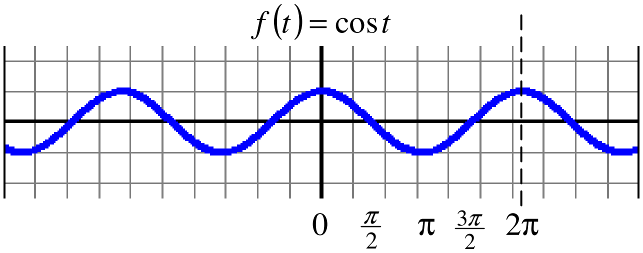
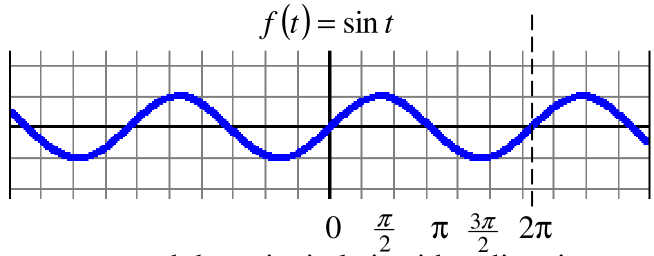
Calculus 130, section 4.6 Derivatives of Trigonometric Functions

notes prepared by Tim Pilachowski

Both $f(t)=\cos t$ and $f(t)=\sin t$ are *periodic functions*, with period equal to 2π. The graphs are pictured below:





The domain of both functions is all real numbers, since we can go around the unit circle in either direction as many times as we want. The range of each is $-1 \le y \le 1$. Recalling that $\cos\left(\frac{\pi}{2}-t\right)=\sin t$ and $\sin\left(\frac{\pi}{2}-t\right)=\cos t$, note that the two graphs are the same shape, just shifted over by $\frac{\pi}{2}$. Examples of applications which sometimes use sine and cosine to model periodic behavior include: temperature fluctuations, tides, seasonal sales, regular breathing, blood pressure (systolic and diastolic), circadian rhythms, and populations of migratory animals.

The graph of sin t has a positive slope at $t = 0$, until slope = 0 at $t = \frac{\pi}{2}$, then a negative slope, until slope = 0 at $t = \frac{3\pi}{2}$, then a positive slope, until slope = 0 at $t = 2\pi+\frac{\pi}{2}=\frac{5\pi}{2}$. In other words, the behavior of the derivative of $y=\sin t$ implies that $\frac{d}{dt}(\sin t)=\cos t$. Likewise, we can match behaviors and state $\frac{d}{dt}(\cos t)=-\sin t$.
The formal proof of these derivative formulae is done in the text, using some identities and theorems which we don't explicitly cover in Math 130. If there's time in Lecture we'll take a look at one of these: $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$. If we don't have time, you can take a look in these notes after Example L.

Example A: Find the first derivative of $f(x)=\cos x+\sin x$. *answer*: $f'(x)=-\sin x+\cos x$

Example B: Differentiate $f(t)=3t^3\cos t$. *answer*: $f'=3t^2(-t\sin t+3\cos t)$

Example C: Given $y = \dfrac{e^{3t}}{\sin t}$, find $\dfrac{dy}{dx}$. *answer*: $y' = \dfrac{e^{3t}(3\sin t - \cos t)}{\sin^2 t}$

Example D: For $y = 5\sin\left(1-t^3\right)$, derive y'. *answer*: $y' = -15t^2 \cos\left(1-t^3\right)$

Example E: Let $f(x) = \cos\sqrt{x}$ and $g(x) = \sqrt{\cos x}$. Find each first derivative.
answers: $f' = -\dfrac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$; $g' = -\dfrac{\sin x}{2\sqrt{\cos x}}$

Example F: Find the equation of the line tangent to $y = 5\sin(4t) - 3\cos(2t)$ at $t = \dfrac{\pi}{2}$. *answer*: $y = 20t + 3 - 10\pi$

Example F extended: Find the slope of the line tangent to $y = 5\sin(4t) - 3\cos(2t)$ at $t = \frac{\pi}{6}$. *answer*: $-10 + 3\sqrt{3}$

Example G: Temperature (F°) during a 24-hour period can be modeled by $T = 72 + 18\sin\left(\frac{\pi(t-8)}{12}\right)$, $t \geq 0$, where $t = 0$ corresponds to midnight. a) Approximate the rate at which temperature is changing at 6 am.
b) When is the temperature at its maximum?

answers: $\frac{3\pi}{2}\cos\left(-\frac{\pi}{6}\right) = \frac{3\pi\sqrt{3}}{4} \approx 4.08°$ per hr; 2 pm

Turning attention now to $y = \tan x$, we use the quotient rule to find the derivative.

$$\tan t = \frac{\sin t}{\cos t} \quad \Rightarrow \quad \frac{d}{dt}[\tan t] = \frac{\cos t(\cos t) - \sin t(-\sin t)}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t$$

Example H: Differentiate $f(t) = t^3 \tan t$. *answer*: $t^2(t\sec^2 t + 3\tan t)$.

Example I: For $y = \tan\left(1 - t^2\right)$, derive y'. *answer*: $-2t\sec^2\left(1 - t^2\right)$.

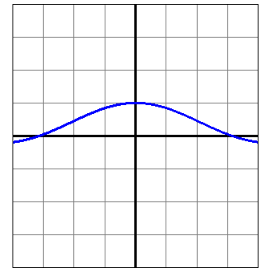
Example J: Let $f(x) = \tan\sqrt{x}$ and $g(x) = \sqrt{\tan x}$. Find each first derivative.
answers: $f' = \dfrac{\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$, $g' = \dfrac{\sec^2 x}{2\sqrt{\tan x}}$

Example K: Find the equation of the line tangent to $y = \tan\left(\dfrac{t}{4}\right)$ at $t = \pi$. *answer*: $y = \dfrac{1}{2}t + 1 - \dfrac{\pi}{2}$.

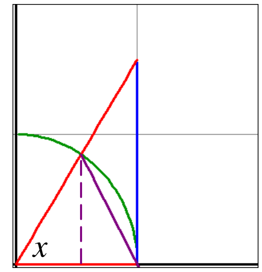
Example L: Find the derivative of $y = \cot(4t)$. *answer*: $y' = \dfrac{-4\sec^2(4t)}{\tan^2(4t)}$.

The only derivative formulae you need to memorize are for $y = \sin x$, $y = \cos x$, and $y = \tan x$.

Example M: Show that $\lim\limits_{x\to 0}\dfrac{\sin}{x}=1$.



From the graph, pictured to the right, the result is intuitively obvious.
For a more formal justification, we begin with the Quadrant I piece of the unit circle, pictured below and to the right. The arc [green] is ¼ of the unit circle. The angle at the origin is x radians. The coordinates of the vertex at the top of the small triangle [purple right side] are therefore (cos x, sin x).



The area of the small triangle $=\frac{1}{2}bh=\frac{1}{2}(1)(\sin x)=\frac{\sin x}{2}$.

The sector of the unit circle from 0 to x radians is the fraction $\frac{x}{2\pi}$ of the full unit circle.

So the area of the sector of the unit circle from 0 to x radians $=\frac{x}{2\pi}*\pi r^2=\frac{x}{2\pi}*\pi\left(1^2\right)=\frac{x}{2}$.

The area of the small triangle is smaller than the area of the sector of the unit circle from 0 to x radians, giving us $\frac{\sin x}{2}\le\frac{x}{2} \quad\Rightarrow\quad \frac{\sin x}{x}\le 1$. ($x\ge 0$; the inequality symbol does not flip.)

The large triangle [blue right side] is similar (technical term) to the right triangle formed by the angle at the origin and the altitude of the small triangle [right side dotted line], so sides of each are proportional. Thus,

$$\frac{\text{height of small right triangle}}{\text{base of small right triangle}}=\frac{\text{height of large triangle}}{\text{base of large triangle}} \quad\Rightarrow\quad \frac{\sin x}{\cos x}=\frac{\text{height of large triangle}}{1}.$$

So the area of the large triangle $=\frac{1}{2}bh=\frac{1}{2}(1)\left(\frac{\sin x}{\cos x}\right)=\frac{\sin x}{2\cos x}$.

The area of the sector of the unit circle from 0 to x radians is smaller than the area of the large triangle, giving us $\frac{x}{2}\le\frac{\sin x}{2\cos x} \quad\Rightarrow\quad \cos x\le\frac{\sin x}{x}$. ($x\ge 0$; the inequality symbol does not flip.)

Putting both inequalities together, we get

$$\frac{1}{\cos x}\le\frac{\sin x}{x}\le 1$$

$$\lim_{x\to 0}\frac{1}{\cos x}\le\lim_{x\to 0}\frac{\sin x}{x}\le\lim_{x\to 0}1$$

$$1\le\lim_{x\to 0}\frac{\sin x}{x}\le 1.$$

By the Squeezing Theorem, $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$.

The text uses this result to show that $\lim\limits_{h\to 0}\frac{\cos h-1}{h}=0$.

Then, using "sum of angle" identities the text shows that $\dfrac{d}{dx}[\sin x]=\lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin(x)}{h}=\cos x$.

Finally, using identities mentioned above, $\sin\left(\dfrac{\pi}{2}-x\right)=\cos x$ and $\cos\left(\dfrac{\pi}{2}-x\right)=\sin x$, the text uses the chain rule to show that $\dfrac{d}{dx}[\cos x]=\dfrac{d}{dx}\left[\sin\left(\dfrac{\pi}{2}-x\right)\right]=-\sin x$.