

# Calculus 131, supplemental sections 11.1–11.2 Logistic Growth

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**Exponential Growth and Decay** In Algebra, you were probably introduced to exponential growth/decay functions. (See 130/131 text section 2.3.) In an exponential growth model,

[rate of change of  $y$ ] is proportional to [current amount].

$$\frac{dy}{dx} = ky$$

Solving via separation of variables (section 11.1) leads to  $y = y_0 e^{kx}$  or  $y = M e^{kx}$ . If  $k > 0$  we have exponential growth. If  $k < 0$  we have exponential decay. The leading coefficient ( $y_0$  or  $M$ ) is the initial amount.

Exponential growth models work when the population can be expected to increase without limitation. One example would be a one-time deposit of money into an account (which could theoretically collect interest forever). Exponential growth models can also be used to describe a population in its beginning stages, before environmental limitations become significant. (Once the environment starts restricting population growth, we'd need to think in terms of *limited growth* functions or *logistic growth functions*).

Exponential decay models can be very useful in life science applications. Examples would include the decay of radioactive isotopes, or a one-time administration of medication which is then metabolized out of the bloodstream.

**Example A:** One injection of 150 mg of a medication is given to a patient. The amount of the drug in the bloodstream decreases by 20% per hour.

Let  $y$  equal the amount of medication in the patient's bloodstream, and  $x$  be time measured in hours. Then the amount of medication is decreasing by 0.2 times the amount present in the bloodstream. That is,

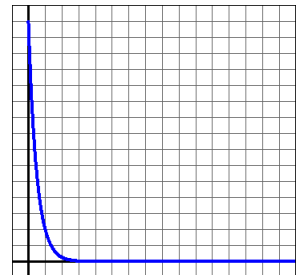
$$\frac{dy}{dx} = -0.2y.$$

Solving via separation of variables, and using the initial condition  $y_0 = 150$ , we would get

$$y = 150 e^{-0.2x}.$$

If we ask, "How much medication is in the patient's bloodstream after a long time?" we can answer by taking a limit.

$$\lim_{x \rightarrow \infty} (150 e^{-0.2x}) = 0.$$



In words: Eventually, all of the medication is absorbed out of the patient's bloodstream and into the patient's body. From the graph to the right, with both axes marked in 10s, we can see that this particular medication is effective for about 24 hours.

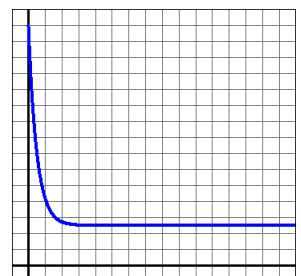
**Example B:** One injection of 150 mg of a medication is given to a patient. In addition, an IV drip is set up to deliver 5 mg per hour. The amount of the drug in the bloodstream decreases by 20% per hour.

Let  $y$  equal the amount of medication in the patient's bloodstream, and  $x$  be time measured in hours. Then at the same time the amount of medication is decreasing by 0.2 times the amount present in the bloodstream, the IV drip is adding 5 mg. That is,

$$\frac{dy}{dx} = -0.2y + 5.$$

Solving via the first order linear process, and using the initial condition  $y_0 = 150$ , we would get

$$y = 25 + 125 e^{-0.2x}.$$



If we ask, “How much medication is in the patient’s bloodstream after a long time?” we can answer by taking a limit.

$$\lim_{x \rightarrow \infty} (25 + 125e^{-0.2x}) = 25.$$

In words: Eventually, the medication reaches an equilibrium level in the patient’s bloodstream at 25 mg, i.e. the amount entering through the IV drip balances the amount being absorbed into the patient’s body. From the graph above, with both axes marked in 10s, we can see that this particular medication reaches equilibrium after about 30 hours.

**Limited Growth Functions** Exponential growth functions grow without bound. In reality, populations will have limitations of space and resources. As time goes on, the population growth rate will slow until the size of the population reaches an equilibrium. Limited growth functions incorporate a maximum population size that is sustainable in a given environment. (See 130/131 text section 2.3.). In a limited growth model, assuming a maximum population size  $N$ ,

[rate of change of  $y$ ] is proportional to [difference between maximum/equilibrium amount and  $y$ ].

$$\frac{dy}{dx} = k(N - y)$$

This DE can be solved via the first order linear process (section 11.2).

Limited growth models apply when population growth can be described in relation to the portion of the environment that remains to be filled. One example would be news spread by media—the information spreads among those who haven’t gotten the news. A life science application would be the spread of a sickness that is not contagious, for example exposure to contaminated food—the illness is contracted by those not yet exposed to the contaminated items. Another model that falls into this category is Newton’s Law of Cooling, referred to in some Math 131 text exercises as a means of determining time of death.

Limited growth models can sometimes be useful in describing a population in its later stages, as the population size is approaching its maximum.

Example C: A local restaurant has 1000 regular customers. On any given day, 25% of those regulars order a breakfast item using eggs in the recipe. The latest shipment of eggs is contaminated with salmonella bacteria.

Let  $y$  equal the number of people who have gotten sick from salmonella, and  $x$  be time measured in days. Then the rate of spread can be described by the DE

$$\frac{dy}{dx} = 0.25(1000 - y).$$

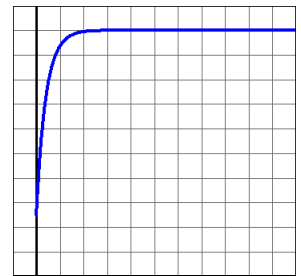
Solving via separation of variables, and using the initial condition  $y(0) = 250$  (25% of 1000), we would get

$$y = 1000 - 750e^{-0.25x}.$$

If we ask, “How many people might get food poisoning from the contaminated eggs?” we can answer by taking a limit.

$$\lim_{x \rightarrow \infty} (1000 - 750e^{-0.25x}) = 1000.$$

In words: Potentially, all 1000 regulars might become ill. From the graph to the right, with the  $x$ -axis marked in 10s and the  $y$ -axis marked in 100s, unless the source of contamination is identified and removed from the kitchen, almost all of the regulars will be sick after about 20 days.



**Logistic Growth Functions** In their beginnings, before environmental limitations become significant, populations will grow in an almost exponential fashion. As time goes on, the population growth rate will slow, in a manner similar to limited growth functions, until the size of the population reaches an equilibrium, a maximum population size that is sustainable in a given environment. A logistic growth function incorporates both of these aspects. (See 130/131 text section 11.1.) Assuming a maximum population size  $N$ ,

[rate of change of  $y$ ] proportional to [ $y$ ] and [difference between equilibrium amount and  $y$ ].

$$\frac{dy}{dx} = \frac{k}{N} y(N - y).$$

In a population scenario, rate of growth is proportional both to the number in the current population (and thus able to produce offspring,  $ky$ ) and to the limitations imposed by environmental factors such as availability of food and the presence of natural predators (the portion of the environment that remains to be filled,  $\frac{N - y}{N}$ ). In this model,  $k$  is the *intrinsic rate of growth* (rate of growth if not limited by outside factors) and  $N$  is called the *carrying capacity* (maximum sustainable population).

Distributing  $\frac{1}{N}$  and using commutativity of multiplication to rearrange gives us the text's version of the logistic growth model,

$$\frac{dy}{dx} = \frac{k}{N} y(N - y) = k \left(1 - \frac{y}{N}\right) y.$$

At the beginning, when  $y$  is near 0,  $\frac{dy}{dx} \approx k(1)y$  (exponential growth) and as  $y \rightarrow N$ ,  $\frac{dy}{dx} \approx k(1 - 1)y = 0$  (like the equilibrium stage of the limited growth model).

The solution method described in section 11.1 exercise #33 uses an integration method called "partial fractions" which we did not cover in Math 131. It is the basis for stating a general solution to a logistic growth DE,

$$\frac{dy}{dx} = k \left(1 - \frac{y}{N}\right) y \Rightarrow y = \frac{N}{1 + be^{-kx}}, \quad b = \frac{N - y_0}{y_0}.$$

Other scenarios that combine a personal contact element and a maximum/equilibrium state include news spread by word of mouth and illnesses spread by contagion (person to person contact)—more contagious people means more sources of infection, but the rate of spread will be slowed down as the number of people who can still be infected decreases.

Example D: Forest Service ecologists have determined that a national forest can sustain a maximum population of 900 deer. When the forest is first designated as a wildlife preserve there are 100 deer. Two years later there are 140 deer.

Let  $y$  equal the number of deer in the population, and  $x$  be time measured in years. From the maximum given, we have  $\frac{dy}{dx} = k \left(1 - \frac{y}{900}\right) y$ . We also have information about growth to use to estimate a value for  $k$ . Since

population increases from 100 to 140 over two years, we can say that when  $y = 140$ ,  $\frac{dy}{dx} \approx 20$  deer per year.

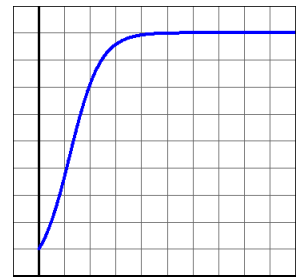
$$\frac{dy}{dx} = k \left(1 - \frac{y}{900}\right) y \Rightarrow 20 \approx k \left(1 - \frac{140}{900}\right) 140 \Rightarrow \frac{20}{(140) \left(1 - \frac{140}{900}\right)} \approx 0.17 \approx k$$

Then we have,

$$\frac{dy}{dx} = 0.17 \left(1 - \frac{y}{900}\right) y \Rightarrow b = \frac{900 - 100}{100} = 8 \Rightarrow y = \frac{900}{1 + 8e^{-0.17x}}.$$

Note that as time passes, the population of deer will reach its equilibrium state,

$$\lim_{x \rightarrow \infty} \left( \frac{900}{1 + 8e^{-0.17x}} \right) = 900.$$



From the graph above, with the  $x$ -axis marked in 10s and the  $y$ -axis marked in 100s, the population of deer will reach its maximum sustainable size after about 40 years.