x is the terminal element of a v -distinguished subexpression with respect to w (see section 5 of [H1]). Recall that an $r+1$ -tuple $[\sigma_0, \dots, \sigma_r]$ is v -distinguished if $\sigma_0 = v$, and if for each $j \geq 1$,

1. $\sigma_j \in \{\sigma_{j-1}, \sigma_{j-1}s_j\}$; and
2. $\sigma_j = \sigma_{j-1}$ only if $\sigma_{j-1} < \sigma_{j-1}s_j$.

We are going to prove a formula for $R_{x,w}^v(Q)$. We need first some more definitions.

For a v -distinguished subexpression $\underline{\sigma} = [\sigma_0, \dots, \sigma_r]$, we set $\pi(\underline{\sigma}) = \sigma_r$. Also define

$$\mathcal{D}(x) = \{\underline{\sigma} \mid \pi(\underline{\sigma}) = x\}.$$

Further define integers $n(\underline{\sigma}) = |\{j \mid \sigma_{j-1} = \sigma_j\}|$, and $m(\underline{\sigma}) = |\{j \mid \sigma_{j-1} > \sigma_j\}|$. We have the following easy lemma.

Lemma 7.4 *If $\underline{\sigma}$ is v -distinguished with respect to w , then*

$$\ell(\pi(\underline{\sigma})) = \ell(v) + \ell(w) - n(\underline{\sigma}) - 2m(\underline{\sigma}).$$

Now write $w' = s_1 \cdots s_{r-1}$, so that $w' = ws_r < w$. The equality

$$\sum_{y \leq v|w'} R_{y,w'}^v(Q) \tilde{T}_y (\tilde{T}_{s_r} + Q) = \sum_{x \leq v|w} R_{x,w}^v(Q) \tilde{T}_x$$

give rise to recurrence relations for the polynomials $R_{x,w}^v$. Using these, we can prove the following proposition by induction on $\ell(w)$.

Proposition 7.5 *If $x \leq v|w$, then*

$$R_{x,w}^v(Q) = \sum_{\underline{\sigma} \in \mathcal{D}(x)} Q^{n(\underline{\sigma})} = \sum_{\underline{\sigma} \in \mathcal{D}(x)} Q^{\ell(v) + \ell(w) - \ell(x) - 2m(\underline{\sigma})}.$$

Taking the parity of the exponents into account, this proposition gives a direct verification of the conclusion of Lemma 6.1 for Wakimoto sheaves.

Proof. The proof is by induction on $\ell(w)$, and uses case-by-case analysis. Write w' as above and write $s_r = s$. We consider the set $\mathcal{D}'(y)$ of v -distinguished subexpressions with respect to w' which end with y . Integers n and m can be attached to each of these in the same way as before.

Note that $x \leq v|w$ implies that $xs \leq v|w'$ or $x \leq v|w'$, or both.

Case 1: $x \leq v|w$, $x \leq v|w'$, and $xs \not\leq v|w'$.

(a) $x < xs$. Then every $\underline{\sigma} \in \mathcal{D}(x)$ is of the form $[?, x, x]$. Given such, write $\underline{\sigma}' = [?, x]$ (the first part of $\underline{\sigma}$). We see $n(\underline{\sigma}) = n(\underline{\sigma}') + 1$, $m(\underline{\sigma}) = m(\underline{\sigma}')$, and $R_{x,w}^v = QR_{x,w'}^v$. Moreover, we have a bijection of sets $S : \mathcal{D}(x) \xrightarrow{\sim} \mathcal{D}'(x)$ given by $\underline{\sigma} \mapsto \underline{\sigma}'$.

(b) $xs < x$. This cannot happen, since the only possible $\underline{\sigma}'$ is of form $[?, x]$, and $\underline{\sigma}$ cannot have form $[?, x, x]$, since then we would need to have $x < xs$ (by definition of v -distinguished).

Case 2: $x \leq v|w$, $x \not\leq v|w'$, and $xs \leq v|w'$.

(a) $x < xs$. Then every $\underline{\sigma}$ is of form $[?, xs, x]$; set $\underline{\sigma}' = [?, xs]$. We have $n(\underline{\sigma}) = n(\underline{\sigma}')$, and $R_{x,w}^v = R_{xs,w'}^v$. Also, $S : \mathcal{D}(x) \xrightarrow{\sim} \mathcal{D}'(xs)$ by $\underline{\sigma} \mapsto \underline{\sigma}'$.

(b) $xs < x$. Then $\underline{\sigma}$ is of form $[?, xs, x]$; let $\underline{\sigma}' = [?, xs]$. We have $n(\underline{\sigma}) = n(\underline{\sigma}')$, and $R_{x,w}^v = R_{xs,ws}^v$, and $S : \mathcal{D}(x) \xrightarrow{\sim} \mathcal{D}'(xs)$ by $\underline{\sigma} \mapsto \underline{\sigma}'$.

Case 3: $x \leq v|w$, $x \leq v|w'$, and $xs \leq v|w'$.

(a) $x < xs$. We define $\mathcal{A}(x) = \{\underline{\sigma} \text{ is of the form } [?, xs, x]\}$, and $\mathcal{B}(x) = \{\underline{\sigma} \text{ is of the form } [?, x, x]\}$. Note that $\mathcal{D}(x) = \mathcal{A}(x) \coprod \mathcal{B}(x)$.

For $\underline{\sigma} = [?, xs, x] \in \mathcal{A}(x)$, let $\underline{\sigma}' = [?, xs]$. The map $\underline{\sigma} \mapsto \underline{\sigma}'$ gives $\mathcal{A}(x) \xrightarrow{\sim} \mathcal{D}'(xs)$, and $n(\underline{\sigma}) = n(\underline{\sigma}')$.

For $\underline{\sigma} = [?, x, x] \in \mathcal{B}(x)$, let $\underline{\sigma}' = [?, x]$. The map $\underline{\sigma} \mapsto \underline{\sigma}'$ gives $\mathcal{B}(x) \xrightarrow{\sim} \mathcal{D}'(x)$, and $n(\underline{\sigma}) = n(\underline{\sigma}') + 1$.

In either case, we have $R_{x,w}^v = QR_{x,ws}^v + R_{xs,ws}^v$.

(b) $xs < x$. Then $\underline{\sigma} = [?, xs, x]$, and setting $\underline{\sigma}' = [?, xs]$, we have $n(\underline{\sigma}) = n(\underline{\sigma}')$, $S : \mathcal{D}(x) \xrightarrow{\sim} \mathcal{D}'(xs)$. We have $R_{x,w}^v = R_{xs,ws}^v$.

The proposition now follows easily by using the induction hypothesis and by taking into account the various cases. \square

In general it is difficult to say anything really concrete concerning the polynomial $R_{x,w}^v(Q)$: even its degree is mysterious. However, suppose $\tilde{T}_v \tilde{T}_{w^{-1}}^{-1}$ has a *minimal expression*, i.e., suppose we can write

$$\tilde{T}_v \tilde{T}_{w^{-1}}^{-1} = \tilde{T}_{t_1}^{\epsilon_1} \cdots \tilde{T}_{t_k}^{\epsilon_k},$$

where $t_1 \cdots t_k = vw$ is a reduced expression, and $\epsilon_i \in \{1, -1\}$, for each i . Letting c_x denote the coefficient of \tilde{T}_x in the expansion of such a minimal expression, one can easily prove by induction on k that the Q -degree of c_x is bounded above:

$$\deg_Q c_x \leq k - \ell(x).$$

We get the following consequence, which we record for later use.

Proposition 7.6 *Suppose $\tilde{T}_v \tilde{T}_{w^{-1}}^{-1}$ has a minimal expression. Then*

$$\deg_Q R_{x,w}^v(Q) \leq \ell(vw) - \ell(x).$$

7.3 The situation at $q = 1$

Recall from section 4.2 that the elements $C_w'' = \text{Tr}(\text{Fr}_q, IC_w) = \text{Tr}(\text{Fr}_q, IC(\mathcal{B}_{\bar{w}})[\ell(w)])$ form a basis of the Iwahori-Hecke algebra \mathcal{H} . Consider the algebra homomorphism $\mathcal{H} \rightarrow \mathbb{Z}[\widehat{W}]$ given by specializing $q^{1/2} = 1$. We will

denote the image of C''_w by $C''_w|$, etc. The elements T_w map to the usual basis of the group ring $\mathbb{Z}[\widetilde{W}]$, and the elements C''_w map to another basis. In fact

$$C''_w| = \sum_{y \leq w} \varepsilon_w P_{y,w}(1) T_y|.$$

We will identify $T_y|$ with y in what follows.

Now fix a coweight λ , and write $\lambda = \lambda_1 - \lambda_2$, where both λ_i are dominant. We have

$$\varepsilon_\lambda q_\lambda^{1/2} \Theta_\lambda = \varepsilon_\lambda q_\lambda^{1/2} \widetilde{T}_{\lambda_1} \widetilde{T}_{\lambda_2}^{-1} = \sum_w a_w C''_w.$$

Since the Wakimoto sheaf giving rise to this function is perverse and \mathcal{B} -equivariant, we see that $a_w \in \mathbb{N}[q^{1/2}, q^{-1/2}]$ for every w , and thus $a_w \neq 0$ if and only if $a_w(1) \neq 0$. We call the set $\{w \mid a_w \neq 0\}$ the *IC-support* of $\varepsilon_\lambda q_\lambda^{1/2} \Theta_\lambda$.

Applying the homomorphism $x \mapsto x|$ to $\varepsilon_\lambda q_\lambda^{1/2} \widetilde{T}_{\lambda_1} \widetilde{T}_{\lambda_2}^{-1}$ gives

$$\varepsilon_\lambda t_{\lambda_1} t_{-\lambda_2} = \varepsilon_\lambda t_\lambda = \sum_{w \leq t_\lambda} Q_{w,t_\lambda}(1) C''_w|.$$

We thus see that

$$a_w(1) = Q_{w,t_\lambda}(1),$$

and that the IC-support of $\varepsilon_\lambda q_\lambda^{1/2} \Theta_\lambda$ is precisely the set $\{w \leq t_\lambda\}$.

Using now the geometric interpretation of inverse Kazhdan-Lusztig polynomials in the affine case (cf. Theorem 3.5), we get the following proposition.

Proposition 7.7 *The IC-support of the function $\varepsilon_\lambda q_\lambda^{1/2} \Theta_\lambda$ is precisely the set $\{w \leq t_\lambda\}$. For each such w , the sum of the multiplicities $\sum_i m(\lambda, w, i)$ is equal to the intersection Euler characteristic of the space*

$$\mathcal{B}_{t_\lambda} \cap \mathcal{B}^{\overline{w}}.$$

8 Applications to Shimura varieties

8.1 Relating nearby cycles on local models and Shimura varieties with Iwahori level structure

In certain cases, the singularities occurring in the bad reduction of Shimura varieties of PEL type with Iwahori level structure are equivalent to the singularities of one of the schemes M_μ/\mathbb{Z}_p that we were considering. More precisely, for minuscule μ , M_μ is a local model in the sense of Rapoport and Zink [RZ2] of a certain Shimura variety.

Consider a Shimura variety $Sh = Sh_K(G, X)$ of PEL type, with Iwahori level structure at p . (The same methods should yield analogous results for arbitrary

parahoric level structure.) Denote by E its reflex field. Since Shimura varieties of PEL type are moduli spaces of abelian varieties with additional structure, there is a natural way to define a model over the ring of integers of the completion of E at a prime \mathfrak{p} over p (see [Ko], [RZ2]).

Furthermore, Rapoport and Zink have defined a so-called local model of the Shimura variety: a scheme defined in terms of linear algebra, which has 'the same' singularities as the model of the Shimura variety.

Now assume that $G_{\mathbb{Q}_p}$ is isomorphic to $GL_{n,\mathbb{Q}_p} \times \mathbb{G}_{m,\mathbb{Q}_p}$ or GS_{p2g,\mathbb{Q}_p} .

The Shimura datum gives rise to a minuscule cocharacter μ of GL_{n,\mathbb{Q}_p} or GS_{p2g,\mathbb{Q}_p} , respectively. The functorial description of the local model shows that it can be embedded into the deformation from $\text{Grass}_{\mathbb{Q}_p}$ to $\mathcal{F}l_{\mathbb{F}_p}$ à la Laumon (see [HN1]), with generic fibre \mathcal{Q}_μ . Since the local model is flat [G1], [G2], it coincides with the scheme-theoretic closure M_μ of \mathcal{Q}_μ in this deformation.

The relation between the model of the Shimura variety over \mathbb{Z}_p and its local model is given by a diagram

$$Sh \xleftarrow{\varphi} \mathcal{M} \xrightarrow{\psi} \mathbf{M}^{\text{loc}}$$

of O_E -schemes, where φ is a torsor under a smooth affine group scheme, and ψ is smooth. The fibres of φ are geometrically connected (more precisely, this holds for the restriction of φ to any geometric connected component of \mathcal{M}). One can show that étale-locally around each point of the special fibre, the model of the Shimura variety and the local model are isomorphic. (See [RZ2].)

The stratification of the special fibre of \mathbf{M}^{loc} induces stratifications of the special fibres of \mathcal{M} and Sh : the inverse images under ψ of strata in $\mathbf{M}_{\mathbb{F}_p}^{\text{loc}}$ define a stratification of \mathcal{M} , and there is a stratification of Sh such that the inverse images of strata under φ are the strata in \mathcal{M} . For details see [GN]. Following loc.cit., we call the resulting stratification on $Sh_{\mathbb{F}_p}$ the *Kottwitz-Rapoport stratification*.

The smooth base change theorem implies that the nearby cycles functor commutes with pull-back under smooth morphisms, so we have (up to shifts by the dimensions)

$$\varphi^* R\Psi_{Sh} \cong R\Psi_{\mathcal{M}} \cong \psi^* R\Psi_{\mathbf{M}^{\text{loc}}}.$$

Furthermore, the intermediate extension commutes with pull-back under smooth morphisms (see the proof of theorem III.11.3 in [KW]), which implies that

$$\varphi^* IC_{Sh,w} \cong IC_{\mathcal{M},w} \cong \psi^* IC_{\mathbf{M}^{\text{loc}},w}{}^3.$$

(Again, this is up to a shift by the dimensions of the strata: these intersection complexes are normalized so that they are perverse.)

³As in the introduction $IC_{Sh,w}$ and $IC_{\mathcal{M},w}$ should be interpreted as direct sums of intersection complexes over geometric connected components of Sh and \mathcal{M} .

So we see that (since ϕ and ψ have the same relative dimension)

$$\varphi^* R\Psi_{Sh}^{ss} \cong \varphi^* \left(\bigoplus_w \bigoplus_i IC_{Sh,w}(-i)^{m(w,i)} \right),$$

and because φ (restricted to any geometrically connected component of \mathcal{M}) is smooth with geometrically connected fibres, this yields

$$R\Psi_{Sh}^{ss} \cong \bigoplus_w \bigoplus_i IC_{Sh,w}(-i)^{m(w,i)}.$$

Hence the sheaf of nearby cycles on the Shimura variety decomposes in the same way as the sheaf of nearby cycles on the local model.

8.2 The weight spectral sequence

By virtue of the Appendix, Theorem 10.1, the nearby cycles $R\Psi := R\Psi_{Sh}$ is a *mixed* perverse sheaf on the special fiber of Sh . Therefore, $R\Psi$ possesses a weight filtration ([BBD], §5), that is, an increasing filtration W such that each graded piece

$$\mathrm{gr}_i^W := \mathrm{gr}_i^W(R\Psi)$$

is a pure perverse sheaf of weight i . Now assume that Sh is proper over \mathcal{O}_E (here E denotes the completion of the reflex field at a prime above p). Then we have a canonical isomorphism of Galois modules

$$H^i(Sh_{\overline{\mathbb{F}}_p}, R\Psi) = H^i(Sh_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell[\ell(\mu)]),$$

and the weight filtration on $R\Psi$ yields the *weight spectral sequence*

$${}_W E_1^{pq} = H^{p+q}(Sh_{\overline{\mathbb{F}}_p}, \mathrm{gr}_{-p}^W) \implies H^{p+q}(Sh_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell[\ell(\mu)]).$$

This spectral sequence degenerates in E_2 , and abuts to the weight filtration on the cohomology group on the right hand side. Deligne's monodromy-weight conjecture asserts that the weight filtration on the right hand side is the same as the monodromy filtration defined with the help of the inertia action.

One might hope that an explicit description of the E_1 -terms might provide a modest first step toward the monodromy-weight conjecture for these Shimura varieties. The methods of this paper allow us to completely describe the semi-simplification $(\mathrm{gr}_i^W)^{ss}$ of each graded piece $\mathrm{gr}_i^W(R\Psi)$, in the category $P(Sh \times_s \eta, \overline{\mathbb{Q}}_\ell)$. This can be used to describe the E_1 -terms, if one forgets the Galois action. The following summarizes our results in this direction. (We will write $m(w, i)$ for $m(R\Psi, w, i)$ here.)

Proposition 8.1 *In the category $P(Sh \times_s \eta, \overline{\mathbb{Q}}_\ell)$, we have the identity*

$$(\mathrm{gr}_i^W)^{ss} = \bigoplus_{\substack{w \in \mathrm{Adm}(\mu) \\ \ell(w) + 2j = i}} IC_{Sh,w}(-j)^{\oplus m(w,j)}.$$

Here $IC_{Sh,w}$ is the intersection complex of the closure of the Kottwitz-Rapoport stratum Sh_w in $Sh_{\overline{\mathbb{F}}_p}$. Consequently, if Sh is proper over \mathcal{O}_E , we deduce (forgetting the Galois actions) isomorphisms of $\overline{\mathbb{Q}}_\ell$ -spaces

$${}_W E_1^{pq} = \bigoplus_{\substack{w \in \text{Adm}(\mu) \\ \ell(w) + 2j = -p}} IH^{p+q+\ell(w)}(\overline{Sh}_w, \overline{\mathbb{Q}}_\ell)^{\oplus m(w,j)}.$$

These results illustrate the fundamental importance of the Kottwitz-Rapoport stratification on $Sh_{\overline{\mathbb{F}}_p}$. That stratification is indexed by the familiar set $\text{Adm}(\mu)$, but the strata themselves are still rather mysterious. In future work we will examine these strata more closely.

9 Examples

In this section we present a few examples for multiplicities of nearby cycles sheaves $R\Psi = R\Psi(IC(\mathcal{Q}_\mu)[\ell(t_\mu)])$. Most of the examples are for $G = GL_n$, but there is one example for each of GSp_4 and GSp_6 , and also one for G of type G_2 .

9.1 The case of a divisor with normal crossings

Proposition 9.1 *In the Drinfeld case, where GL_n and $\mu = (1, 0, \dots, 0)$, we have $m(x, i) = 1$ for all $x \in \text{Adm}(\mu)$, $0 \leq i \leq \ell(t_\mu) - \ell(x)$, and $m(x, i) = 0$ otherwise.*

Proof. This is a combinatorial exercise using the following facts. In this case all Kazhdan-Lusztig polynomials are 1 (since the special fiber is a union of smooth divisors with normal crossings) and $\text{Tr}(\text{Fr}_q, R\Psi_x) = \varepsilon_\mu(1 - q)^{\ell(t_\mu) - \ell(x)}$ (cf. [H3], Prop. 5.2). \square

9.2 GL_2

Proposition 9.2 *In the case of GL_2 and an arbitrary coweight μ , all multiplicities $m(x, i) = 1$ for $x \in \text{Adm}(\mu)$ and $0 \leq i \leq \ell(t_\mu) - \ell(x)$, and $m(x, i) = 0$ otherwise.*

Proof. Argue as above, using the explicit formula for $\text{Tr}(\text{Fr}_q, R\Psi_x)$ given in [H1], Prop. 10.3. \square

9.3 Other examples

Further examples can be computed in the following way: the Kottwitz conjecture (see section 2.7) describes the semi-simple trace of Frobenius on $R\Psi$, and since the inertia action is unipotent, this is the same as the usual trace. So the function of $R\Psi$ is known, and thus the multiplicities can be computed by applying the base change from the T_w -basis to the C_w'' -basis in the Hecke algebra. In order to carry this out, though, Bernstein functions and lots of Kazhdan-Lusztig polynomials have to be computed, so it is almost impossible to do this manually. The following examples were computed with the help of a computer program.

Because in all the interesting examples the number of alcoves is quite large, we did not print out the multiplicities for each single alcove, but we subsumed the information for several alcoves into one line.

Besides the length l and the number of alcoves that have certain multiplicities, we also put the 'Bruhat configuration' in the tables, i.e. the number of alcoves of length $l + 1$, $l + 2$, \dots which are greater than the given alcove with respect to the Bruhat order; cf. the remarks below.

The multiplicities listed below are always the numbers $m(w, i)$ where i is understood to range from 0 to $\ell(t_\mu) - \ell(w)$ (the multiplicities for i not in this range always happen to vanish, cf. section 9.4 for some discussion of this point).

9.3.1 $G = GL_4, \mu = (1, 1, 0, 0)$

Number of admissible alcoves: 33

Length	#Alcoves	Multiplicities	Bruhat configuration
$l=0$	1	1,1,2,1,1	4, 10, 12, 6
$l=1$	4	1,2,2,1	5, 8, 5
$l=2$	8	1,1,1	3, 3
	2	1,2,1	4, 4
$l=3$	12	1,1	2
$l=4$	6	1	-

This is the simplest non-trivial case. We see already that the multiplicities for w are not determined by the length of w .

9.3.2 $G = GL_5, \mu = (1, 1, 0, 0, 0)$

Number of admissible alcoves: 131

Length	#Alcoves	Multiplicities	Bruhat configuration
$l=0$	1	1, 1, 2, 2, 2, 1, 1	5, 15, 30, 40, 30, 10
$l=1$	5	1, 2, 3, 3, 2, 1	6, 17, 28, 24, 9
$l=2$	5	1, 1, 2, 1, 1	4, 10, 12, 6
	5	1, 2, 2, 2, 1	6, 14, 15, 7
	5	1, 3, 4, 3, 1	7, 17, 18, 8
$l=3$	5	1, 1, 1, 1	4, 6, 4
	20	1, 2, 2, 1	5, 8, 5
	5	1, 2, 2, 1	5, 9, 6
$l=4$	30	1, 1, 1	3, 3
	10	1, 2, 1	4, 4
$l=5$	30	1, 1	2
$l=6$	10	1	-

9.3.3 $G = GL_6, \mu = (1, 1, 0, 0, 0, 0)$

Number of admissible alcoves: 473

Length	#Alcoves	Multiplicities	Bruhat configuration
$l=0$	1	1, 1, 2, 2, 3, 2, 2, 1, 1	6, 21, 50, 90, 120, 110, 60, 15
$l=1$	6	1, 2, 3, 4, 4, 3, 2, 1	7, 24, 55, 86, 88, 52, 14
$l=2$	6	1, 1, 2, 2, 2, 1, 1	5, 15, 30, 40, 30, 10
	6	1, 2, 3, 3, 3, 2, 1	7, 24, 48, 58, 39, 12
	9	1, 3, 5, 6, 5, 3, 1	8, 28, 56, 67, 44, 13
$l=3$	30	1, 2, 3, 3, 2, 1	6, 17, 28, 24, 9
	6	1, 2, 2, 2, 2, 1	7, 20, 30, 24, 9
	12	1, 3, 4, 4, 3, 1	8, 25, 39, 31, 11
	2	1, 4, 7, 7, 4, 1	9, 30, 47, 36, 12
$l=4$	15	1, 1, 2, 1, 1	4, 10, 12, 6
	6	1, 1, 1, 1, 1	5, 10, 10, 5
	30	1, 2, 2, 2, 1	6, 14, 15, 7
	6	1, 2, 2, 2, 1	6, 14, 16, 8
	3	1, 2, 3, 2, 1	6, 15, 18, 9
	30	1, 3, 4, 3, 1	7, 17, 18, 8
$l=5$	30	1, 1, 1, 1	4, 6, 4
	60	1, 2, 2, 1	5, 8, 5
	30	1, 2, 2, 1	5, 9, 6
$l=6$	80	1,1,1	3, 3
	30	1,2,1	4, 4
$l=7$	60	1,1	2
$l=8$	15	1	-

9.3.4 $G = GL_3, \mu = (2, 2, 0)$

Number of admissible alcoves: 19

Length	#Alcoves	Multiplicities	Bruhat configuration
l=0	1	1, 1, 2, 1, 1	3, 6, 6, 3
l=1	3	1,2,2,1	4, 5, 3
l=2	3	1,1,1	2, 2
	3	1,1,1	3, 3
l=3	6	1,1	2
l=4	3	1	-

9.3.5 $G = GL_3, \mu = (3, 1, 0)$

Number of admissible alcoves: 49

Length	#Alcoves	Multiplicities	Bruhat configuration
l=0	1	1, 2, 3, 3, 3, 2, 1	3, 6, 9, 12, 12, 6
l=1	3	1, 3, 5, 5, 3, 1	4, 8, 12, 12, 6
l=2	3	1, 2, 3, 2, 1	4, 9, 11, 6
	3	1, 3, 4, 3, 1	4, 9, 11, 6
l=3	3	1,2,2,1	4, 6, 4
	3	1,2,2,1	4, 8, 6
	3	1,4,4,1	6, 10, 6
l=4	3	1,1,1	2, 2
	9	1,2,1	4, 4
l=5	12	1,1	2
l=6	6	1	-

This example shows that the Bruhat configuration *numbers* given in the table do not determine the multiplicities (look at the elements of length 2). On the other hand, in this example it is still quite easy to check that the Bruhat *graphs* for the two types of elements are different.

9.3.6 $G = GL_4, \mu = (2, 0, 0, 0)$

Number of admissible alcoves: 65

Length	#Alcoves	Multiplicities	Bruhat configuration
l=0	1	1, 1, 2, 2, 2, 1, 1	4, 10, 16, 18, 12, 4
l=1	4	1, 2, 3, 3, 2, 1	5, 11, 15, 11, 4
l=2	4	1, 1, 2, 1, 1	3, 6, 6, 3
	4	1, 2, 2, 2, 1	5, 10, 9, 4
	2	1, 3, 4, 3, 1	6, 12, 10, 4
l=3	12	1,2,2,1	4, 5, 3
	4	1,1,1,1	4, 6, 4
l=4	6	1,1,1	2, 2
	12	1,1,1	3, 3
l=5	12	1,1	2
l=6	4	1	-

9.3.7 $G = GL_4, \mu = (2, 1, 0, 0)$

Number of admissible alcoves: 143

Length	#Alcoves	Multiplicities	Bruhat configuration
$l=0$	1	1, 2, 3, 4, 4, 3, 2, 1	4, 10, 20, 30, 36, 30, 12
$l=1$	4	1, 3, 5, 6, 5, 3, 1	5, 14, 25, 33, 29, 12
$l=2$	4	1, 3, 4, 4, 3, 1	5, 12, 20, 21, 10
	4	1, 2, 3, 3, 2, 1	5, 14, 25, 26, 12
	2	1, 6, 11, 11, 6, 1	8, 20, 30, 28, 12
$l=3$	4	1,2,2,2,1	3, 6, 9, 6
	4	1,2,3,2,1	5, 13, 17, 9
	12	1,3,4,3,1	6, 15, 19, 10
$l=4$	12	1,2,2,1	4, 8, 6
	8	1,2,2,1	5, 9, 6
	8	1,3,3,1	6, 11, 7
	2	1,3,3,1	6, 12, 8
$l=5$	8	1,1,1	3, 3
	28	1,2,1	4, 4
$l=6$	30	1,1	2
$l=7$	12	1	-

9.3.8 $G = GSp_4, \mu = (1, 1, 0, 0)$

Number of admissible alcoves: 13

Length	#Alcoves	Multiplicities	Bruhat configuration
$l=0$	1	1, 1, 1, 1	3, 5, 4
$l=1$	2	1, 1, 1	3, 3
	1	1, 2, 1	4, 4
$l=2$	5	1,1	2
$l=3$	4	1	-

9.3.9 $G = GSp_6$, $\mu = (1, 1, 1, 0, 0, 0)$

Number of admissible alcoves: 79

Length	#Alcoves	Multiplicities	Bruhat configuration
l=0	1	1,1,1,2,1,1,1	4, 9, 17, 22, 18, 8
l=1	2	1,1,2,2,1,1	4, 10, 16, 15, 7
	2	1,2,3,3,2,1	5, 13, 19, 17, 8
l=2	1	1,1,2,1,1	4, 10, 12, 6
	4	1,2,2,2,1	5, 10, 11, 6
	2	1,2,2,2,1	5, 11, 14, 8
	2	1,3,4,3,1	6, 13, 14, 7
l=3	6	1,1,1,1	3, 5, 4
	2	1,1,1,1	4, 6, 4
	4	1,2,2,1	5, 8, 5
	4	1,2,2,1	5, 9, 6
	1	1,3,3,1	6, 12, 8
l=4	14	1,1,1	3, 3
	8	1,2,1	4, 4
l=5	18	1,1	2
l=6	8	1	-

9.3.10 G of type G_2 , $\mu = (2, 1, 0)$

Number of admissible alcoves: 41

Length	#Alcoves	Multiplicities	Bruhat configuration
l=0	1	1, 1, 1, 1, 1, 1, 1	3, 5, 7, 9, 10, 6
l=1	1	1, 1, 1, 1, 1, 1	3, 5, 7, 9, 6
	1	1, 1, 1, 1, 1, 1	3, 6, 9, 10, 6
	1	1, 2, 2, 2, 2, 1	4, 7, 9, 10, 6
l=2	2	1,1,1,1,1	3, 5, 7, 5
	1	1,2,2,2,1	4, 7, 9, 6
	2	1,2,2,2,1	4, 8, 10, 6
l=3	3	1,1,1,1	3, 5, 4
	2	1,2,2,1	4, 7, 5
	1	1,2,2,1	4, 8, 6
	1	1,2,2,1	5, 9, 6
l=4	4	1,1,1	3, 3
	5	1,2,1	4, 4
l=5	10	1,1	2
l=6	6	1	-

9.4 Numerical Observations and Conjectures

Looking at the tables, one can make several empirical observations:

- (A) We have $m(w, i) \in [0, \ell(\mu) - \ell(w)]$, or equivalently, the function $\text{Tr}(\text{Fr}_q, R\Psi_w)$ is a polynomial in q of degree $\leq \ell(\mu) - \ell(w)$;

- (B) The sequence $m(w, 0), \dots, m(w, \ell(\mu) - \ell(w))$ is palindromic; it increases to the middle, then decreases again.
- (C) We have $m(w, 0) = m(w, \ell(\mu) - \ell(w)) = 1$.

From some more detailed information not contained in the tables, we also made the following striking observation:

- The multiplicity polynomial $m(w) = \sum_i m(w, i)q^i$ is determined by the isomorphism type of the Bruhat graph of the set $\{x \in \text{Adm}(\mu) \mid x \geq w\}$ (although not by the Bruhat configuration numbers alone). It is remarkable that the multiplicity polynomials seem to be determined by data that is independent of the underlying root system.

Using the explicit formula for $\text{Tr}(\text{Fr}_q, R\Psi_w)$ in the minuscule case, proved in [H3],[HP], we can prove this last observation in that case. Based on our data we conjecture it holds in complete generality. We offer the following explanations for (A)-(C).

Proof of (A) for GL_n , or for μ minuscule.

From the definition, we get a recursive procedure to compute the multiplicities:

$$\varepsilon_w \sum_i m(w, i)q^i = \text{Tr}(\text{Fr}_q, R\Psi_w) - \sum_{\substack{x \in \text{Adm}(\mu) \\ x > w}} \varepsilon_x m(x) P_{w,x}.$$

By downward induction on $\ell(w)$, we deduce that $\text{Tr}(\text{Fr}_q, R\Psi_w)$ is a polynomial of degree $\leq \ell(\mu) - \ell(w)$ for all $w \in \text{Adm}(\mu)$ if and only if $m(w)$ is a polynomial of degree $\leq \ell(\mu) - \ell(w)$, for all such w .

Now suppose μ is either a minuscule coweight, or an arbitrary coweight for the group GL_n . In [H3] and [HP] it is proved that for any coweight $\nu \in \Omega(\mu)$, the function Θ_ν has a minimal expression. Therefore by Proposition 7.6, we see that

$$\Theta_\nu(w) = q_w^{-1/2} R(Q),$$

where $R(Q) \in \mathbb{N}[Q]$, and $\deg_Q R(Q) \leq \ell(t_\nu) - \ell(w)$. From this it is easy to see (via the Kottwitz conjecture), that $\text{Tr}(\text{Fr}_q, R\Psi_w)$ is a polynomial in q of degree $\leq \ell(\mu) - \ell(w)$.

We remark that the conclusion also holds when every dominant coweight $\lambda \leq \mu$ is a sum of dominant minuscule coweights.

Some time after this article had been submitted, we found a proof which works without using minimal expressions and hence works without the above restriction on G or μ [GH]. Furthermore, it follows from recent results of Ram [Ra] that in fact minimal expressions always exist; hence the proof above is applicable to the general case.

Proof of (B).

Recall that $DR\Psi = R\Psi(\ell(\mu))$. Using Lemma 4.5, we deduce immediately that the multiplicity function $m(R\Psi, w) := \sum_i m(R\Psi, w, i)q^i$ is *palindromic*, i.e., $\overline{m(R\Psi, w)} = q_\mu^{-1}q_w m(R\Psi, w)$.

The multiplicities being palindromic is also a consequence of the conjecture that the monodromy and the weight filtration on nearby cycles agree. In the function field case this is known to be true (cf. [BB] §5); since the multiplicities in the unequal characteristic case and in the function field case coincide, we get the same consequences in the unequal characteristic case. In particular this proves also that the multiplicities increase to the middle, and then decrease again.

Proof of (C) in the minuscule case.

It is easy to see that, for minuscule μ , $m(w, l(\mu) - l(w)) = 1$ for all w because then $\text{Tr}(\text{Fr}_q, R\Psi_w)$ is essentially an R -polynomial ([H3], [HP]), and thus the coefficient of $q^{l(\mu)-l(w)}$ in $\text{Tr}(\text{Fr}, R\Psi_w)$ is ε_w .

This also implies $m(w, 0) = 1$ for all w , since the multiplicities are palindromic.

If μ is minuscule, the identity $m(x, 0) = 1$ for all $x \in \text{Adm}(\mu)$ is equivalent to the following combinatorial identity for such x :

$$\sum_{\substack{w \in \text{Adm}(\mu) \\ w \geq x}} \varepsilon_w = \varepsilon_\mu.$$

This has certain geometric consequences. For example, it shows that any codimension 1 Iwahori-orbit in a local model is contained in at least two irreducible components. It follows from this and properties of the Bruhat order that the smooth locus of a local model is precisely the union of the extreme strata. Consequently, we get the analogous results for certain Shimura varieties. (See [GN] for related results, proved by a different method.)

Proposition 9.3 *Let Sh be a Shimura variety as in section 8. Then the smooth locus of the special fiber of Sh is precisely the union of the Kottwitz-Rapoport strata indexed by the translation elements in the set $\text{Adm}(\mu)$. Here μ is the minuscule coweight in the Shimura datum for Sh .*

10 Appendix

Part I: Nearby cycles preserve mixedness

Due to fundamental results of Deligne [Weil2], it is known that nearby cycles preserve mixedness in the function-field setting (i.e., when working over a trait which is the Henselization at a point of a smooth curve over a finite field). The aim of this appendix is to prove an analogous result in the setting of unequal characteristic traits. This result applies to nearby cycles on local models and integral models for Shimura varieties over p -adic number rings, ensuring that they possess weight filtrations, as postulated in the introduction.

Let (S, s, η) denote a Henselian trait (i.e, the spectrum of a complete discrete valuation ring). We assume that the residue field $k(s)$ is finite. Let $k(\bar{\eta})$ be a separable closure of $k(\eta)$, and let \bar{S} denote the normalization of S in $\bar{\eta}$, with closed point \bar{s} . Let $X \rightarrow S$ be a separated, finite-type scheme over S .

We will use the notion of S -variety (cf. [deJ2], 2.15). A separated finite-type S -scheme X will be called an S -variety if $X \rightarrow S$ is flat and X is integral.

For $A \in D_c^b(X_\eta, \bar{\mathbb{Q}}_\ell)$, define $R\Psi^X(A) = \bar{i}^* R\bar{j}_* A_{\bar{\eta}}$, an object in $D_c^b(X \times_s \eta, \bar{\mathbb{Q}}_\ell)$.

Let Λ_η denote the constant ℓ -adic sheaf $\bar{\mathbb{Q}}_\ell$ on X_η (or some λ -adic constant sheaf E_λ , for $\lambda|\ell$) We will prove the following theorem.

Theorem 10.1 *Suppose that X_η is smooth. Then the complex $R\Psi^X(\Lambda_\eta)$ is mixed.*

The proof of the theorem consists of several steps which eventually reduce it to the calculations of Rapoport-Zink ([RZ1], [I]) for the case of a proper strictly semi-stable S -variety (for the precise definition of *strictly semi-stable*, see [deJ2], 2.16). We will use de Jong's alterations [deJ2] to reduce the general case to that case. Roughly, if X/S is a proper S -variety, one takes a strictly semi-stable alteration $\pi : X' \rightarrow X$, and then applies the push-forward functor π_* to appropriate nearby cycles on the alteration. Thanks to Rapoport-Zink and properties of the push-forward functor, the resulting complex is mixed. One proves using the trace map that it contains the original nearby cycles as a subobject, thus essentially completing the proof of the theorem in that case. (An extra difficulty arises because in the end we have to descend from a finite extension $k(s')$ back down to $k(s)$; see Corollary 10.5.) It is convenient to work with perverse sheaves throughout this process. In particular, we will work with the perverse push-forward ${}^p\pi_*$.

10.1 Alterations

The following theorem is a direct consequence of [deJ2], Theorem 6.5.

Theorem 10.2 (de Jong) *Let X/S be an S -variety. Then there exists a trait $S' = (S', s', \eta')$ finite over S , an S' -variety X' for which there is an alteration $X' \rightarrow X$ of S -varieties, and an open immersion $j : X' \hookrightarrow \overline{X'}$ of S' -varieties such that $\overline{X'}$ is a proper strictly semi-stable S' -variety:*

$$\begin{array}{ccccc} \overline{X'} & \xleftarrow{j} & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ S' & \xleftarrow{\text{id}} & S' & \longrightarrow & S. \end{array}$$

10.2 Proof for X'/S'

Let X', S' etc. be as in Theorem 10.2. The special fiber $\overline{X'}_{s'}$ is globally a union of reduced divisors with normal crossings and the calculations of Rapoport-Zink are valid (cf. [I], Th. 3.2). The sheaves $R^q \Psi^{X'}(\Lambda_{\eta'})$ are explicitly computed in loc. cit, and are easily seen to be mixed. Thus Theorem 10.1 holds in this case.

This implies the theorem for X'/S' . By Theorem 10.2 the morphism j embeds X' as an open subscheme in $\overline{X'}$. Then the formula

$$j^* R\Psi^{\overline{X'}}(\Lambda_{\eta'}) = R\Psi^{X'}(\Lambda_{\eta'})$$

shows that the right hand side is mixed.

10.3 Proof for X_η smooth and geometrically integral

Assume that X/S is finite-type and separated, with X_η smooth and geometrically integral. The nearby cycles are supported on the scheme-theoretic closure of X_η in X_s . We may therefore assume X/S is flat by replacing X with the scheme-theoretic closure of X_η in X (which is flat with the same generic fiber since X_η is reduced). Thus, we may assume X/S is an S -variety whose generic fiber is smooth and geometrically integral.

Let S' and X' be as in Theorem 10.2.

Let us denote by $\pi : X' \rightarrow X_{S'}$ the resulting alteration of S' -varieties. Let us also denote by $\pi : X'_{\eta'} \rightarrow X_{\eta'}$ the morphism on the generic fibers. This is an alteration, hence by definition is generically finite, say of degree n . Since $X_{\eta'}$ is integral, π is also generically flat. More precisely, there exists a non-empty open subset $U \subset X_{\eta'}$ such that $\pi : \pi^{-1}(U) \rightarrow U$ is flat and finite.

Let us denote ${}^p\pi^* := {}^pH^0\pi^*$ (resp. ${}^p\pi_* := {}^pH^0\pi_*$), the perverse versions of the pull-back and push-forward functors via π . Since π is proper, we have $\pi_* = \pi_!$ and so ${}^p\pi_* = {}^p\pi_!$.

Let $\Lambda_{\eta'}^X$ denote the constant sheaf on $X_{\eta'}$, and let $\Lambda_{\eta'}^{X'}$ denote the constant sheaf on $X'_{\eta'}$. For $d = \dim(X_\eta)$, let $A^X = \Lambda_{\eta'}^X[d](\frac{d}{2})$, and $A^{X'} = \Lambda_{\eta'}^{X'}[d](\frac{d}{2})$; the smoothness of $X'_{\eta'}$ (resp. $X_{\eta'}$) ensures that $A^{X'}$ (resp. A^X) is perverse of weight zero. We will need the following lemma.

Lemma 10.3 *The perverse sheaf A^X is a subobject of ${}^p\pi_*(A^{X'})$, in the category $P(X_{\eta'}, \overline{\mathbb{Q}}_\ell)$.*

Proof. The adjunction morphism $A^X \rightarrow \pi_*\pi^*A^X$ and the perversity of A^X yield a morphism of perverse sheaves $A^X \rightarrow {}^p\pi_*A^{X'}$. Since A^X is simple, it is sufficient to prove this morphism is not the zero morphism. This can be detected by showing that for some dense open $U \subset X_{\eta'}$, the restriction of the morphism to U is nonzero. Now choose U over which π is flat and finite, and use the trace map

$$\mathrm{Tr} : (\pi_!\pi^*(\Lambda_{\eta'}^X))|_U \rightarrow (\Lambda_{\eta'}^X)|_U,$$

which composed with the canonical adjunction map $\Lambda_{\eta'}^X \rightarrow \pi_*\pi^*(\Lambda_{\eta'}^X) = \pi_!\pi^*(\Lambda_{\eta'}^X)$ is just multiplication by the degree n . For the definition and basic properties of this trace map, see [SGA 4, Exp. XVII, p. 553-554]. \square

Now let $r : X_{S'} \rightarrow X$ denote the projection morphism. Since r arises from a finite ‘‘change of trait’’ morphism, the invariance of nearby cycles under change of trait ([SGA 4 1/2, Thm. finitude 3.7]) implies that we have a canonical isomorphism $r^*R\Psi^X(\Lambda_\eta[d](\frac{d}{2})) = R\Psi^{X_{S'}}(A^X)$ in the category $D_c^b(X \times_s \eta', \overline{\mathbb{Q}}_\ell)$. Here the meaning of this last category and of the left hand side of the equality is the following. The nearby cycles $R\Psi^X(\Lambda_\eta[d](\frac{d}{2}))$ live on the geometric special fiber $X_{\bar{s}} = X_{\bar{s}'}$, and naturally carry a compatible action of $\mathrm{Gal}(\bar{\eta}/\eta)$. The sheaf $r^*R\Psi^X(\Lambda_\eta[d](\frac{d}{2}))$ is just that same sheaf together with the compatible action of $\mathrm{Gal}(\bar{\eta}'/\eta')$ via the canonical restriction map $\mathrm{Gal}(\bar{\eta}'/\eta') \rightarrow \mathrm{Gal}(\bar{\eta}/\eta)$. By Lemma 10.3, this is a subobject (in the category $P(X \times_{s'} \eta', \overline{\mathbb{Q}}_\ell)$) of

$$R\Psi^{X_{S'}}({}^p\pi_*(A^{X'})) = {}^p\pi_*R\Psi^{X'}(A^{X'}).$$

(The last equality follows from the compatibility of nearby cycles and proper push-forwards together with the fact that $R\Psi$ commutes with the perverse truncation functors ${}^p\tau_{\leq 0}$ and ${}^p\tau_{\geq 0}$, hence with the functor ${}^pH^0$. The compatibility of $R\Psi$ and the perverse truncation functors follows easily from the fact that $R\Psi$ preserves ${}^pD_c^{\leq 0}$ and commutes with Verdier duality.)

We have used here the basic fact that the functor $R\Psi$, being t -exact, induces an exact functor on categories of perverse sheaves.

We can now complete the argument. By the case of X'/S' treated above, the complex $R\Psi^{X'}(A^{X'})$ is mixed. Thus by [BBD] §5, ${}^p\pi_*R\Psi^{X'}(A^{X'})$ is mixed. Its

subobject $r^*R\Psi^X(\Lambda_\eta[d](\frac{d}{2}))$ is mixed too. By the lemma and corollary in the following section on descent, we deduce finally that $R\Psi^X(\Lambda_\eta)$ is also mixed. This proves Theorem 10.1 in the case of finite-type separated S -schemes X where X_η is smooth and geometrically integral.

10.4 Proof for X_η smooth

We consider finally the general case, where X_η is only assumed smooth. Since $X_{\bar{\eta}}$ is smooth, its connected components are integral. If η'/η is a finite extension over which all the connected components are defined, then it follows that each connected component of $X_{\eta'}$ is smooth and geometrically integral. Let S' denote the normalization of S in η' ; then S' is a trait, finite over S . As above let $r : X_{S'} \rightarrow X$ denote the projection. For each connected component $X_{\eta'}^{(\alpha)}$, $\alpha = 1, 2, \dots, m$, let $X^{(\alpha)}$ denote its scheme-theoretic closure in $X_{S'}$. We have closed immersions of S' -schemes $i_\alpha : X^{(\alpha)} \hookrightarrow X_{S'}$. Each $X^{(\alpha)}$ is a separated finite-type S' -scheme whose generic fiber is smooth and geometrically integral; hence the results of the previous section obtain for it.

It is clear that

$$R\Psi^{X_{S'}}(\Lambda_{\eta'}) = \bigoplus_{\alpha} i_{\alpha,*} R\Psi^{X^{(\alpha)}}(\Lambda_{\eta'});$$

by the previous section the right hand side is mixed, hence so is the left hand side. Further, as before, $r^*R\Psi^X(\Lambda_\eta) = R\Psi^{X_{S'}}(\Lambda_{\eta'})$ is just the complex $R\Psi^X(\Lambda_\eta)$ endowed with its compatible action of $\text{Gal}(\bar{\eta}'/\eta')$ via $\text{Gal}(\bar{\eta}'/\eta') \rightarrow \text{Gal}(\bar{\eta}/\eta)$. Now Corollary 10.5 finally proves that $R\Psi^X(\Lambda_\eta)$ is mixed, as desired.

10.5 Descent step

The following results are well-known (cf. [KW], p.14); we include them for the reader's convenience.

Lemma 10.4 *Let $r : S' \rightarrow S$ denote a finite surjective morphism of traits with finite residue fields. Let X/S be finite-type, and by abuse denote also by r the morphism $X_{S'} \rightarrow X_s$ on special fibers deduced from $r \times_S \text{id}_X : X_{S'} \rightarrow X$.*

If \mathcal{G} is a smooth étale Weil sheaf on X_s such that $r^\mathcal{G}$ is a mixed sheaf on $X'_{s'}$ with weights $\leq w$, then \mathcal{G} is also mixed with weights $\leq w$.*

We suppose $k(s)$ has q elements. As noted above in section 10.3, because $X_{\bar{s}} = X_{\bar{s}'}$, we can regard \mathcal{G} as a sheaf on $X_{\bar{s}}$ endowed with a compatible action of a geometric Frobenius $\text{Frob}_q \in \text{Gal}(\bar{\eta}/\eta)$, and we can regard $r^*\mathcal{G}$ as this same sheaf endowed with a compatible action of a geometric Frobenius in $\text{Gal}(\bar{\eta}'/\eta')$ which maps to $\text{Frob}_q^n = \text{Frob}_{q^n}$ under the restriction map $\text{Gal}(\bar{\eta}'/\eta') \rightarrow \text{Gal}(\bar{\eta}/\eta)$, where $n = [k(s') : k(s)]$.

Proof. Let $X_0 = X_s = s \times_S X$. Note that the “change of trait” morphism r above is the morphism

$$s' \times_{S'} X' \rightarrow s \times_S X$$

which arises via the canonical identifications $s' \times_{S'} (S' \times_S X) = s' \times_S X = s' \times_s X_s$ from the projection

$$r : X_0 \times_\kappa \kappa' \rightarrow X_0,$$

where $X_0 = X_s$, $\kappa = \mathbb{F}_q$, and $\kappa' = \mathbb{F}_{q^n}$.

The sheaf $r^*\mathcal{G}$ is by hypothesis a smooth mixed Weil sheaf on $X_0 \times_\kappa \kappa'$. We want to prove that \mathcal{G} is a mixed Weil sheaf on X_0 .

Since r^* is exact, we can assume without loss of generality that \mathcal{G} is simple. We will prove that in this case \mathcal{G} is pure.

Since $r^*\mathcal{G}$ is mixed, there is a non-zero smooth Weil sheaf \mathcal{F} on $X_0 \times_\kappa \kappa'$ which is a simple (thus pure) Weil subsheaf of $r^*\mathcal{G}$.

Let now F denote the Frobenius automorphism Fr_q for X_0 , so that F^n is the Frobenius automorphism for $X_0 \times_\kappa \kappa'$. We are given an isomorphism $(F^n)^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. Using this and the fact that the category of Weil sheaves is closed under the formation of kernels and cokernels, one can show that

$$\sum_{i=0}^{n-1} (F^i)^*\mathcal{F}$$

is a nonzero Weil subsheaf of \mathcal{G} . Since the latter is simple, we have in fact

$$\bigoplus_{i \in I} (F^i)^*\mathcal{F} = \mathcal{G},$$

where $I \subset \{0, 1, \dots, n-1\}$ is a nonempty subset. (The sum can be made a direct sum by omitting certain i 's.)

Now fix any isomorphism $\tau : \bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. Note that the summands above are pure Weil sheaves on $X_0 \times_\kappa \kappa'$, whose τ -weights are all identical. Suppose each is τ -pure of weight w . For any geometric point \bar{x} over a closed point $x \in X_0$, we set $F_x = F^{[\kappa(x):\kappa]}$; then for any eigenvalue α of F_x^n on an irreducible summand of $\mathcal{G}_{\bar{x}}$, we have $|\tau(\alpha)| = q^{nd(x)w/2}$. A similar statement thus holds for F_x replacing F_x^n and $q^{wd(x)/2}$ replacing $q^{wd(x)n/2}$. This holds for every choice of x , and w is independent of that choice. It follows that \mathcal{G} is pure of weight w . \square

Corollary 10.5 *Let r, X be as above. Let $\mathcal{G} \in D_c^{b, \text{Weil}}(X_s, \bar{\mathbb{Q}}_\ell)$. If $r^*\mathcal{G}$ is a mixed complex in $D_c^{b, \text{Weil}}(X_{s'}, \bar{\mathbb{Q}}_\ell)$, then \mathcal{G} is also mixed.*

Proof. By passing to cohomology sheaves, we can reduce to the case where \mathcal{G} and $r^*\mathcal{G}$ are constructible Weil sheaves on X_s and $X_{s'}$, respectively. By Noetherian induction and gluing of constructible sheaves, we can reduce further to the case where \mathcal{G} and thus $r^*\mathcal{G}$ is a smooth Weil sheaf, where the result was proved in the foregoing lemma. \square

10.6 A variant

The following variant applies to the nearby cycles appearing in the local models and deformations of affine flag varieties treated in this paper. It does not apply to Shimura varieties: while the generic fibers of these are smooth, they are usually not geometrically integral.

Theorem 10.6 *Suppose X_η is geometrically integral, with intersection complex ${}^0IC(X_\eta)$. Then $R\Psi^X({}^0IC(X_\eta))$ is mixed.*

Here, ${}^0IC(X_\eta)$ denotes the intersection complex of X_η , suitably Tate-twisted and shifted to make it a pure self-dual perverse sheaf of weight zero. This normalization is convenient but of course not necessary for the theorem.

Proof. The proof works the same way as before. First, replacing X with the scheme-theoretic closure of X_η , we may assume X/S is flat, hence integral, i.e., we may assume X is an S -variety with geometrically integral generic fiber.

Now consider the alterations $\pi : X' \rightarrow X_{S'}$ and $\pi : X'_{\eta'} \rightarrow X_{\eta'}$ associated to the diagram in Theorem 10.2. Let $r : X_{S'} \rightarrow X$ denote the projection morphism. Arguing almost as before (see the Lemma below) we can use the trace map to prove that ${}^0IC(X_{\eta'}) = {}^0IC(X_\eta)_{\eta'}$ is a subquotient of ${}^p\pi_* A^{X'}$. By invariance of nearby cycles under change of traits, $r^* R\Psi^X({}^0IC(X_\eta)) = R\Psi^{X_{S'}}({}^0IC(X_{\eta'}))$, which is a subquotient of $R\Psi^{X_{S'}}({}^p\pi_* A^{X'}) = {}^p\pi_* R\Psi^{X'}(A^{X'})$. The latter is mixed by section 10.2. Applying descent as in section 10.5, the result follows. \square

Lemma 10.7 *The perverse sheaf ${}^0IC(X_{\eta'})$ is a subquotient of ${}^p\pi_*(A^{X'})$, in the category $P(X_{\eta'}, \overline{\mathbb{Q}}_\ell)$.*

Proof. Choose the open set $U \subset X_{\eta'}$ as in Lemma 10.3. Let $j' : U' := \pi^{-1}(U) \rightarrow X'_{\eta'}$ be the open immersion. Note that $IC := {}^0IC(X_{\eta'})$ is just $j_{!*} A^U$, and $A^{X'} = j'_{!*} A^{U'}$, where A^U , resp. $A^{U'}$ has the obvious meaning.

By using adjunction maps and the morphism of functors $j_{!*} \rightarrow j_*$, we get morphisms

$$IC \rightarrow {}^p\pi_* \pi^* j_{!*} A^U \rightarrow {}^p\pi_* \pi^* j_* A^U \rightarrow {}^p\pi_* j'_{!*} \pi^* A^U = {}^p\pi_* j'_* A^{U'}$$

and

$${}^p\pi_* A^{X'} = {}^p\pi_* j'_{!*} A^{U'} \rightarrow {}^p\pi_* j'_* A^{U'}$$

Now restricting to U with j^* and using the trace map as before shows that the first map is non-zero, hence a monomorphism. Also, restricting shows that the image of IC under the first map lies in the image of the second map (if not, then IC embeds into the cokernel of the second map, which is ruled out after restricting both to U). Thus, IC is a subquotient of ${}^p\pi_* A^{X'}$. \square

Part II: A bound on the weights of nearby cycles

The method of proof of Theorem 10.6 gives an estimate on the weights:

$$w(R\Psi^X({}^0IC(X_\eta))) \leq \dim(X_\eta).$$

(One is reduced to the proper strictly semi-stable case, where it is easy).

Now let $R\Psi = R\Psi(IC(\bar{\mathcal{Q}}_\mu)[\ell(t_\mu)])$, as usual.

The fact (proved in section 7.2) that for μ either minuscule, or a coweight for GL_n , the function $\text{Tr}(\text{Fr}_q, R\Psi_x)$ is polynomial in q of degree $\leq \ell(\mu) - \ell(x)$, is equivalent to the following much stronger bound on the weights of ${}^0R\Psi := R\Psi(\frac{\ell(\mu)}{2})$: for each $x \in \text{Adm}(\mu)$, and each closed point $z \in \mathcal{B}_x$, the weights of the complex ${}^0R\Psi_z$ are $\leq \ell(\mu) - \ell(x)$. In all the examples we computed, it is true that the trace of Frobenius is always a polynomial in q with this degree. In all the cases we have proved this (thus far), it follows from the existence of minimal expressions for the functions Θ_λ in the Iwahori-Hecke algebra (cf. section 9.4). It would be interesting to find a direct geometric argument to prove this sharper bound on the weights in all cases.⁴

The following gives a sufficient condition for the sharp bound to hold. In this statement, we assume there is the action of a geometrically connected smooth group scheme G/S on X/S , and we say a $G_{\bar{s}}$ -equivariant Weil perverse sheaf \mathcal{K} on $X_{\bar{s}}$ satisfies the sharp bound if for each locally closed immersion $i : \mathcal{O} \rightarrow X_{\bar{s}}$ of a $G_{\bar{s}}$ -orbit \mathcal{O} , we have $w(i^*\mathcal{K}) \leq \dim(X_{\bar{s}}) - \dim(\mathcal{O})$.

Proposition 10.8 *Let X/S be an S -variety such that X_η is geometrically integral. Suppose that X/S carries an action of a geometrically connected smooth group scheme G/S . Suppose that there exists a trait S' , finite over S , and an S -alteration $X' \rightarrow X$ with compactification \bar{X}' as in Theorem 10.2, each carrying an action of $G_{S'}$, such that the inclusion $j : X' \hookrightarrow \bar{X}'$ and the S' -alteration $\pi : X' \rightarrow X_{S'}$ are $G_{S'}$ -equivariant. Then $R\Psi^X({}^0IC(X_\eta))$ satisfies the sharp bound on its weights.*

Proof. This is similar to the proof of Theorem 10.6. The sharp bound holds for the strictly semi-stable case, and thus also for $R\Psi^{X'}(A^{X'})$. It is enough to prove that the sharp bound holds for ${}^p\pi_1 R\Psi^{X'}(A^{X'})$. So, let $\mathcal{O} \subset X_{S'}$ denote a $G_{S'}$ -orbit of dimension m , and suppose $z \in \mathcal{O}$ is a closed point. Using equivariance, one can show that any $G_{S'}$ -orbit contained in $\pi^{-1}(\mathcal{O})$ dominates \mathcal{O} , hence has dimension $\geq m$. Thus, the weights of $\pi_1 R\Psi^{X'}(A^{X'})_z = R\Gamma_c(\pi^{-1}(z), R\Psi^{X'}(A^{X'}))$ are $\leq \dim(X'_{S'}) - m$, and we are done. \square

There are \mathcal{B} -equivariant semi-stable resolutions for the local models $\mathbf{M}^{\text{loc}} = M_\mu$, in the cases $GS p_4$, $\mu = (1, 1, 0, 0)$ [deJ1], and $GS p_6$, $\mu = (1, 1, 1, 0, 0, 0)$ [Ge]. Together with the above considerations, we get the following corollary.

⁴In the meantime, we have found such a proof, see [GH].

Corollary 10.9 *Let μ be the minuscule coweight $(1, 1, 0, 0)$ for GSp_4 or the minuscule coweight $(1, 1, 1, 0, 0, 0)$ for GSp_6 . Then the sharp bound holds for the weights of ${}^0R\Psi$. In particular, in each of these cases the multiplicity functions*

$$\sum_i m(R\Psi, w, i)q^i$$

are polynomials of degree at most $\ell(\mu) - \ell(w)$, for every $w \in \text{Adm}(\mu)$.

This Corollary also follows from results already announced in section 9, but the approach given here is perhaps more conceptual.

Acknowledgements This work was initiated during our visits to the Institute for Advanced Study in 2000-2001. We are grateful to the IAS for its hospitality and financial support. The research of the second author was also partially supported by NSF grant DMS-0303605.

References

- [AB] S. Arkhipov, R. Bezrukavnikov, *Perverse sheaves on affine flags and Langlands dual group*, Preprint math.RT/0201073 (2002).
- [BB] A. Beilinson, J. Bernstein, *A proof of Jantzen conjectures*, Adv. Soviet Math. **16**, Part 1, (1993), 1-50.
- [BBD] A. Beilinson, I.N. Bernstein, P. Deligne, *Faisceaux Pervers*, Astérisque **100**, (1981).
- [Del] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, Antwerp II, Springer Lecture Notes in Math. **349**, 1973, 501–595.
- [deJ1] A. de Jong, *The moduli spaces of principally polarized abelian varieties with $\Gamma_0(p)$ -level structure*, J. Algebraic Geom. **2** (1993), 667-688.
- [deJ2] A. de Jong, *Smoothness, semi-stability and alterations*, Publ. Math. I.H.E.S, **83**, (1996) 51-93.
- [Weil2] P. Deligne, *La conjecture de Weil II*, Publ. Math. IHES **52**, 137–252 (1980)
- [F] G. Faltings, *Algebraic loop groups and moduli spaces of bundles*, J. Eur. Math. Soc. (JEMS) **5** (2003), no. 1, 41–68.
- [Ga] D. Gaitsgory. *Construction of central elements in the affine Hecke algebra via nearby cycles*, Invent. Math. **144** (2001), 253-280.
- [Ge] A. Genestier, *Un modèle semi-stable de la variété de Siegel de genre 3 avec structures de niveau de type $\Gamma_0(p)$* , Compositio Math. **123** (2000), no. 3, 303-328.
- [GN] A. Genestier, B.-C. Ngô, *Alcoves et p -rang des variétés abéliennes*, Ann. Inst. Fourier **52** (2002), no. 6, 1665–1680.
- [Gi] V. Ginzburg, *Perverse sheaves on a loop group and Langlands duality*, Preprint math.AG/9511007 (1995).
- [GM] M. Goresky and R. MacPherson, *Intersection Homology II*, Invent. Math. **71** (1983), 77-129.
- [G1] U. Görtz, *On the flatness of local models for certain Shimura varieties of PEL-type*, Math. Ann. **321** (2001), 689–727.
- [G2] U. Görtz, *On the flatness of local models for the symplectic group*, Adv. in Math. **176** (2003), 89–115.
- [GH] U. Görtz, T. Haines, *Bounds on weights of nearby cycles and Wakimoto sheaves on affine flag manifolds*, to appear in manuscripta math., math.AG/0502499

- [Had] Z. Haddad, *A Coxeter group approach to Schubert varieties* in: Infinite dimensional groups with applications, Publ., Nath. Sci. Res. Inst. **4**, 157–165 (1985).
- [H1] T. Haines, *The combinatorics of Bernstein functions*, Trans. Amer. Math. Soc. **353**, No. 3, 1251-1278 (2001).
- [H2] T. Haines, *Drinfeld-Plücker relations for homogeneous spaces, (affine) flag varieties, and Rapoport-Zink models*, In preparation.
- [HKP] T. Haines, R. Kottwitz, and A. Prasad, *Iwahori-Hecke Algebras*, math.RT/0309168.
- [HN1] T. Haines, B. C. Ngô, *Nearby cycles for local models of some Shimura varieties*, Compositio Math. **133** (2002), 117-150.
- [HN2] T. Haines, B.C. Ngô, *Alcoves associated to special fibers of local models*, Amer. J. Math. **124** (2002), 1125-1152.
- [HP] T. Haines, A. Pettet, *Formulae relating the Bernstein and Iwahori-Matsumoto presentations of an affine Hecke algebra*, J. of Algebra **252** (2002), 127-149.
- [H3] T. Haines *Test functions for Shimura varieties: the Drinfeld case*, Duke Math. J. **106** (2001), 19-40.
- [I] L. Illusie, *Autour du théorème de monodromie locale*, Astérisque **223** (1994), 9-57.
- [IM] N. Iwahori, H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p -adic Chevalley groups*, Inst. Hautes Études Sci. Publ. math. **25** (1965), 5-48.
- [KL1] D. Kazhdan, G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.
- [KL2] D. Kazhdan, G. Lusztig, *Schubert varieties and Poincaré duality*, Proc. Symp. Pure Math. **36** (1980) 185–203.
- [KT] M. Kashiwara, T. Tanisaki, *Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebra. II. Intersection cohomologies of Schubert varieties*, in: Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris 1989). Progr. Math. **92**, Birkhäuser 1990.
- [KW] R. Kiehl, R. Weissauer, *Weil conjectures, Perverse Sheaves and ℓ -adic Fourier transform*, Springer Erg. Math. 3. Folge, **42** (2001).
- [Ko] R. Kottwitz, *Points on some Shimura varieties over finite fields*, J. AMS **5** (1992), 373–444.
- [L1] G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. **2** (1989), 599–635.

- [L2] G. Lusztig, *Singularities, characters formulas and a q -analogue of weight multiplicities* in: *Analyse et topologie sur les espaces singuliers*, *Astérisque* **101-102** (1983), 200-229.
- [L3] G. Lusztig, *Cells in Affine Weyl Groups and Tensor Categories*, *Adv. Math* **129** (1997), 85–98.
- [Ma] B. Mazur, *Notes on étale cohomology of number fields*, *Ann. Sci. Ecole Norm. Sup.* **6** (1973), 521-556.
- [Mi] J.S. Milne, *Étale Cohomology*, Princeton Univ. Press, 1980.
- [MV] I. Mirkovic, K. Vilonen, *Perverse sheaves on loop grassmannians and Langlands duality*, *Math. Res. Lett.* **7** (2000), no. 1, 13–24.
- [NP] B. C. Ngô, P. Polo, *Résolutions de Demazure affines et formule de Casselman-Shalika géométrique*, *J. Algebraic Geom.* **10** (2001), no. 3, 515–547.
- [Ra] A. Ram, *Alcove walks, Hecke algebras, spherical functions, crystals and column strict tableaux*, math.RT/0601343
- [RZ1] M. Rapoport, Th. Zink, *Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, *Invent. Math.* **68** (1982), 21-101.
- [RZ2] M. Rapoport, T. Zink, *Period domains for p -divisible groups*, *Ann. of Math. Stud.* **141**, Princeton University Press (1996).
- [SGA 1] A. Grothendieck et. al., *Revêtements étales et groupe fondamental*, *Séminaire de Géométrie Algébrique 1960–61*, Springer Lecture Notes in Math. **224** (1971).
- [SGA 5] A. Grothendieck, *Cohomologie ℓ -adique et fonctions L* , *Séminaire de Géométrie Algébrique 1965–66*, Springer Lecture Notes in Math. **589** (1977).
- [SGA 7] A. Grothendieck et. al./P.Deligne, N. Katz, *Groupes de monodromie en géométrie algébrique*, *Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969*. Springer Lecture Notes in Math. **288** (1972), **340** (1973).
- [S] O. Schiffmann, *On the center of affine Hecke algebras of type A* , *J. of Algebra* **253** (2002), no. 2, 392–416.
- [ST] J.-P. Serre, J. Tate, *Good reduction of abelian varieties*, *Ann. of Math.* (2) **88** (1968), 492-517.

Thomas Haines: Mathematics Department, University of Maryland, College Park, MD 20742-4015, USA. Email: tjh@math.umd.edu