MAPPING CLASS GROUP DYNAMICS ON SURFACE GROUP REPRESENTATIONS

WILLIAM M. GOLDMAN

ABSTRACT. Deformation spaces $\mathsf{Hom}(\pi, G)/G$ of representations of the fundamental group π of a surface Σ in a Lie group G admit natural actions of the mapping class group Mod_Σ, preserving a Poisson structure. When G is compact, the actions are ergodic. In contrast if G is noncompact semisimple, the associated deformation space contains open subsets containing the Fricke-Teichmüller space upon which Mod_Σ acts properly. Properness of the Mod_Σ-action relates to (possibly singular) locally homogeneous geometric structures on Σ. We summarize known results and state open questions about these actions.

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Introduction

A natural object associated to a topological surface Σ is the *deformation space* of representations of its fundamental group $\pi = \pi_1(\Sigma)$ in a Lie group G. These spaces admit natural actions of the mapping class group Mod_Σ of Σ, and therefore determine linear representations of Mod_Σ.

The purpose of this paper is to survey recent results on the dynamics of these actions, and speculate on future directions in this subject.

The prototypes of this theory are two of the most basic spaces in Riemann surface theory: the Jacobian and the Fricke-Teichmüller space. The *Jacobian* $\mathsf{Jac}(M)$ of a Riemann surface M homeomorphic to Σ identifies with the deformation space $\mathsf{Hom}(\pi, G)/G$ when G is the circle $\mathsf{U}(1)$. The Jacobian parametrizes topologically trivial holomorphic complex line bundles over M, but its topological type (and symplectic structure) are invariants of the underlying topological surface Σ. The action of Mod_Σ is the action of the integral symplectic group $\mathsf{Sp}(2g, \mathbb{Z})$ on the torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$, which is a measure-preserving chaotic (ergodic) action.

In contrast, the *Teichmüller space* $\mathfrak{T}_\Sigma$ (*Fricke space* if $\partial\Sigma \neq \emptyset$) is comprised of equivalence classes of *marked conformal structures* on Σ. A marked conformal structure is a pair (M, f) where f is a homeomorphism and M is a Riemann surface. Marked conformal structures (f_1, M_1) and (f_2, M_2) are *equivalent* if there is a biholomorphism $M_1 \xrightarrow{h} M_2$ such that $h \circ f_1$ is homotopic to f_2. Denote the equivalence class of a marked conformal structure (f, M) by

$$\langle f, M \rangle \in \mathfrak{T}_\Sigma.$$

A marking f determines a representation of the fundamental group:

$$\pi = \pi_1(\Sigma) \xrightarrow{f_*} \pi_1(M) \subset \mathsf{Aut}(\tilde{M}).$$

By the uniformization theorem (at least when $\chi(\Sigma) < 0$), these identify with *marked hyperbolic structures* on Σ, which in turn identify with conjugacy classes of *discrete embeddings* of the fundamental group π in the group $G = \mathsf{PGL}(2, \mathbb{R})$ of isometries of the hyperbolic plane. These classes form a connected component of $\mathsf{Hom}(\pi, G)/G$, which is homeomorphic to a cell of dimension $-3\chi(\Sigma)$ [35]. The mapping class group Mod_Σ acts properly on $\mathfrak{T}_\Sigma$. The quotient orbifold

$$\mathfrak{M}_\Sigma := \mathfrak{T}_\Sigma / \mathsf{Mod}_\Sigma$$

is the *Riemann moduli space,* consisting of biholomorphism classes of (unmarked) conformal structures on Σ.

Summarizing:

- When G is compact, $\mathsf{Hom}(\pi, G)/G$ has nontrivial homotopy type, and the action of the mapping class group exhibits nontrivial dynamics;
- When $G = \mathsf{PGL}(2, \mathbb{R})$ (or more generally a noncompact semisimple Lie group), $\mathsf{Hom}(\pi, G)/G$ contains open sets (like Teichmüller space) which are contractible and admit a proper Mod_Σ-action. Often these open sets correspond to locally homogeneous geometric structures uniformizing Σ.

Thus dynamically complicated mapping class group actions accompany nontrivial homotopy type of the deformation space. In general the dynamics exhibits properties of these two extreme cases, as will be described in this paper.

1. Generalities

Let π be a finitely generated group and G a real algebraic Lie group. The set $\mathsf{Hom}(\pi, G)$ of homomorphisms $\pi \longrightarrow G$ has the natural structure of an affine algebraic set. The group

$$\mathsf{Aut}(\pi) \times \mathsf{Aut}(G)$$

acts on $\mathsf{Hom}(\pi, G)$ by left- and right- composition, preserving the algebraic structure: if $\alpha \in \mathsf{Aut}(\pi)$ and $h \in \mathsf{Aut}(G)$ are automorphisms, then the action of (α, h) on $\rho \in \mathsf{Hom}(\pi, G)$ is the composition $h \circ \rho \circ \alpha^{-1}$:

$$\pi \xrightarrow{\alpha^{-1}} \pi \xrightarrow{\rho} G \xrightarrow{h} G$$

The *deformation space* is the quotient space of $\mathsf{Hom}(\pi, G)$ (with the classical topology) by the subgroup $\mathsf{Inn}(G)$ of *inner automorphisms* of G, and is denoted $\mathsf{Hom}(\pi, G)/G$. The action of the inner automorphism ι_γ determined by an element $\gamma \in \pi$ equals $\iota_{\rho(\gamma^{-1})}(\rho)$. Therefore $\mathsf{Inn}(\pi)$ acts trivially on $\mathsf{Hom}(\pi, G)/G$ and the induced action of $\mathsf{Aut}(\pi)$ on $\mathsf{Hom}(\pi, G)/G$ factors through the quotient

$$\mathsf{Out}(\pi) := \mathsf{Aut}(\pi)/\mathsf{Inn}(\pi).$$

When Σ is a closed orientable surface with $\chi(\Sigma) < 0$, then the natural homomorphism

$$\pi_0(\mathsf{Diff}(\Sigma)) \longrightarrow \mathsf{Out}(\pi)$$

is an isomorphism. The *mapping class group* Mod_Σ is the subgroup of $\mathsf{Out}(\pi)$ corresponding to *orientation-preserving* diffeomorphisms of Σ.

When Σ has nonempty boundary with components $\partial_i \Sigma$, this deformation space admits a *boundary restriction map*

$$(1.1) \qquad \mathsf{Hom}(\pi_1(\Sigma), G)/G \longrightarrow \prod_{i \in \pi_0(\partial\Sigma)} \mathsf{Hom}\big(\pi_1(\partial_i\Sigma)), G\big)/G.$$

The fibers of the boundary restriction map are the *relative character varieties.* This action of Mod_Σ preserves this map.

1.1. **The Symplectic Structure.** These spaces possess *algebraic symplectic structures,* invariant under Mod_Σ. For the moment we focus on the *smooth part* of $\mathsf{Hom}(\pi, G)$, which we define as follows. When G is reductive, the subset $\mathsf{Hom}(\pi, G)^{--}$ consisting of representations whose image does not lie in a parabolic subgroup of G is a smooth submanifold upon which $\mathsf{Inn}(G)$ acts properly and freely. The quotient $\mathsf{Hom}(\pi, G)^{--}/G$ is then a smooth manifold, with a Mod_Σ-invariant symplectic structure.

The symplectic structure depends on a choice of a nondegenerate Ad-invariant symmetric bilinear form $\mathbb{B}$ on the Lie algebra $\mathfrak{g}$ of G and an orientation on Σ. The composition

$$\pi \xrightarrow{\ \rho\ } G \xrightarrow{\ \mathsf{Ad}\ } \mathsf{Aut}(\mathfrak{g})$$

defines a π-module $\mathfrak{g}_{\mathsf{Ad}\rho}$. The Zariski tangent space to $\mathsf{Hom}(\pi, G)$ at a representation ρ is the space $Z^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho})$ of *1-cocycles.* The tangent space to the orbit $G\rho$ equals the subspace $B^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho})$ of *1-coboundaries.* These facts are due to Weil [102], see also Raghunathan [88]. If G acts properly and freely on a neighborhood of ρ in $\mathsf{Hom}(\pi, G)$, then $\mathsf{Hom}(\pi, G)/G$ is a manifold near $[\rho]$ with tangent space $H^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho})$. In that case a nondegenerate symmetric $\mathsf{Ad}(G)$-invariant bilinear form

$$\mathfrak{g} \times \mathfrak{g} \xrightarrow{\ \mathbb{B}\ } \mathbb{R}$$

defines a pairing of π-modules

$$\mathfrak{g}_{\mathsf{Ad}\rho} \times \mathfrak{g}_{\mathsf{Ad}\rho} \xrightarrow{\ \mathbb{B}\ } \mathbb{R}.$$

Cup product using $\mathbb{B}$ as coefficient pairing defines a nondegenerate skew-symmetric pairing

$$H^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho}) \times H^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho}) \xrightarrow{\ \mathbb{B}_*(\cup)\ } H^2(\pi, \mathbb{R}) \cong \mathbb{R}$$

on each tangent space

$$T_{[\rho]}\mathsf{Hom}(\pi, G)/G \cong H^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho}).$$

Here the isomorphism $H^2(\pi, \mathbb{R}) \cong \mathbb{R}$ arises from the orientation on Σ. The resulting exterior 2-form $\omega_{\mathbb{B}}$ is closed [36], and defines a symplectic structure on the smooth part $\mathsf{Hom}(\pi, G)^{--}/G$ of $\mathsf{Hom}(\pi, G)/G$. This topological definition makes it apparent that $\omega_{\mathbb{B}}$ is Mod_Σ-invariant. In particular the action preserves the measure μ defined by $\omega_{\mathbb{B}}$. When G is compact, the total measure is finite (Jeffrey-Weitsman [61, 62], Huebschmann [60]).

1.2. **The Complex Case.** When G is a complex Lie group, $\mathsf{Hom}(\pi, G)$ has a complex algebraic structure preserved by the $\mathsf{Aut}(\pi) \times \mathsf{Aut}(G)$-action. When G is a complex semisimple Lie group, the above construction, applied to a nondegenerate Ad-invariant *complex-bilinear form*

$$\mathfrak{g} \times \mathfrak{g} \xrightarrow{\ \mathbb{B}\ } \mathbb{C},$$

determines a *complex-symplectic structure* on $\mathsf{Hom}(\pi, G)^{--}/G$, that is, a closed nondegenerate holomorphic $(2, 0)$-form. This complex-symplectic structure is evidently Mod_Σ-invariant. For a discussion of this structure when $G = \mathsf{SL}(2, \mathbb{C})$, see [45].

The choice of a marked conformal structure on Σ determines a *hyper-Kähler structure* on $\mathsf{Hom}(\pi, G)/G$ subordinate to this complex-symplectic structure.

A complex-symplectic structure on a $4m$-dimensional real manifold V is given by an integrable almost complex structure J and a closed nondegenerate skew-symmetric bilinear form

$$TM \times TM \xrightarrow{\Omega} \mathbb{C}$$

which is complex-bilinear with respect to J. Alternatively, it is defined by a reduction of the structure group of the tangent bundle TV from $\mathsf{GL}(4m, \mathbb{R})$ to the subgroup

$$\mathsf{Sp}(2m, \mathbb{C}) \subset \mathsf{GL}(4m, \mathbb{R}).$$

A hyper-Kähler structure further reduces the structure group of the tangent bundle from $\mathsf{Sp}(2m, \mathbb{C})$ to its maximal compact subgroup $\mathsf{Sp}(2m) \subset \mathsf{Sp}(2m, \mathbb{C})$. All of these structures are required to satisfy certain integrability conditions. A hyper-Kähler structure subordinate to a complex-symplectic structure (Ω, J) is defined by a Riemannian metric g and integrable almost complex structures I, K such that:

- g is Kählerian with respect to each of I, J, K,
- the complex structures I, J, K satisfy the quaternion identities,
- $\Omega(X, Y) = -g(IX, Y) + i\, g(KX)$ for $X, Y \in TM$.

Goldman-Xia [54], §5 describes this structure in detail when $G = \mathsf{GL}(1, \mathbb{C})$.

From this we can associate to every point in Teichmüller space $\mathfrak{T}_\Sigma$ a compatible hyper-Kähler structure on the complex-symplectic space $\mathsf{Hom}(\pi, G)^{--}/G$. However the hyper-Kähler structures are not Mod_Σ-invariant.

1.3. Singularities of the deformation space.
In general the spaces $\mathsf{Hom}(\pi, G)$ and $\mathsf{Hom}(\pi, G)/G$ are not manifolds, but their local structure admits a very explicit cohomological description. For convenience assume that G is reductive algebraic and that ρ is a *reductive representation,* that is, its image $\rho(\pi)$ is Zariski dense in a reductive subgroup of G. For $\rho \in \mathsf{Hom}(\pi, G)$, denote the centralizer of $\rho(\pi)$ by $\mathfrak{Z}(\rho)$ and the center of G by $\mathfrak{Z}$.

A representation $\rho \in \mathsf{Hom}(\pi, G)$ is a *singular point* of $\mathsf{Hom}(\pi, G)$ if and only if

$$\dim(\mathfrak{Z}(\rho)/\mathfrak{Z}) > 0.$$

Equivalently, the isotropy group of $\mathsf{Inn}(G)$ at ρ is not discrete, that is, the action of $\mathsf{Inn}(G)$ at ρ is not *locally free.*

The Zariski tangent space $T_\rho \mathsf{Hom}(\pi, G)$ equals the space $Z^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho})$ of $\mathfrak{g}_{\mathsf{Ad}\rho}$-valued 1-cocycles on π. The tangent space to the orbit $G \cdot \rho$ equals the subspace $B^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho})$ of coboundaries. Thus the *Zariski normal space* at ρ to the orbit $G \cdot \rho$ in $\mathsf{Hom}(\pi, G)$ equals the cohomology group $H^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho})$.

Here is a heuristic interpretation. Consider an analytic path $\rho_t \in \mathsf{Hom}(\pi, G)$ with $\rho_0 = \rho$. Expand it as a power series in t:

$$(1.2) \qquad \rho_t(x) = \exp\big(u_0(x)t + u_2(x)t^2 + u_3(x)t^3 + \dots\big)\rho(x)$$

where

$$\pi \xrightarrow{u_n} \mathfrak{g}$$

for $n \geq 0$. The condition

$$(1.3) \qquad\qquad\qquad \rho_t(xy) = \rho_t(x)\rho_t(y)$$

implies that the tangent vector $u = u_0$ satisfies the *cocycle condition*

$$(1.4) \qquad\qquad\qquad u(xy) = u(x) + \mathsf{Ad}\rho(x)u(y),$$

(the *linearization* of (1.3). The vector space of solutions of (1.4) is the space $Z^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho})$ of $\mathfrak{g}_{\mathsf{Ad}\rho}$-valued *1-cocycles* of π.

The Zariski tangent space to the orbit $G \cdot \rho$ equals the subspace $B^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho}) \subset Z^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho})$ consisting of *1-coboundaries*. Suppose that a path ρ_t in $\mathsf{Hom}(\pi, G)$ is induced by a conjugation by a path g_t

$$\rho_t(x) = g_t\rho(x)g_t^{-1},$$

where g_t admits a power series expansion

$$g_t = \exp(v_1 t + v_2 t^2 + \dots),$$

where $v_1, v_2, \dots \in \mathfrak{g}$. Thus the tangent vector to ρ_t is tangent to the orbit $G \cdot \rho$. Expanding the power series, this tangent vector equals

$$u(x) = v_1 - \mathsf{Ad}\rho(x)v_1,$$

that is, $u = \delta v_1 \in B^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho})$ is a coboundary.

Let $u \in T_\rho \mathsf{Hom}(\pi, G) = Z^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho})$ be a tangent vector to $\mathsf{Hom}(\pi, G)$ at ρ. We give necessary and sufficient conditions that u be tangent to an analytic path of representations.

Solving the equation (1.3) to second order gives:

$$(1.5) \qquad u_2(x) - u_2(xy) + \mathsf{Ad}\rho(x)u_2(y) = \frac{1}{2}[u(x), \mathsf{Ad}\rho(x)u(y)].$$

Namely, the function,

$$\pi \times \pi \longrightarrow \mathfrak{g}$$

$$(1.6) \qquad\qquad (x, y) \longmapsto \frac{1}{2}[u(x), \mathsf{Ad}\rho(x)u(y)]$$

is a $\mathfrak{g}_{\mathsf{Ad}\rho}$-valued 2-cochain on π, This 2-cochain is the coboundary δu_2 of the 1-cochain $\pi \xrightarrow{u_2} \mathfrak{g}_{\mathsf{Ad}\rho}$. Similarly there are conditions on the coboundary of u_n in terms of the lower terms in the power series expansion (1.2).

The operation (1.6) has a cohomological interpretation as follows. π acts on $\mathfrak{g}$ by Lie algebra automorphisms, so that Lie bracket defines a pairing of π-modules

$$\mathfrak{g}_{\mathsf{Ad}\rho} \times \mathfrak{g}_{\mathsf{Ad}\rho} \xrightarrow{[,]} \mathfrak{g}_{\mathsf{Ad}\rho}.$$

The Lie algebra of $\mathfrak{Z}(\rho)$ equals $H^0(\pi; \mathfrak{g}_{\mathsf{Ad}\rho})$. The linearization of the action of $\mathfrak{Z}(\rho)$ is given by the cup product on $H^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho})$ with $[,]$ as coefficient pairing:

$$H^0(\pi; \mathfrak{g}_{\mathsf{Ad}\rho}) \times H^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho}) \xrightarrow{[,]_*(\cup)} H^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho}).$$

Now consider the cup product of 1-dimensional classes. The bilinear form

$$H^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho}) \times H^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho}) \xrightarrow{[,]_*(\cup)} H^2(\pi; \mathfrak{g}_{\mathsf{Ad}\rho}).$$

is symmetric; let Q_ρ be the associated quadratic form.

Suppose u is tangent to an analytic path. Solving (1.2) to second order (as in (1.5) and (1.6)) implies that

$$[,]_*(\cup)([u],[u]) = \delta u_2,$$

that is,

(1.7) $Q_\rho([u]) = 0.$

Under the above hypotheses, the necessary condition (1.7) is also sufficient. In fact, by Goldman-Millson [50], ρ has a neighborhood N in $\mathsf{Hom}(\pi, G)$ analytically equivalent to a neighborhood of 0 of the cone C_ρ in $Z^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho})$ defined by the homogeneous quadratic function

$$Z^1(\pi; \mathfrak{g}_{\mathsf{Ad}\rho}) \longrightarrow H^2(\pi; \mathfrak{g}_{\mathsf{Ad}\rho})$$

$$u \longmapsto Q_\rho([u]).$$

Then the germ of $\mathsf{Hom}(\pi, G)/G$ at $[\rho]$ is the quotient of this cone by the isotropy group $\mathfrak{Z}(\rho)$. (These spaces are special cases of *symplectic stratified spaces* of Sjamaar-Lerman [79].)

An explicit *exponential mapping*

$$N \xrightarrow{\mathsf{Exp}_\rho} \mathsf{Hom}(\pi, G)$$

was constructed by Goldman-Millson [49] using the Green's operator of a Riemann surface M homeomorphic to Σ.

The subtlety of these constructions is underscored by the following false argument, which seemingly proves that the Torelli subgroup of

Mod_Σ acts identically on the whole component of $\mathsf{Hom}(\pi, G)/G$ containing the trivial representation. This is easily seen to be false, for G semisimple.

Here is the fallacious argument. The trivial representation ρ_0 is fixed by all of Mod_Σ. Thus Mod_Σ acts on the analytic germ of $\mathsf{Hom}(\pi, G)/G$ at ρ_0. At ρ_0, the coefficient module $\mathfrak{g}_{\mathsf{Ad}\rho}$ is trivial, and the tangent space corresponds to ordinary (untwisted) cohomology:

$$T_{\rho_0}\mathsf{Hom}(\pi, G) = Z^1(\pi; \mathfrak{g}) = Z^1(\pi) \otimes \mathfrak{g}.$$

The quadratic form is just the usual cup-product pairing, so any homologically trivial automorphism ϕ fixes the quadratic cone N pointwise. By Goldman-Millson [50], the analytic germ of $\mathsf{Hom}(\pi, G)$ at ρ_0 is equivalent to the quadratic cone N. *Therefore $[\phi]$ acts trivially on an open neighborhood of ρ in* $\mathsf{Hom}(\pi, G)$. By analytic continuation, $[\phi]$ acts trivially on the whole component of $\mathsf{Hom}(\pi, G)$ containing ρ.

The fallacy arises because the identification Exp_ρ of a neighborhood N in the quadratic cone with the germ of $\mathsf{Hom}(\pi, G)$ at ρ depends on a choice of Riemann surface M. Each point $\langle f, M \rangle \in \mathfrak{T}_\Sigma$ determines an exponential map $\mathsf{Exp}_{\rho,\langle f,M\rangle}$ from the germ of the quadratic cone to $\mathsf{Hom}(\pi, G)$, and these are *not* invariant under Mod_Σ. In particular, no family of isomorphisms of the analytic germ of $\mathsf{Hom}(\pi, G)$ at ρ_0 with the quadratic cone N is Mod_Σ-invariant.

Problem 1.1. *Investigate the dependence of* $\mathsf{Exp}_{\rho,\langle f,M\rangle}$ *on the marked Riemann surface* $\langle f, M \rangle$.

1.4. **Surfaces with boundary.** When Σ has nonempty boundary, an Ad-invariant inner product $\mathbb{B}$ on $\mathfrak{g}$ and an orientation on Σ determines a *Poisson structure* (Fock-Rosly [32], Guruprasad-Huebschmann-Jeffrey-Weinstein [55]). The *symplectic leaves* of this Poisson structure are the level sets of the boundary restriction map (1.1).

For each component $\partial_i\Sigma$ of $\partial\Sigma$, fix a conjugacy class $C_i \subset G$. The subspace

$$(1.8) \qquad \mathsf{Hom}(\pi, G)/G_{(C_1,\dots,C_b)} \subset \mathsf{Hom}(\pi, G)/G$$

consisting of $[\rho]$ such that

$$(1.9) \qquad \rho(\partial_i\Sigma) \subset C_i$$

has a symplectic structure. (To simplify the discussion we assume that it is a smooth submanifold.) De Rham cohomology with twisted coefficients in $\mathfrak{g}_{\mathsf{Ad}\rho}$ is naturally isomorphic with group cohomology of π. In terms of De Rham cohomology, the tangent space at $[\rho]$ to

$\mathsf{Hom}(\pi, G)/G_{(C_1,...,C_b)}$ identifies with

$$\mathsf{Ker}\bigg(H^1(\Sigma; \mathfrak{g}_{\mathsf{Ad}\rho}) \to H^1(\partial\Sigma; \mathfrak{g}_{\mathsf{Ad}\rho})\bigg)$$

$$\cong \mathsf{Image}\bigg(H^1(\Sigma, \partial\Sigma; \mathfrak{g}_{\mathsf{Ad}\rho}) \to H^1(\Sigma; \mathfrak{g}_{\mathsf{Ad}\rho})\bigg).$$

The cup product pairing

$$H^1(\Sigma; \mathfrak{g}_{\mathsf{Ad}\rho}) \times H^1(\Sigma, \partial\Sigma; \mathfrak{g}_{\mathsf{Ad}\rho}) \xrightarrow{\mathbb{B}_*(\cup)} H^2(\Sigma, \partial\Sigma; \mathbb{R})$$

induces a symplectic structure on $\mathsf{Hom}(\pi, G)/G_{(C_1,...,C_b)}$.

Given a (possibly singular) foliation $\mathfrak{F}$ of a manifold X by symplectic manifolds, the Poisson structure is defined as follows. For functions $f, g \in C^\infty(X)$, their Poisson bracket is a function $\{f, g\}$ on X defined as follows. Let $x \in X$ and let L_x be the leaf of $\mathfrak{F}$ containing x. Define the value of $\{f, g\}$ at x as the Poisson bracket

$$\{f|_{L_x}, g|_{L_x}\}_{L_x},$$

where $\{,\}_{L_x}$ denotes the Poisson bracket operation on the symplectic manifold L_x, and $f|_{L_x}, g|_{L_x} \in C^\infty(L_x)$, are the restrictions of f, g to L_x.

The examples below exhibit *exterior bivector fields* ξ representing the Poisson structure. If $f, g \in C^\infty(X)$, their Poisson bracket $\{f, g\}$ is expressed as an interior product of ξ with the exterior derivatives of f, g:

$$\{f, g\} = \xi \cdot (df \otimes dg).$$

In local coordinates $(x^1, \ldots, x^n)$, write

$$\xi = \sum_{i,j} \xi^{i,j} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

with $\xi^{i,j} = -\xi^{j,i}$. Then

$$\{f, g\} = \sum_{i,j} \left(\xi^{i,j} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right) \cdot \left(\frac{\partial f}{\partial x_i} dx^i \otimes \frac{\partial g}{\partial x_j} dx^j \right)$$

$$= \sum_{i,j} \xi^{i,j} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right)$$

1.5. **Examples of relative $\mathsf{SL}(2, \mathbb{C})$-character varieties.** We give a few explicit examples, when $G = \mathsf{SL}(2, \mathbb{C})$, and Σ is a three-holed or four-holed sphere, or a one-holed or two-holed torus. Since generic conjugacy classes in $\mathsf{SL}(2, \mathbb{C})$ are determined by the trace function

$$\mathsf{SL}(2, \mathbb{C}) \xrightarrow{\mathsf{tr}} \mathbb{C}$$

the relative character varieties are level sets of the mapping

$$\mathsf{Hom}(\pi, G)/G \longrightarrow \mathbb{C}^b$$

$$[\rho] \longmapsto \left[\mathsf{tr}\big(\rho(\partial_i(\Sigma))\big)\right]_{i=1,\ldots,b}$$

1.5.1. *The three-holed sphere.* When Σ is a three-holed sphere, its fundamental group admits a presentation

$$\pi = \langle A, B, C \mid ABC = 1 \rangle$$

where A, B, C correspond to the three components of $\partial\Sigma$. Here is the fundamental result for $\mathsf{SL}(2, \mathbb{C})$-character varieties of a rank two free group:

Theorem (Vogt [101]-Fricke [33]). *The map*

$$\mathsf{Hom}(\pi, G)/G \longrightarrow \mathbb{C}^3$$

$$[\rho] \longmapsto \begin{bmatrix} \mathsf{tr}\big(\rho(A)\big) \\ \mathsf{tr}\big(\rho(B)\big) \\ \mathsf{tr}\big(\rho(C)\big) \end{bmatrix}$$

is an isomorphism of affine varieties.

In particular, the symplectic leaves are just points. See [46] for an elementary proof..

1.5.2. *The one-holed torus.* When Σ is a one-holed torus, its fundamental group admits a presentation

$$\pi = \langle X, Y, Z, K \mid XYZ = 1, K = XYX^{-1}Y^{-1} \rangle$$

where X, Y are simple loops intersecting once, and $K = XYX^{-1}Y^{-1}$ corresponds to the boundary. Presenting the interior of Σ as the quotient

$$\mathsf{int}(\Sigma) = \big(\mathbb{R}^2 - \mathbb{Z}^2\big)/\mathbb{Z}^2$$

the curves X, Y correspond to the $(1, 0)$ and $(0, 1)$-curves respectively. Once again the Vogt-Fricke theorem implies that $\mathsf{Hom}(\pi, G)/G \cong \mathbb{C}^3$, with coordinates

$$x = \mathsf{tr}\big(\rho(X)\big)$$
$$y = \mathsf{tr}\big(\rho(Y)\big)$$
$$z = \mathsf{tr}\big(\rho(Z)\big).$$

The boundary trace $\mathsf{tr}\big(\rho(K)\big)$ is:

$$\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$$

so the relative character varieties are the level sets $\kappa^{-1}(t)$. This mapping class group Mod_Σ acts by polynomial transformations of $\mathbb{C}^3$, preserving the function κ (compare [44]). The Poisson structure is given by the bivector field

$$d\kappa \cdot \big(\partial_x \wedge \partial_y \wedge \partial_z\big) = \big(2x - yz\big)\partial_y \wedge \partial_z$$
$$+ \big(2y - zx\big)\partial_z \wedge \partial_x$$
$$+ \big(2z - xy\big)\partial_x \wedge \partial_y$$

(where ∂_x denotes $\frac{\partial}{\partial x}$, etc.).

1.5.3. *The four-holed sphere.* When Σ is a four-holed sphere, the relative character varieties admit a similar description. Present the fundamental group as

$$\pi = \langle A, B, C, D \mid ABCD = 1 \rangle$$

where the generators A, B, C, D correspond to the components of $\partial\Sigma$. The elements

$$X = AB, Y = BC, Z = CA$$

correspond to simple closed curves on Σ. Denoting the trace functions $\mathsf{Hom}(\pi, G)/G \longrightarrow \mathbb{C}$ corresponding to elements $A, B, C, D, X, Y, Z \in \pi$ by lower-case, the relative character varieties are defined by:

$$x^2 + y^2 + z^2 + xyz = (ab + cd)x + (bc + ad)y$$
$$+ (ac + bd)z + (4 - a^2 - b^2 - c^2 - d^2 - abcd)$$

with Poisson structure

$$\xi = \big(ab + cd - 2x - yz\big)\partial_y \wedge \partial_z$$
$$+ \big(bc + da - 2y - zx\big)\partial_z \wedge \partial_x$$
$$+ \big(ca + bd - 2z - xy\big)\partial_x \wedge \partial_y.$$

1.5.4. *The two-holed torus.* Presenting the fundamental group of a two-holed torus as

$$\pi = \langle A, B, X, Y \mid AXY = YXB \rangle,$$

where $A, B \in \pi$ correspond to the two components of $\partial\Sigma$, the elements

$$Z := Y^{-1}X^{-1},$$
$$U := AXY = BYX,$$
$$V := BY,$$
$$W := AX$$

are represented by simple closed curves. Using the same notation for trace coordinates as above, the relative character varieties are defined by the equations:

$$a + b = xw + yv + uz - xyu$$
$$ab = x^2 + y^2 + z^2 + u^2 + v^2 + w^2$$
$$+ vwz - xyz - xuv - yuw - 4$$

and the Poisson structure is

$$(2z - xy)\partial_x \wedge \partial_y + (2x - yz)\partial_y \wedge \partial_z + (2y - yx)\partial_z \wedge \partial_x +$$
$$(2u - vx)\partial_v \wedge \partial_x + (2v - xu)\partial_x \wedge \partial_u + (2x - uv)\partial_u \wedge \partial_v +$$
$$(2u - wy)\partial_w \wedge \partial_y + (2w - yu)\partial_y \wedge \partial_u + (2y - uw)\partial_u \wedge \partial_w +$$
$$(2(xy - z) - vw)\partial_v \wedge \partial_w + (2(xu - v) - wz)\partial_w \wedge \partial_z$$
$$+ (2(yu - w) - zv)\partial_z \wedge \partial_v.$$

These formulas are derived by applying the formulas for the Poisson bracket of trace functions developed in Goldman [38] in combination with the trace identities in $\mathsf{SL}(2, \mathbb{C})$ (see [46]).

2. Compact Groups

The simplest case occurs when $G = \mathsf{U}(1)$. Then

$$\mathsf{Hom}(\pi, G)/G = \mathsf{Hom}(\pi, G) \cong \mathsf{U}(1)^{2g} \cong H^1(\Sigma; \mathbb{R}/\mathbb{Z})$$

is a $2g$-dimensional torus. If M is a closed Riemann surface diffeomorphic to Σ, then $\mathsf{Hom}(\pi, G)/G$ identifies with the *Jacobi variety* of M, parametrizing topologically trivial holomorphic line bundles over M. Although the complex structures on $\mathsf{Hom}(\pi, G)/G$ vary with the complex structures on Σ, the symplectic structure is independent of M.

2.1. **Ergodicity.** The mapping class group action in this case factors through the symplectic representation

$$\mathsf{Mod}_\Sigma \longrightarrow \mathsf{Sp}(2g, \mathbb{Z})$$

(since the representation variety is just the ordinary cohomology group with values in $\mathbb{R}/\mathbb{Z}$), which is easily seen to be ergodic. This generalizes to arbitrary compact groups:

Theorem 2.1. *Let G be a compact group. The $\mathsf{Out}(\pi)$-action on $\mathsf{Hom}(\pi, G)/G$ is ergodic.*

When the simple factors of G are locally isomorphic to $\mathsf{SU}(2)$ and Σ is orientable, this was proved in Goldman [43]. For general G, this theorem is due to Pickrell-Xia [82] when Σ is closed and orientable, and Pickrell-Xia [83] for compact orientable surfaces of positive genus.

The following conjecture generalizes the above ergodicity phenomenon:

Conjecture 2.2. *Let $\Omega^*(\mathsf{Hom}(\pi,G)/G)$ be the de Rham algebra consisting of all measurable differential forms on $\mathsf{Hom}(\pi,G)/G$. Then the symplectic structures $\omega_{\mathbb{B}}$ generate the subalgebra of $\Omega^*(\mathsf{Hom}(\pi,G)/G)$ consisting of Mod_Σ-invariant forms.*

Since the μ-measure of $\mathsf{Hom}(\pi,G)/G$ is finite, the representation of Mod_Σ on

$$\mathfrak{H} := L^2(\mathsf{Hom}(\pi,G)/G,\mu))$$

is unitary. Andersen has informed me that he has proved vanishing of the first cohomology group $H^1(\mathsf{Mod}_\Sigma, \mathfrak{H})$, and has raised the following conjecture generalizing Conjecture 2.2::

Conjecture 2.3. *Suppose*

$$C^\infty(\mathsf{Hom}(\pi,G)/G) \xrightarrow{D} C^\infty(\mathsf{Hom}(\pi,G)/G)$$

is a differential operator which commutes with the Mod_Σ-action on $\mathsf{Hom}(\pi,G)/G$. Then D is a scalar multiple of the identity operator.

2.2. **The unitary representation.** Ergodicity means that the only trivial subrepresentation of $\mathfrak{H}$ is the subspace $\mathbb{C}$ consisting of constants. Furthermore the action is *weak mixing,* by which we mean that $\mathbb{C}$ is the only *finite-dimensional* invariant subspace [43]. On the other hand the orthogonal complement $\mathfrak{H}_0$ to $\mathbb{C}$ in $\mathfrak{H}$ contains invariant subspaces. For example the closure of the span of trace functions of *nonseparating simple closed curves on Σ* is an invariant subspace [48].

Problem 2.4. *Decompose the representation on $\mathfrak{H}_0$ into irreducible representations of Mod_Σ.*

When $G = \mathsf{U}(1)$, and Σ is the 2-torus, $\mathsf{Hom}(\pi,G)/G$ naturally identifies with T^2, by the functions α, β corresponding to a basis of $\pi_1(\Sigma)$. The functions

$$\phi_{m,n} := \alpha^m \beta^n,$$

forms a Hilbert basis of $\mathfrak{H}$, indexed by $(m,n) \in \mathbb{Z}^2$. The Mod_Σ-representation on $\mathfrak{H}$ arises from the linear $\mathsf{GL}(2,\mathbb{Z})$-action on its basis $\mathbb{Z}^2$. The $\mathsf{GL}(2,\mathbb{Z})$-orbits on $\mathbb{Z}^2$ are indexed by integers $d \geq 0$. The orbit of $(d,0)$ consists of all $(m,n) \in \mathbb{Z}^2$ with $\mathsf{gcd}(m,n) = d$. These are Hilbert bases for irreducible constituents C_d of $\mathfrak{H}$.

The irreducible constituents C_d admit an alternate description, as follows. The d-fold covering homomorphism

$$G \xrightarrow{\Phi_d} G$$

induces a covering space

$$\mathsf{Hom}(\pi, G)/G \longrightarrow \mathsf{Hom}(\pi, G)/G.$$

Let L_d denote the closure of the image of the induced map $\mathfrak{H} \longrightarrow \mathfrak{H}$. Then

$$L_d = \widehat{\bigoplus_{d'|d}} C_{d'}$$

so C_d consists of the orthocomplement in L_d of the sum of all $L_{d'}$ for $d'|d$ but $d' \neq d$.

Problem 2.5. *Find a similar geometric interpretation for the irreducible constituents for compact nonabelian groups G.*

2.3. **Holomorphic objects.** By Narasimhan-Seshadri [78], and Ramanathan [89], a marked conformal structure (f, M) on Σ interprets $\mathsf{Hom}(\pi, G)/G$ as a moduli space of *holomorphic objects* on M. To simplify the exposition we only consider the case $G = \mathsf{U}(n)$, for which $\mathsf{Hom}(\pi, G)/G$ identifies with the moduli space $\mathfrak{U}_n(M)$ of *semistable holomorphic $\mathbb{C}^n$-bundles* over M of zero degree [78]. The union of all $\mathfrak{U}_n(M)$ over $\langle f, M \rangle$ in $\mathfrak{T}_\Sigma$ forms a holomorphic fiber bundle

$$\mathfrak{U}_n \longrightarrow \mathfrak{T}_\Sigma$$

with an action of Mod_Σ. The quotient $\mathfrak{U}_n/\mathsf{Mod}_\Sigma$ fibers holomorphicly over $\mathfrak{M}_\Sigma$. The Narasimhan-Seshadri theorem gives a (non-holomorphic) map

$$\mathfrak{U}_n \xrightarrow{\text{hol}} \mathsf{Hom}(\pi, G)/G$$

which on the fiber $\mathfrak{U}_n(M)$ is the bijection associating to an equivalence class of semistable bundles the equivalence class of the holonomy representation of the corresponding flat unitary structure. $\mathfrak{U}_n/\mathsf{Mod}_\Sigma$ inherits a foliation $\mathfrak{F}_\mathfrak{U}$ from the the foliation of $\mathfrak{U}_n$ by level sets of hol. The dynamics of this foliation are equivalent to the dynamics of the Mod_Σ-action on $\mathsf{Hom}(\pi, G)$.

Go one step further and replace $\mathfrak{T}_\Sigma$ by its unit sphere bundle $U\mathfrak{T}_\Sigma$ and $\mathfrak{M}_\Sigma$ by its (orbifold) unit sphere bundle

$$U\mathfrak{M}_\Sigma = (U\mathfrak{T}_\Sigma)/\mathsf{Mod}_\Sigma.$$

Pull back the fibration $\mathfrak{U}^k(\Sigma)$ to $U\mathfrak{M}_\Sigma$, to obtain a flat $\mathsf{Hom}(\pi, G)/G$-bundle $U\mathfrak{U}_n$ over $U\mathfrak{M}_\Sigma$,

The *Teichmüller geodesic flow* is a vector field on $U\mathfrak{M}_\Sigma$ generating the geodesics for the Teichmüller metric on $\mathfrak{T}_\Sigma$. (Masur citeMasur) Its horizontal lift with respect to the flat connection is an vector field on the total space whose dynamics mirrors the dynamics of the Mod_Σ-action on $\mathsf{Hom}(\pi, G)/G$.

As the Mod_Σ-action on $\mathsf{Hom}(\pi, G)/G$ is weak-mixing, the unitary representation on $L^2\big(\mathsf{Hom}(\pi, G)/G, \mu\big)$ provides no nontrivial finite-dimensional representations. Thus these representations markedly differ from the representations obtained by Hitchin [58] and Axelrod, Della-Pietra, and Witten [4] obtained from projectively flat connections on $\mathsf{Hom}(\pi, G)/G$. Recently Andersen [2] has proved that these finite-dimensional projective representations of Mod_Σ are *asymptotically* faithful.

2.4. **Automorphisms of free groups.** Analogous questions arise for the outer automorphism group of a free group π of rank r. Let G be a compact connected Lie group. Then Haar measure on G defines an $\mathsf{Out}(\pi)$-invariant probability measure on $\mathsf{Hom}(\pi, G)$.

Conjecture 2.6. *If $r \geq 3$, the action of $\mathsf{Out}(\pi)$ on $\mathsf{Hom}(\pi, G)$ is ergodic.*

Using calculations in [43], this conjecture has been proved [47] when all of the simple factors of G are locally isomorphic to $\mathsf{SU}(2)$.

2.5. **Topological dynamics.** The topological theory is more subtle, since no longer may we ignore invariant subsets of measure zero. For example, if $F \subset G$ is a finite subgroup, then $\mathsf{Hom}(\pi, F)$ is finite and its image in $\mathsf{Hom}(\pi, G)/G$ is an invariant closed subset.

One might expect that if a representation $\rho \in \mathsf{Hom}(\pi, G)$ has dense image in $\mathsf{SU}(2)$, that the Mod_Σ-orbit of $[\rho]$ is dense in $\mathsf{Hom}(\pi, G)/G$. This is true if Σ is a one-holed torus (Previte-Xia [84]) and if the genus of Σ is positive (Previte-Xia [85]). In genus 0, representations ρ exist with dense image but $\mathsf{Mod}_\Sigma \cdot [\rho]$ consists of only two points.

Similar examples exist when Σ is a four-holed sphere. Benedetto and I showed [5], that when $-2 < a, b, c, d < 2$, the set of $\mathbb{R}$-points of the relative character variety has one compact component. This component is diffeomorphic to S^2. Depending on the boundary traces (a, b, c, d), this component corresponds to either $\mathsf{SL}(2, \mathbb{R})$-representations or $\mathsf{SU}(2)$-representations. Previte and Xia [86] found representations ρ in the components corresponding to $\mathsf{SL}(2, \mathbb{R})$-representations having dense image, but whose orbit $\big(\mathsf{Mod}_\Sigma \cdot [\rho]\big)$ has two points. On the other hand, in both cases, Previte and Xia [87] showed the action is *minimal* (every orbit is dense) for a dense set of boundary traces in $[-2, 2]^4$.

Problem 2.7. *Determine necessary and sufficient conditions on a general representation ρ for its orbit $\mathsf{Mod}_\Sigma \cdot [\rho]$ to be dense.*

The case when $G = \mathsf{SU}(2)$ and Σ an n-holed sphere for $n > 4$ remains open.

2.6. **Individual elements.** For a closed surface of genus one, an individual element is ergodic on the $\mathsf{SU}(2)$-character variety if and only if it is *hyperbolic.* In his doctoral thesis [13, 14], Brown used KAM-theory to show this no longer holds for actions on relative $\mathsf{SU}(2)$-character varieties over the one-holed torus. Combining Brown's examples with a branched-cover construction suggests:

Problem 2.8. *Construct an example of a pseudo-Anosov mapping class for a closed surface which is not ergodic on the $\mathsf{SU}(2)$-character variety.*

3. Noncompact Groups and Uniformizations

For noncompact G, one expects less chaotic dynamics. Trivial dynamics – in the form of *proper* Mod_Σ-actions — occur for many invariant open subsets corresponding to *locally homogeneous geometric structures,* (in the sense of Ehresmann [27]) or *uniformizations.* Such structures are defined by local coordinate charts into a homogeneous space G/H with coordinate changes which are restrictions of transformations from G. Such an atlas globalizes to a *developing map,* an immersion $\tilde\Sigma \longrightarrow G/H$ of the universal covering space $\tilde\Sigma \longrightarrow \Sigma$ which is equivariant with respect to a homomorphism $\pi \xrightarrow{\rho} G$.

To obtain a deformation space of such structures with an action of the mapping class group, one introduces markings for a fixed topological surface Σ, just as in the definition of Teichmüller space. The *deformation space* $\mathsf{Def}_{(G,G/H)}(\Sigma)$ consists of equivalence classes of marked $(G, G/H)$-structures with a *holonomy map*

$$\mathsf{Def}_{(G,G/H)}(\Sigma) \xrightarrow{\mathsf{hol}} \mathsf{Hom}(\pi, G)/G$$

which is Mod_Σ-equivariant. The *Ehresmann-Thurston theorem* asserts that, with respect to an appropriate topology on $\mathsf{Def}_{(G,G/H)}(\Sigma)$, the mapping hol is a local homeomorphism. (This theorem is implicit in Ehresmann [28] and first explicitly stated by Thurston [98]. More detailed proofs were given by Lok [70], Canary-Epstein-Green [19], and Goldman [41]. Bergeron and Gelander [6] give a detailed modern proof with applications to discrete subgroups.)

If $G = \mathsf{PGL}(2,\mathbb{R})$ and $G/H = \mathsf{H}^2$ is the hyperbolic plane, then $\mathsf{Def}_{(G,G/H)}(\Sigma) = \mathfrak{T}_\Sigma$.

Examples of uniformizations with proper Mod_Σ-actions include:

- $G = \mathsf{PSL}(2,\mathbb{R})$: The Teichmüller space $\mathfrak{T}_\Sigma$, regarded as the component of discrete embeddings in $\mathsf{Hom}(\pi, G)/G$;
- $G = \mathsf{PSL}(2,\mathbb{C})$: Quasi-fuchsian space $\mathcal{Q}_\Sigma$ is an open subset of $\mathsf{Hom}(\pi, G)/G$ which is equivariantly biholomorphic to $\mathfrak{T}_\Sigma \times \overline{\mathfrak{T}_\Sigma}$;
- $G = \mathsf{SL}(3,\mathbb{R})$. The deformation space $\mathfrak{C}_\Sigma$ of convex $\mathbb{R}\mathsf{P}^2$-structures is a connected component of $\mathsf{Hom}(\pi, G)/G$ (Choi-Goldman [20]) and the Mod_Σ-action is proper. More generally if G is a split $\mathbb{R}$-form of a semisimple group, Labourie [69] has shown that Mod_Σ acts properly on the contractible component of $\mathsf{Hom}(\pi, G)/G$ discovered by Hitchin [59].

3.1. **Fricke-Teichmüller space.** A *Fuchsian representation* of π into $G = \mathsf{PSL}(2,\mathbb{R})$ is an isomorphism ρ of $\pi = \pi_1(\Sigma)$ onto a discrete subgroup of G. Since π is torsionfree and ρ is injective, $\rho(\pi)$ is torsionfree. Hence it acts freely on H^2 and the quotient $\mathsf{H}^2/\rho(\pi)$ is a complete hyperbolic surface. The representation ρ defines a homotopy equivalence

$$\Sigma \longrightarrow \mathsf{H}^2/\rho(\pi)$$

which is homotopic to a homeomorphism. Thus ρ is the holonomy homomorphism of a hyperbolic structure on Σ. The collection of $\mathsf{PGL}(2,\mathbb{R})$-conjugacy classes of such homomorphisms identifies (via the Uniformization Theorem) with the *Teichmüller space* $\mathfrak{T}_\Sigma$ of Σ. When $\partial\Sigma \neq \emptyset$, then the *Fricke space* is defined as the deformation space of complete hyperbolic structures on $\mathsf{Int}(\Sigma)$ such that each end is either a cusp or a complete collar on a simple closed geodesic (a *funnel*). These representations map each component of $\partial\Sigma$ to either a parabolic or a hyperbolic element of $\mathsf{PSL}(2,\mathbb{R})$ respectively. For details on Fricke spaces see Bers-Gardiner [8].

The Mod_Σ-action on $\mathfrak{T}_\Sigma$ is proper. This fact seems to have first been noted by Fricke [33] (see Bers-Gardiner [8] or Farb-Margalit [29]). It follows from two facts:

- Mod_Σ preserve a metric on $\mathfrak{T}_\Sigma$;
- The *simple marked length spectrum*

$$(3.1) \qquad \left\{ \text{simple closed curves on } \Sigma \right\} \Big/ \mathsf{Diff}^0(\Sigma) \longrightarrow \mathbb{R}_+$$

is a proper map.

See Abikoff [1], Bers-Gardiner [8], Farb-Margalit [29] or Harvey [56] for a proof. Another proof follows from Earle-Eels [24], and the closely

related fact (proved by Palais and Ebin [25]) that the full diffeomorphism group of a compact smooth manifold acts properly on the space of Riemannian metrics. Compare [53].

3.2. **Other components and the Euler class.** Consider the case $G = \mathsf{PSL}(2, \mathbb{R})$. Then the components of $\mathsf{Hom}(\pi, G)/G$ are indexed by the *Euler class*

$$\mathsf{Hom}(\pi, G)/G \xrightarrow{\;e\;} H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}$$

whose image equals

$$\mathbb{Z} \cap [2 - 2g - b, 2g - 2 + b]$$

where Σ has genus g and b boundary components. Thus $\mathsf{Hom}(\pi, G)/G$ has $4g+2b-3$ connected components ([40] and Hitchin [57] when $b = 0$). The main result of [35] is that the two extreme components $e^{-1}\big(\pm(2 - 2g - b)\big)$ consist of discrete embeddings. These two components differ by the choice of orientation, each one corresponding to $\mathfrak{T}_\Sigma$, upon which Mod_Σ acts properly. In contrast,

Conjecture 3.1. *Suppose that $b = 0$ (Σ is closed). For each integer $1 \le k \le 2g+b-2$, the Mod_Σ-action on the component $e^{-1}(2-2g+b+k)$ of $\mathsf{Hom}(\pi, G)$ is ergodic.*

When $b = 0$, the component

$$e^{-1}(3 - 2g) \;\approx\; \Sigma \times \mathbb{R}^{6g-8}$$

represents a $6g - 6$-dimensional *thickening* of Σ, upon which Mod_Σ acts. However, Morita [81] showed that Mod_Σ cannot act smoothly on Σ itself inducing the homomorphism $\mathsf{Diff}(\Sigma) \longrightarrow \mathsf{Mod}_\Sigma$. (Recently Markovic [80] has announced that if Σ is a closed surface of genus > 5, then Mod_Σ cannot even act on Σ by homeomorphisms inducing $\mathsf{Homeo}(\Sigma) \longrightarrow \mathsf{Mod}_\Sigma$.)

Problem 3.2. *Determine the smallest dimensional manifold homotopy-equivalent to Σ upon which Mod_Σ acts compatibly with the outer action of Mod_Σ on $\pi_1(\Sigma)$.*

3.3. **The one-holed torus.** For surfaces with nonempty boundary, the dynamics appears more complicated. When Σ is a one-holed torus ($g = b = 1$) and $G = \mathsf{PSL}(2, \mathbb{R})$ or $\mathsf{SU}(2)$, this was completely analyzed in [44].

As in §1.5.2, the $\mathsf{SL}(2, \mathbb{C})$-character variety identifies with $\mathbb{C}^3$, where the three coordinates (x, y, z) are traces of three generators of π corresponding to the generators X, Y, XY. In these coordinates, the trace

of the element $K = XYX^{-1}Y$ of π corresponding to $\partial\Sigma$ equals

$$\kappa(x, y, z) := x^2 + y^2 + z^2 - xyz - 2.$$

The *relative* $\mathsf{SL}(2, \mathbb{C})$-*character variety* of Σ is then the family of level sets $\kappa^{-1}(t)$ of $\mathbb{C}^3 \xrightarrow{\kappa} \mathbb{C}$.

The set $\kappa^{-1}(t) \cap \mathbb{R}^3$ of $\mathbb{R}$-points of $\kappa^{-1}(t)$, for boundary trace $t \in \mathbb{R}$, are of two types:

- The $\mathsf{SU}(2)$-characters, with $x, y, z \in [-2, 2]$ and $t < 2$;
- The $\mathsf{SL}(2, \mathbb{R})$-characters, with either:
  - at least one of x, y, z lies in $(-\infty, -2] \cup [2, \infty)$, or
  - each x, y, z lies in $[-2, 2]$ and $t \geq 2$.

If $|t| > 2$, no $\mathsf{SU}(2)$-characters lie in $\kappa^{-1}(t) \cap \mathbb{R}^3$. If $t \neq 2$, these two subsets of $\kappa^{-1}(t) \cap \mathbb{R}^3$ are disjoint. If $t = 2$, these two subsets intersect on the subset

$$[-2, 2]^3 \cap \kappa^{-1}(2)$$

corresponding to $\mathsf{SO}(2)$-representations. The space of $\mathsf{SO}(2)$-characters is 2-sphere with 4 branch points of cone angle π (a tetrahedron with smoothed edges).

The Mod_Σ-action determines a dynamical system on each level set. By Keen [65], the Fricke space of Σ is the subset

$$\big\{ (x, y, z) \in \mathbb{R}^3 \mid \kappa(x, y, z) \leq -2 \big\}$$

with a proper Mod_Σ-action. Each level set $\mathbb{R}^3 \cap \kappa^{-1}(t)$, for $t < -2$, is homeomorphic to a disjoint union of four discs; the four components are distinguished by different lifts of the representation from $\mathsf{PSL}(2, \mathbb{R})$ to $\mathsf{SL}(2, \mathbb{R})$.

The level set $\mathbb{R}^3 \cap \kappa^{-1}(-2)$ has one notable feature. It has five components, four of which correspond to the Teichmüller space of Σ, and the other component $\{(0, 0, 0)\}$ consists of just the origin. The Teichmüller space (corresponding to the deformation space of complete *finite area* hyperbolic structures) corresponds to representations taking the boundary element of π to a parabolic transformation of trace -2. On the other hand, $\{(0, 0, 0)\}$ corresponds the *quaternion representation* in $\mathsf{SU}(2)$:

$$X \longmapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

$$Y \longmapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The peripheral element $K \in \pi$ maps to the nontrivial *central* element $-I \in \mathsf{SU}(2)$.

Here we see — for the first time — the coexistence of two extremes of dynamical behavior:

- The proper action on the $\mathsf{SL}(2, \mathbb{R})$-characters;
- The entire mapping class group Mod_Σ fixes a point, in a sense, the "most chaotic" action.

This dichotomy persists for $-2 < t < 2$. The origin deforms to a compact component, consisting of characters of $\mathsf{SU}(2)$-representations with an ergodic Mod_Σ-action. Four contractible components, correspond to holonomy representations of hyperbolic structures on a torus with a cone point. The cone angle θ relates to the boundary trace by

$$t = -2\cos(\theta/2).$$

The Mod_Σ-action on these components is proper.

Although Mod_Σ acts properly, none of the corresponding representations are discrete embeddings. The key property seems to be that nonseparating simple loops are mapped to hyperbolic elements, so the simple marked length spectrum (3.1) is a proper map.

Problem 3.3. *Find general conditions which ensure that* (3.1) *is proper.*

The level set $\mathbb{R}^3 \cap \kappa^{-1}(2)$ consists of characters of abelian representations, and Mod_Σ is ergodic on each of the four connected components of the smooth part of $\mathbb{R}^3 \cap \kappa^{-1}(2)$. When $2 < t \le 18$, the Mod_Σ-action on $\mathbb{R}^3 \cap \kappa^{-1}(t)$ is ergodic.

For $t > 18$, the level sets $\mathbb{R}^3 \cap \kappa^{-1}(t)$ display both proper dynamics and chaotic dynamics. The region $(-\infty, -2]^3$ consists of characters of discrete embeddings ρ where the quotient hyperbolic surface $\mathsf{H}^2/\rho(\pi)$ is homeomorphic to a three-holed sphere. Every homotopy equivalence $\Sigma \longrightarrow P$, where P is a hyperbolic surface homeomorphic to a three-holed sphere, determines such a character. Furthermore these determine closed triangular regions which are freely permuted by Mod_Σ. On the complement of these wandering domains the action is ergodic.

When $G = \mathsf{PGL}(2, \mathbb{R})$, the group of (possibly orientation-reversing) isometries of H^2, a similar analysis was begun by Stantchev [94, 52]. One obtains similar dynamical systems, where Mod_Σ acts now on the space of representations into the group

$$G_\pm = \mathsf{SL}(2, \mathbb{C}) \cap \big(\mathsf{GL}(2, \mathbb{R}) \cup i\,\mathsf{GL}(2, \mathbb{R})\big)$$

which doubly covers the two-component group $\mathsf{PGL}(2, \mathbb{R})$. These $G_\pm$-representations are again parametrized by traces. They comprise four components, one of which is the subset of $\mathbb{R}^3$ parametrizing $\mathsf{SL}(2, \mathbb{R})$-representations discussed above. The other three components are

$$\mathbb{R} \times i\mathbb{R} \times i\mathbb{R}, \quad i\mathbb{R} \times \mathbb{R} \times i\mathbb{R}, \quad i\mathbb{R} \times i\mathbb{R} \times \mathbb{R}$$

respectively. Consider $i\mathbb{R}\times\mathbb{R}\times i\mathbb{R}$. For $-14 \leq t < 2$, the Mod_Σ-action is ergodic, but when $t < -14$, wandering domains appear. The wandering domains correspond to homotopy-equivalences $\Sigma \longrightarrow P$, where P is a hyperbolic surface homeomorphic to a two-holed projective plane. The action is ergodic on the complement of the wandering domains.

Problem 3.4. *Determine the ergodic behavior of the Mod_Σ-action on the level sets*

$$\left(i\mathbb{R} \times \mathbb{R} \times i\mathbb{R}\right) \cap \kappa^{-1}(t)$$

where $t > 2$. The level sets for $t > 6$ contains wandering domains corresponding to Fricke spaces of a one-holed Klein bottle.

3.4. **Hyperbolic 3-manifolds.** When $G = \mathsf{PSL}(2,\mathbb{C})$, the subset $\mathcal{Q}_\Sigma$ of $\mathsf{Hom}(\pi, G)/G$ corresponding to embeddings of π onto quasi-Fuchsian subgroups of G is open and Mod_Σ-invariant. Furthermore the Bers isomorphism [7] provides a Mod_Σ-invariant biholomorphism

$$\mathcal{Q}_\Sigma \longrightarrow \mathfrak{T}_\Sigma \times \overline{\mathfrak{T}_\Sigma}.$$

Properness of the action of Mod_Σ on $\mathfrak{T}_\Sigma$ implies properness on $\mathcal{Q}_\Sigma$.

Points on the boundary of $\mathcal{Q}_\Sigma$ also correspond to discrete embeddings, but the action is much more complicated. Recently Souto and Storm [93] have proved that $\partial \mathcal{Q}_\Sigma$ contains a Mod_Σ-invariant closed nowhere dense topologically perfect set upon which the action is topologically transitive. From this they deduce that every continuous Mod_Σ-invariant function on $\partial \mathcal{Q}_\Sigma$ is constant.

While for representations into $G = \mathsf{PSL}(2,\mathbb{R})$, the Mod_Σ-orbits of discrete embeddings are themselves *discrete,* the situation becomes considerably more complicated for larger G. For $G = \mathsf{PSL}(2,\mathbb{C})$, representations corresponding to the fiber of a hyperbolic mapping torus furnish points with infinite stabilizer. This is one of the easiest ways to see that Mod_Σ does not act properly on characters of discrete embeddings. Namely, if M^3 is a hyperbolic 3-manifold which admits a fibration $M^3 \xrightarrow{f} S^1$, then the class of the restriction ρ of the holonomy representation

$$\pi_1(M^3) \longrightarrow \mathsf{PSL}(2,\mathbb{C})$$

to the surface group

$$\pi \;:=\; \pi_1(f^{-1}(s_0)) \;\cong\; \mathsf{Ker}\big(\pi_1(M) \xrightarrow{f_*} \mathbb{Z}\big)$$

is invariant under the monodromy automorphism $h \in \mathsf{Aut}(\pi)$ of M^3. That is, there exists $g \in \mathsf{PSL}(2,\mathbb{C})$ such that

$$\rho\big(h(\gamma)\big) = g\rho(\gamma)g^{-1}$$

for all $\gamma \in \pi$. Furthermore $[\rho]$ is a smooth point of $\mathsf{Hom}(\pi, G)/G$. Kapovich [64] proved McMullen's conjecture [75] that the derivative of the mapping class $[h]$ at $[\rho]$ is hyperbolic, that is, no eigenvalue has norm 1. This contrasts the case of *abelian representations,* since homologically trivial pseudo-Anosov mapping classes act trivially on $\mathsf{Hom}(\pi, G)$.

Thus Mod_Σ does not act properly on the set of characters of discrete embeddings. Let $[\rho]$ (as above) be the character of a discrete embedding of π as the fiber of a hyperbolic mapping torus. The stabilizer of $[\rho]$ contains the infinite cyclic group generated by the mapping class corresponding to $[h]$. (In fact $\langle [h] \rangle$ has finite index in the stabilizer of $[\rho]$.) Since stabilizers of proper actions of discrete groups are finite, Mod_Σ does not act properly.

The Souto-Storm theorem shows that this chaotic dynamical behavior pervades the entire boundary of quasi-Fuchsian space $\mathcal{Q}_\Sigma$.

In another direction, using ideas generalizing those of Bowditch [10], Tan, Wong and Zhang [97] have shown that the action of Mod_Σ on the representations satisfying the analogue of *Bowditch's Q-conditions* is proper. This also generalizes the properness of the action on the space of quasi-Fuchsian representations.

At present little is known about the dynamics of Mod_Σ acting on the $\mathsf{SL}(2,\mathbb{C})$-character variety. Conversations with Dumas led to the following problem:

Problem 3.5. *Find a point $\rho \in \mathsf{Hom}(\pi, \mathsf{SL}(2,\mathbb{C}))$ such that the closure of its orbit $\overline{\mathsf{Mod}_\Sigma \cdot [\rho]}$ meets both the image of the unitary characters $\mathsf{Hom}(\pi, \mathsf{SU}(2))$ and the closure $\overline{\mathcal{Q}_\Sigma}$ of the quasi-Fuchsian characters.*

3.4.1. *Homological actions.* The action of Mod_Σ on the homology of $\mathsf{Hom}(\pi, G)/G$ furnishes another source of possibly interesting linear representations of Mod_Σ. With Neumann [51], we proved that for the relative $\mathsf{SL}(2,\mathbb{C})$-character varieties of the one-holed torus and four-holed sphere, the action of Mod_Σ factors through a finite group.

Atiyah-Bott [3] use infinite-dimensional Morse theory to analyze the algebraic topology of $\mathsf{Hom}(\pi, G)/G$, when G is compact. For the non-singular components their techniques imply that the Mod_Σ-action on the rational cohomology of $\mathsf{Hom}(\pi, G)/G$ factors through the symplectic representation of Mod_Σ on $H^*(\Sigma)$. In particular Biswas [9] proved that the Torelli group acts trivially on nonsingular components. In contrast, Cappell-Lee-Miller [17, 18] proved the surprising result that that the Torelli group acts *nontrivially* on the homology of the $\mathsf{SU}(2)$-character variety when Σ is closed.

3.5. Convex Projective Structures and Hitchin representations.

When $G = \mathsf{SL}(3,\mathbb{R})$, the mapping class group Mod_Σ acts properly on the component $\mathfrak{C}_\Sigma$ of $\mathsf{Hom}(\pi,G)/G$ corresponding to convex $\mathbb{R}\mathsf{P}^2$-structures (Goldman [42], Choi-Goldman [20, 21]). This component is homeomorphic to a cell of dimension $-8\chi(\Sigma)$, and, for a marked Riemann surface M homeomorphic to Σ, admits the natural structure of a holomorphic vector bundle over $\mathfrak{T}_\Sigma$. The work of Labourie [68] and Loftin [71, 72, 73] gives a more intrinsic holomorphic structure on $\mathfrak{C}_\Sigma$.

The existence of this contractible component is a special case of a general phenomenon discovered by Hitchin [59]. Hitchin finds, for *any* split real form of a semisimple group G, a contractible component in $\mathsf{Hom}(\pi,G)/G$. For $G = \mathsf{SL}(n,\mathbb{R})$ this component is characterized as the component containing the composition of discrete embeddings

$$\pi \longrightarrow \mathsf{SL}(2,\mathbb{R})$$

with the irreducible representation

$$\mathsf{SL}(2,\mathbb{R}) \longrightarrow \mathsf{SL}(n,\mathbb{R}).$$

Recently, Labourie has found a dynamical description [69] of representations in Hitchin's component, and has proved they are discrete embeddings. Furthermore he has shown that Mod_Σ acts properly on this component. (These closely relate to the *higher Teichmüller spaces* of Fock-Goncharov [30, 31].)

When G is the automorphism group of a Hermitian symmetric space of noncompact type, Bradlow, Garcia-Prada, and Gothen have found other components of $\mathsf{Hom}(\pi,G)/G$, for which the *Toledo invariant,* is maximal [11, 12]. Their techniques involve Morse theory along the lines of Hitchin [57]. Recently, Burger, Iozzi, and Wienhard have shown [15] that the representations of maximal Toledo invariant consist of discrete embeddings. Using results from [16], Wienhard has informed me that Mod_Σ acts properly on these components,

3.6. The energy of harmonic maps.

An interesting invariant of surface group representations arises from the theory of *twisted harmonic maps* of Riemann surfaces, developed in detail in collaboration with Wentworth [53]. Namely to each *reductive representation* $\pi \xrightarrow{\rho} G$, one associates an *energy function*

$$\mathfrak{T}_\Sigma \xrightarrow{E_\rho} \mathbb{R}$$

whose qualitative properties reflect the Mod_Σ-action. Assuming that the Zariski closure of $\rho(\pi)$ in G is reductive, for every marked Riemann

surface $\Sigma \longrightarrow M$, there is a ρ-equivariant harmonic map

$$\tilde{M} \longrightarrow G/K$$

where $K \subset G$ is a maximal compact subgroup (Corlette [22], Donaldson [23], Labourie [67], and Jost-Yau [63], following earlier work by Eels-Sampson [26]). Its energy density determines an exterior 2-form on Σ, whose integral is defined as $E_\rho(\langle f, M \rangle)$.

When $\rho(\pi)$ lies in a compact subgroup of G, then the twisted harmonic maps are constant, and the energy function is constantly zero. At the other extreme is the following result, proved in [53]:

Theorem 3.6. *Suppose that ρ is an embedding of π onto a convex cocompact discrete subgroup of G. Then the energy function E_ρ is a proper function on $\mathfrak{T}_\Sigma$.*

Here a discrete subgroup $\Gamma \subset G$ is *convex cocompact* if there exists a geodesically convex subset $N \subset G/K$ such that $\Gamma \backslash N$ is compact. For $\mathsf{PSL}(2, \mathbb{C})$, these are just the quasi-Fuchsian representations. This result was first proved by Tromba [99] for Fuchsian representations in $\mathsf{PSL}(2, \mathbb{R})$, and the ideas go back to Sacks-Uhlenbeck [90] and Schoen-Yau [91].

It is easy to prove (see [53]) that if $\Omega \subset \mathsf{Hom}(\pi, G)/G$ is a Mod_Σ-invariant open set for which each function E_ρ is proper, for $[\rho] \in \Omega$, then the action of Mod_Σ on Ω is proper. This gives a general analytic condition implying properness.

Unfortunately, convex cocompactness is extremely restrictive; Kleiner and Leeb have proved [66] that in rank > 1 such groups are *never* Zariski dense. However, we know many examples (the deformation space $\mathfrak{C}_\Sigma$ of convex $\mathbb{R}\mathsf{P}^2$-structures, the Hitchin representations by Labourie [69], other components of maximal representations [11, 12, 16]) where we expect the Mod_Σ-action to be proper. The only use of geodesic convexity in the above result is that the images of harmonic maps are constrained to lie in the set N.

Problem 3.7. *Find a substitute for convex cocompactness in higher rank which includes the above examples of proper Mod_Σ-actions, and for which E_ρ is proper.*

The work of Bonahon-Thurston on geometric tameness, and its recent extensions, implies that the energy function of a discrete embedding $\pi \longrightarrow \mathsf{PSL}(2, \mathbb{C})$ is proper if and only if it is quasi-Fuchsian [53].

3.7. **Singular uniformizations and complex projective structures.** When $G = \mathsf{PSL}(2, \mathbb{R})$, the other components of $\mathsf{Hom}(\pi, G)$ may

be studied in terms of hyperbolic structures with singularities as follows. Instead of requiring all of the coordinate charts to be local homeomorphisms, one allows charts which at isolated points look like the map

$$\mathbb{C} \longrightarrow \mathbb{C}$$
$$z \mapsto z^k$$

that is, the geometric structure has an isolated singularity of *cone angle* $\theta = 2k\pi$. Such a singular hyperbolic structure may be alternatively described as a singular Riemannian metric g whose curvature equals -1 plus Dirac distributions weighted by $2\pi - \theta_i$ at each singular point p_i of cone angle θ_i. The structure is nonsingular on the complement $\Sigma - \{p_1, \ldots, p_k\}$, and that hyperbolic structure has holonomy representation

$$\pi_1\big(\Sigma - \{p_1, \ldots, p_k\}\big) \xrightarrow{\hat{\rho}} \mathsf{PSL}(2, \mathbb{R})$$

such that the holonomy of a loop γ_i encircling p_i is elliptic with rotation angle θ_i.

In particular if each $\theta_i \in 2\pi\mathbb{Z}$, then $\hat{\rho}(\gamma_i) = 1$. The representation $\hat{\rho}$ extends to a representation ρ of $\pi_1(\Sigma)$:

$$
\begin{array}{c}
\pi_1\big(\Sigma - \{p_1, \ldots, p_k\}\big) \\
\Big\downarrow \qquad \searrow^{\hat{\rho}} \\
\pi_1(\Sigma) \dashrightarrow_{\rho} \mathsf{PSL}(2, \mathbb{R}).
\end{array}
$$

Applying Gauss-Bonnet to g implies that ρ has Euler class

$$e(\rho) = \chi(M) + \frac{1}{2\pi} \sum_{i=1}^{k} (\theta_i - 2\pi).$$

It is convenient to assume that each $\theta_i = 4\pi$ and the points p_i are not necessarily distinct — a cone point of cone angle 4π with multiplicity m is then a cone point with cone angle $2(m+1)\pi$. The uniformization theorem of McOwen [77], Troyanov [100], and Hitchin [57] asserts: given a Riemann surface $M \approx \Sigma$, there exists a unique singular hyperbolic structure in the conformal class of M with cone angle θ_i at x_i for $i = 1, \ldots, k$ as long as

$$\chi(\Sigma) + \frac{1}{2\pi} \sum_{i=1}^{k} (\theta_i - 2\pi) < 0.$$

(Hitchin only considers the case when θ_i are multiples of 2π, while McOwen and Troyanov deal with arbitrary positive angles.) The resulting *uniformization map* assigns to the collection of points $\{p_1, \ldots, p_k\}$ (where $0 \leq k \leq |\chi(\Sigma)|$) the singular hyperbolic structure with cone angles 4π (counted with multiplicity) at the p_i. The equivalence class of the holonomy representation in the component

$$e^{-1}(\chi(\Sigma) + k) \subset \mathsf{Hom}(\pi, G)/G$$

defines a map from the symmetric product $\mathsf{Sym}^k(M)$ to $e^{-1}(\chi(M)+k)$. The following result follows from Hitchin [35]:

Theorem 3.8. *Let M be a closed Riemann surface. The above map*

$$\mathsf{Sym}^k(M) \longrightarrow e^{-1}(\chi(M) + k)$$

is a homotopy equivalence.

The union of the symmetric powers $\mathsf{Sym}^k(M)$, one for each marked Riemann surface M, over $\langle f, M \rangle \in \mathfrak{T}_\Sigma$, can be given the structure of a holomorphic fiber bundle $\mathfrak{S}^k(\Sigma)$ over $\mathfrak{T}_\Sigma$, to which the action of Mod_Σ on $\mathfrak{T}_\Sigma$ lifts. The above maps define a homotopy equivalence

$$\mathfrak{S}^k(\Sigma) \overset{\mathbb{U}}{\longrightarrow} e^{-1}(\chi(\Sigma) + k).$$

which is evidently Mod_Σ-equivariant. However, since Mod_Σ acts properly on $\mathfrak{T}_\Sigma$, it also acts properly on the $(6g - 6 + 2k)$-dimensional space $\mathfrak{S}^k(\Sigma)$. The quotient $\mathfrak{S}^k(\Sigma)/\mathsf{Mod}_\Sigma$ is the total space of an (orbifold) $\mathsf{Sym}^k(\Sigma)$-bundle over the Riemann moduli space

$$\mathfrak{M}_\Sigma := \mathfrak{T}_\Sigma/\mathsf{Mod}_\Sigma.$$

The fibers of $\mathbb{U}$ define a (non-holomorphic) foliation of $\mathfrak{S}^k(\Sigma)/\mathsf{Mod}_\Sigma$, a *flat* $\mathsf{Sym}^k(\Sigma)$-*bundle,* whose dynamics mirrors the dynamics of the Mod_Σ-action on the component $e^{-1}(\chi(\Sigma) + k)$.

In general, $\mathbb{U}$ is *not* onto: if $\Sigma \overset{f}{\to} \Sigma'$ is a degree one map to a closed surface Σ' of smaller genus, and ρ' is a Fuchsian representation of $\pi_1(\Sigma')$, then $\rho := \rho' \circ f_*$ lies outside $\mathsf{Image}(\mathbb{U})$. The following conjecture arose in discussions with Neumann:

Conjecture 3.9. *If $k = 1$, then $\mathbb{U}$ is onto. In general a $\mathsf{PSL}(2, \mathbb{R})$-representation with dense image lies in $\mathsf{Image}(\mathbb{U})$.*

3.8. **Complex projective structures.** A similar construction occurs with the deformation space $\mathbb{CP}^1(\Sigma)$ of *marked* $\mathbb{CP}^1$-*structures* on Σ. A $\mathbb{CP}^1$-manifold is a manifold with a coordinate atlas modeled on $\mathbb{CP}^1$, with coordinate changes in $G = \mathsf{PSL}(2, \mathbb{C})$. The space $\mathbb{CP}^1(\Sigma)$ consists

of equivalence classes of marked $\mathbb{CP}^1$-structures, that is, homeomorphisms $\Sigma \longrightarrow N$ where N is a $\mathbb{CP}^1$-manifold. Since $\mathsf{PSL}(2,\mathbb{C})$ acts holomorphicly, the $\mathbb{CP}^1$-atlas is a holomorphic atlas and N has a underlying Riemann surface M. The resulting Mod_Σ-equivariant map

$$\mathbb{CP}^1(\Sigma) \longrightarrow \mathfrak{T}_\Sigma$$

is a holomorphic affine bundle, whose underlying vector bundle is the holomorphic cotangent bundle of $\mathfrak{T}_\Sigma$. In particular Mod_Σ acts properly on $\mathbb{CP}^1(\Sigma)$ with quotient a holomorphic affine bundle over $\mathfrak{M}_\Sigma$.

The map which associates to a marked $\mathbb{CP}^1$-structure on Σ its holonomy representation is a local biholomorphism

$$\mathbb{CP}^1(\Sigma) \xrightarrow{\mathsf{hol}} \mathsf{Hom}(\pi, G)/G$$

which is known to be very complicated. Gallo-Kapovich-Marden [34] have shown that its image consists of all equivalence classes of representations ρ for which:

- ρ lifts to a representation $\pi \longrightarrow \mathsf{SL}(2,\mathbb{C})$;
- The image $\rho(\pi)$ is not precompact;
- The image $\rho(\pi)$ is not solvable.

The latter two conditions are equivalent to $\rho(\pi)$ not leaving invariant a finite subset of $\mathsf{H}^3 \cup \partial\mathsf{H}^3$. (The cardinality of this finite subset can be taken to be either 1 or 2.)

The holonomy map hol is Mod_Σ-equivariant. The action of Mod_Σ on $\mathbb{CP}^1(\Sigma)$ is proper, since it covers the action of Mod_Σ on $\mathfrak{T}_\Sigma$. The quotient $\mathbb{CP}^1(\Sigma)/\mathsf{Mod}_\Sigma$ affinely fibers over $\mathfrak{M}_\Sigma$. As before hol defines a foliation of $\mathbb{CP}^1(\Sigma)/\mathsf{Mod}_\Sigma$ orbit equivalent to the Mod_Σ-action on $\mathsf{Hom}(\pi, G)/G$. Thus hol may be regarded as a *resolution* of the Mod_Σ-action.

As a simple example, the Mod_Σ-action is proper on the quasi-Fuchsian subset $\mathcal{Q}_\Sigma \subset \mathsf{Hom}(\pi, G)/G$. As noted above, it is a maximal open set upon which Mod_Σ acts properly. Its restriction

$$\mathsf{hol}^{-1}(\mathcal{Q}_\Sigma) \longrightarrow \mathcal{Q}_\Sigma$$

is a covering space ([39]). However, the bumping phenomenon discovered by McMullen [76] implies that hol is not a covering space on any open neighborhood strictly containing $\mathcal{Q}_\Sigma$.

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Department of Mathematics, University of Maryland, College Park, MD 20742

   E-mail address: `wmg@math.umd.edu`