THE DEFORMATION SPACES OF CONVEX $\mathbb{RP}^2$-STRUCTURES ON 2-ORBIFOLDS

SUHYOUNG CHOI AND WILLIAM M. GOLDMAN

Abstract. We determine that the deformation space of convex real projective structures, that is, projectively flat torsion-free connections with the geodesic convexity property on a compact 2-orbifold of negative Euler characteristic is homeomorphic to a cell of certain dimension. The basic techniques are from Thurston's lecture notes on hyperbolic 2-orbifolds, the previous work of Goldman on convex real projective structures on surfaces, and some classical geometry.

Introduction

Let S be a topological space and $n > 0$ an integer. An *n-dimensional orbifold* is a structure Σ on S which is modeled locally on quotient spaces of domains in $\mathbb{R}^n$ by finite group actions. That is, S is covered by a coordinate patches U with coordinate charts $\psi : U \longrightarrow U'/\Gamma_U$ which are homeomorphisms to quotient spaces of domains $U' \subset \mathbb{R}^n$ by finite groups Γ_U acting on U'. Quotient spaces of manifolds by proper actions of discrete groups have natural orbifold structures.

When a discrete group acts properly and freely on a differentiable manifold, its quotient space inherits a natural differentiable structure. Thus orbifolds generalize quotient spaces by proper discrete actions which are not necessarily free. Orbifolds arising as quotient spaces of manifolds are called *good orbifolds,* and we only consider good orbifolds in this paper. In particular, such an orbifold is the quotient of a simply-connected manifold (the *orbifold universal covering* $\tilde{\Sigma}$ by a discrete group $\pi_1(\mathcal{O})$ (the *orbifold fundamental group* of homeomorphisms acting properly on $\tilde{\Sigma}$.

Let G be a Lie group acting transitively on a manifold X. An *(X,G)-structure on* Σ is defined by an orbifold atlas where the domains U' lie in X, the groups Γ are restrictions of finite subgroups of G, and the coodinate changes lie in G.

An $\mathbb{RP}^2$*-structure* is an (X,G)-structure where X is the real projective plane $\mathbb{RP}^2$ and G is its group $\mathsf{PGL}(3,\mathbb{R})$ of collineations.. If $\Omega \subset \mathbb{RP}^2$ is a convex domain and Γ is a discrete group of collineations acting properly on Ω, then the induced $\mathbb{RP}^2$-structure on Ω/Γ is said to be *convex.*

When $\partial\Sigma \neq \emptyset$, boundaries are required to be principal geodesic (see §3).

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Particularly notable among the convex $\mathbb{RP}^2$-structures are *hyperbolic structures.* Namely the interior Ω of a conic is a projective model for hyperbolic geometry in the sense that the collineations preserving Ω are the isometries of a Riemannian metric of constant negative curvature on Ω.

The purpose of this paper is the classification of convex $\mathbb{RP}^2$-structures on 2-dimensional orbifolds.

We state our results in terms of a *deformation space,* whose points parametrize geometric structures, up to a certain equivalence relation. Let Σ be a compact 2-dimensional orbifold. A *marked convex* $\mathbb{RP}^2$*-structure on* Σ is an isomorphism of the orbifold Σ with a convex $\mathbb{RP}^2$-orbifold Ω/Γ. Such a structure is equivalent to an isomorphism

$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathsf{PGL}(3, \mathbb{R})$$

and a compatible diffeomorphism

$$\tilde{\Sigma} \xrightarrow{\approx} \Omega \subset \mathbb{RP}^2$$

where Ω is a convex domain. Two such structures are *equivalent* if the represenations are conjugate, that is, if they differ by composition inner automorphism of $\mathsf{PGL}(3, \mathbb{R})$.

In Chapter 5: "Orbifolds and Seifert fibered spaces" by W. Thurston [37] proved that the deformation space of hyperbolic structures on closed 2-orbifolds of negative Euler characteristic, that is, the Teichmüller space, is homeomorphic to a cell of dimension

$$-3\chi(X_\Sigma) + 2k + l$$

where k is the number of cone-points and l is the number of corner-reflectors. (We give a detailed proof of this, Theorem 6.9, while a sketchy proof is in some versions of his notes.) (Kulkarni-Lee-Raymond [33] have worked out the Teichmüller spaces also using different methods.)

Recall the orbifold Euler characteristic of orbifolds, a signed sum of the number of cells with weights given by 1 divided by the orders of groups associated to cells. Let Σ be a compact 2-orbifold with $\chi(\Sigma) < 0$. (Here, $\chi(\Sigma)$ is the orbifold Euler characteristic.) The subspace of the deformation space $\mathbb{RP}^2(\Sigma)$ of $\mathbb{RP}^2$-structures on Σ corresponding to convex ones is denoted by $\mathcal{C}(\Sigma)$ and the subspace corresponding to hyperbolic ones is denoted by $\mathcal{T}(\Sigma)$, identified as the Teichmüller space of Σ as defined by Thurston [37]. Then we see that $\mathcal{T}(\Sigma)$ is a subspace of $\mathcal{C}(\Sigma)$, and $\mathcal{C}(\Sigma)$ is an open subset of $\mathbb{RP}^2(\Sigma)$.

Theorem A . *Let* Σ *be a compact* 2*-orbifold with* $\chi(\Sigma) < 0$ *and* $\partial\Sigma = \emptyset$*. Then the deformation space* $\mathcal{C}(\Sigma)$ *of convex* $\mathbb{RP}^2$*-structures on* Σ *is homeomorphic to a cell of dimension*

$$-8\chi(X_\Sigma) + (6k_c - 2b_c) + (3k_r - b_r)$$

where X_Σ *is the underlying space of* Σ*,* k_c *is the number of cone-points,* k_r *the number of corner-reflectors,* b_c *the number of cone-points of order two, and* b_r *the number of corner-reflectors of order two.*

The terms should be familiar to geometric topologists but they will be explained in §1. Loosely, a cone-point has a neighborhood which is an orbit space of a cyclic action on an open disk with a fixed point, with the order being the order of the group and a corner-reflector has a neighborhood which is an orbit space of a dihedral group acting on an open disk. The order is the half of the group order.

We remark that the deformation spaces of convex $\mathbb{RP}^2$-structures on closed surfaces of negative Euler characteristic are very interesting spaces admitting complex structures, which were proved using Mönge-Ampere equations by Labourie [29] and Loftin [30], and they might have Kähler structures as conjectured by one of us (Goldman) and even stronger structures as Labourie suggests.

Loftin [30] has estimated the dimension of subspaces of fixed points of automorphism groups acting on the deformations spaces of convex $\mathbb{RP}^2$-surfaces using techniques from differential geometry and the Riemann-Roch theorem.

Assigning a geometric structure to an orbifold Σ is equivalent to giving its universal cover $\tilde{\Sigma}$ an immersion $\mathbf{dev} : \tilde{\Sigma} \to \mathbb{RP}^2$ equivariant with respect to the homomorphism h from the (orbifold) fundamental group $\pi_1(\Sigma)$ to $\mathrm{PGL}(3, \mathbb{R})$. The pair $(\mathbf{dev}, h)$ is called a *development pair*, $\mathbf{dev}$ a *developing map*, and h the associated *holonomy homomorphism*. The pair $(\mathbf{dev}, h)$ is only defined up to the action of $g \in \mathrm{PGL}(3, \mathbb{R})$ so that $g(\mathbf{dev}, h(\cdot)) = (g \circ \mathbf{dev}, g \circ h(\cdot) \circ g^{-1})$.

The map assigning a geometric structure to the conjugacy class of the associated holonomy homomorphism induces the following map, so-called holonomy map,

$$\mathcal{H} : \mathbb{RP}^2(\Sigma) \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbb{R}))/\mathrm{PGL}(3, \mathbb{R})$$

where $\mathrm{PGL}(3, \mathbb{R})$ acts on

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbb{R}))$$

by conjugation.

Let us denote by $C_{\mathcal{T}}(\Sigma)$ the unique component of

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbb{R}))$$

containing the holonomy homomorphisms of hyperbolic $\mathbb{RP}^2$-structures on Σ. Then $C_{\mathcal{T}}(\Sigma)$ is also a component of the part

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbb{R}))^{st}$$

of

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbb{R}))$$

where $\mathrm{PGL}(3, \mathbb{R})$ acts properly. $C_{\mathcal{T}}/\mathrm{PGL}(3, \mathbb{R})$ is said to be a *Hitchin-Teichmüller component* (see [22]). We prove:

Theorem B . *Let Σ be a closed 2-orbifold with negative Euler characteristic. Then*

$$\mathcal{H} : \mathcal{C}(\Sigma) \to C_{\mathcal{T}}(\Sigma)/\mathrm{PGL}(3, \mathbb{R})$$

is a diffeomorphism, and $C_{\mathcal{T}}(\Sigma)$ consists of discrete faithful representations of $\pi_1(\Sigma)$.

Corollary A . *The Hitchin-Teichmüller component $C_{\mathcal{T}}(\Sigma)/\mathrm{PGL}(3, \mathbb{R})$ is homeomorphic to a cell of the dimension as above.*

This gives us a partial classification of discrete representations of fundamental groups using topological ideas: Benoist [1] characterized the group of projective transformations acting on a convex domain for general dimensions. (We only consider the three-dimensional cases here.) An element of $\mathrm{GL}(3, \mathbb{R})$ is *proximal* if it has an attracting fixed point in $\mathbb{RP}^2$ for its standard action. A proximal element is *positive proximal* if the eigenvalue corresponding to the fixed point is positive. A subgroup Γ of $\mathrm{GL}(3, \mathbb{R})$ is *positive proximal* if all proximal elements of Γ are positive proximal. Proposition 1.1 of [1] shows that if Γ is an irreducible subgroup of $\mathrm{GL}(3, \mathbb{R})$, then Γ preserves a properly convex cone in $\mathbb{R}^3$ if and only if Γ is positive proximal. Such a subgroup Γ of $\mathrm{GL}(3, \mathbb{R})$, if discrete, acts on a convex domain Ω in an affine patch of $\mathbb{RP}^2$ so that Ω/Γ is a 2-orbifold. Suppose that Ω/Γ is compact, and Γ contains a free subgroup of two generators. Then Ω/Γ is an orbifold of negative Euler-characteristic. By Theorem A, such groups are parametrized by cells.

We discuss the rigidity of 2-orbifolds with hyperbolic or convex $\mathbb{RP}^2$-structures in a way related to the interesting recent work of Dunfield and Thurston [15]. A rigid hyperbolic 2-orbifold must be an orbifold with empty boundary, of negative Euler characteristic, and contains no 1-dimensional suborbifold cutting it into smaller orbifolds of negative Euler characteristic. From the classification of such orbifolds, we see that such an orbifold is a sphere with three cone-points, a disk with one cone-point and one corner-reflector, and a disk with three corner-reflectors.

The (omitted) proofs of the following corollaries follow directly from Corollary A and Theorem 6.9.

Corollary B . *The sphere Σ with cone-points of order p, q, r satisfying $p \leq q \leq r, 1/p + 1/q + 1/r < 1$ has as its Teichmüller space a single point. If $p = 2$, then so is $\mathcal{C}(\Sigma)$. If $p > 2$, then $\mathcal{C}(\Sigma)$ is homeomorphic to $\mathbb{R}^2$.*

Corollary C . *Let Σ be a 2-orbifold whose underlying space is a disk and with one cone point of order p and a corner-reflector of order q so that $1/p + 1/2q < 1/2$ has as its Teichmüller space a single point. If $q = 2$, then so is $\mathcal{C}(\Sigma)$. If $q > 2$, then $\mathcal{C}(\Sigma)$ is homeomorphic to $\mathbb{R}$.*

Corollary D . *Let Σ be a 2-orbifold whose underlying space is a disk and with three corner-reflectors of order $p \leq q \leq r$, $1/p + 1/q + 1/r < 1/2$. Then $\mathcal{T}(\Sigma)$ is a single point. If $p = 2$, then so is $\mathcal{C}(\Sigma)$. If $p > 2$, then $\mathcal{C}(\Sigma)$ is homeomorphic to $\mathbb{R}$.*

The proof of Theorem A follows [19] closely: given a compact 2-orbifold of negative Euler characteristic, we find "essential" 1-orbifolds decomposing it into twelve types of "elementary" 2-orbifolds of negative Euler characteristic which can no longer be decomposed. Given one of these elementary ones, we determine the deformation space with the projective structures on the boundary 1-orbifolds fixed. The deformation space fibers over the deformation space of the union of the boundary 1-orbifolds. Finally as we rebuild the original orbifold by various splitting and sewing constructions, we simultaneously rebuild the deformation using the fibrations.

To prove Theorem B, we follow [10]: An $\mathbb{RP}^2$-structure on $\mathcal{O}$ determines a conjugacy class of homomorphisms

$$\pi_1(\Sigma) \longrightarrow \mathsf{PGL}(3, \mathbb{R}).$$

The deformation space of $\mathbb{RP}^2$-structures is locally homeomorphic to the quotient of $\mathrm{Hom}(\pi_1(\Sigma), \mathsf{PGL}(3, \mathbb{R}))$. The image of the deformation space of convex $\mathbb{RP}^2$-structures is an open and closed subset. Theorem A implies this space is connected, implying Theorem B.

§1 contains preliminary material on topological 2-orbifolds. We discuss the inverse topological processes of splitting and sewing 2-orbifolds along 1-orbifolds. §2 concerns 2-dimensional orbifolds with $\mathbb{RP}^2$-structures, henceforth called $\mathbb{RP}^2$*-orbifolds* We review classical projective plane geometry. We discuss the deformation space and its relationship to

$$\mathrm{Hom}(\pi_1(\Sigma), \mathsf{PGL}(3,\mathbb{R}))//\mathsf{PGL}(3,\mathbb{R}).$$

We define *the Hitchin-Teichmüller component* of $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ where $G = \mathrm{PGL}(3,\mathbb{R})$). The deformation space of convex $\mathbb{RP}^2$-structures on Σ identifies with an open and closed subset of $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

§3 details the geometric processes of splitting and sewing $\mathbb{RP}^2$-orbifolds. We discuss how these geometric processes affect the deformation spaces of $\mathbb{RP}^2$-structures.

§4 describes the decomposition of convex $\mathbb{RP}^2$-orbifolds of negative Euler characteristic into elementary ones of twelve types. They are *elementary* in that they cannot be further split along 1-orbifolds into ones of negative Euler characteristic. This follows the work of Thurston [37] for hyperbolic 2-orbifolds.

§5 determines the Teichmüller space of elementary 2-orbifolds, that is, the deformation spaces of hyperbolic structures. This is used later to show the existence of convex structures on elementary orbifolds.

§6 computes the deformation space of convex $\mathbb{RP}^2$-structures on elementary 2-orbifolds. Some of the elementary 2-orbifolds are classified by decomposing them into unions of triangles, and reducing these to configurations of triangles and the corresponding easily solvable algebraic relations, following [19]. For others, we will identify the deformation spaces with other types of configuration spaces.

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1. Preliminaries on orbifolds

We define orbifolds, covering maps, orbifold maps, suborbifolds, and Euler characteristics of orbifolds. We discuss how to obtain orbifolds by cutting along 1-orbifolds and how to sew along 1-orbifolds to obtain bigger orbifolds. Euler characteristic zero 2-orbifolds and regular neighborhoods of 1-orbifolds are defined and classified. The topological operations will be given three different interpretations.

The material here can be found principally in Chapter 5 of various versions of Thurston's note [37] and an expository paper on 2-dimensional orbifolds by Scott [36]. See also Ratcliffe [34], Bridson-Haefliger [3], and Kapovich [25]. Most of the technical details for this paper can be found in another paper [9] on geometric structures on orbifolds, which should be read ahead of this paper.

In this paper, we will only work with differentiable objects although we won't require differentiability for the spaces of such objects. Moreover, we assume that group actions are strongly effective. That is, if an element g pointwise fixes a nonempty open set, then g equals the identity element of G.

1.1. **Definition of orbifolds.** Let Q be a Hausdorff, second countable space. An *orbifold atlas* is a open covering $\{U_i\}_{i\in I}$ that for each U_i, there is an open subset $\tilde{U}_i$ of $\mathbb{R}^n$ and a finite group Γ_i of diffeomorphisms acting on $\tilde{U}_i$ with a homeomorphism $\phi_i : \tilde{U}_i/\Gamma_i \to U_i$. Given an inclusion map $U_i \to U_j$, there is an injective homomorphism $f_{ij} : \Gamma_i \to \Gamma_j$ and an embedding $\tilde{\phi}_{ij} : \tilde{U}_i \to \tilde{U}_j$ equivariant with respect to f_{ij} (that is, $\tilde{\phi}(gx) = f_{ij}(g) \circ \tilde{\phi}_{ij}(x)$ for all $g \in \Gamma_i, x \in \tilde{U}_i$) with the following commutative diagram:

$$\begin{array}{ccc} \tilde{U}_i & \xrightarrow{\tilde{\phi}_{ij}} & \tilde{U}_j \\ \downarrow & & \downarrow \\ \tilde{U}_i/\Gamma_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j/f_{ij}\Gamma_i \\ & & \downarrow \end{array}$$

$$\phi_i \downarrow \qquad \qquad \tilde{U}_j/\Gamma_j \qquad (1)$$

$$\downarrow \phi_j$$

$$U_i \quad \subset \quad U_j \qquad (2)$$

where ϕ_{ij} is induced from $\tilde{\phi}_{ij}$. Actually, $(\tilde{\phi}_{ij}, f_{ij})$ is determined up to the action given by

$$g(\tilde{\phi}_{ij}, f_{ij}(\cdot)) = (g \circ \tilde{\phi}_{ij}, g f_{ij}(\cdot) g^{-1}) \text{ for } g \in \Gamma_j.$$

That is, the equivalence class of the pair is given in the information about the orbifold structure. An *orbifold structure* is a maximal family of coverings satisfying the above conditions. The space Y with the structure is an *n-dimensional orbifold* or *n-orbifold*. Given an orbifold Q the underlying space is denoted X_Q. Clearly, a smooth structure on a manifold is an orbifold structure.

An *orbifold* is a topological space with a maximal orbifold atlas.

Given a Lie group G acting on a space X, we define an *(X, G)-structure* on an orbifold Σ to be a maximal collection

$$\{(U_i, \tilde{U}_i, \Gamma_i, \phi_i), (f_{ij}, \tilde{\phi}_{ij})\}$$

in the orbifold structure where $\tilde{U}_i$ is identified with an open subset of X and Γ_i and f_{ij} are required to be restrictions of elements of G and $\tilde{\phi}_{ij}$ conjugation homomorphisms by f_{ij}s.

Here is the basic example. Suppose X is a homogeneous Riemannian manifold with isometry group G and let $\Gamma \subset G$ be a discrete subgroup acting on an open subset Ω/Γ. Then the quotient orbifold Ω/Γ carries an (X, G)-structure.

A point of X_Q is said to be *regular* if it has a neighborhood with trivial associated group. Otherwise it is *singular*. Let p be a singular point. Then for every chart $(\tilde{U}, \Gamma)$ in the orbifold atlas about p, the point p corresponds to a fixed point of a nontrivial element of Γ.

An *orbifold with boundary* has neighborhoods modeled on open subsets of the closed upper half space $\mathbb{R}^{n,+}$. A *suborbifold* Q' on a subspace $X_{Q'} \subset X_Q$ is the subspace so that each point of $X_{Q'}$ has a neighborhood in X_Q modeled on an open subset U of $\mathbb{R}^n$

with a finite group Γ preserving $U \cap \mathbb{R}^d$ where $\mathbb{R}^d \subset \mathbb{R}^n$ is a proper subspace, so that $(U \cap \mathbb{R}^d, \Gamma')$ is in the orbifold structure of Q'. Here Γ' denotes the restricted group of Γ to $U \cap \mathbb{R}^d$, which is in general a quotient group.

The *interior* of Q is defined as the set of points with neighborhoods modeled on open subsets of $\mathbb{R}^n$. The *boundary* of Q is the complement of the interior. The boundary is denoted by ∂Q. The boundary of an n-orbifold is clearly an $(n-1)$-suborbifold without boundary. (It is a subset of the boundary of the underlying space X_Q but is not necessarily all of it.)

A singular point of a 1-orbifold has always a group $\mathbf{Z}_2$ associated with it which acts as a reflection. We call this singular point a *mirror point*.

We can easily classify 1-orbifolds with compact connected underlying spaces. Each of them is diffeomorphic to a circle, a segment with both endpoints a mirror point, a segment with two endpoints one of which is a mirror point, or a segment without singular points. The second one is said to be a *full* 1-*orbifold*, the third *half* 1-orbifold, and the fourth a *segment*.

Definition 1.1. The singular points and the boundary points of 1-orbifolds are said to be *endpoints*.

Note that all 1-orbifolds are very good:

The singular points of a two-dimensional orbifold fall into three types:

(i) The mirror point: $\mathbb{R}^2/\mathbf{Z}_2$ where $\mathbf{Z}_2$ acts by reflections on the y-axis.
(ii) The cone-points of order n: $\mathbb{R}^2/\mathbf{Z}_n$ where $\mathbf{Z}_n$ acting by rotations by angles $2\pi m/n$ for integers m.
(iii) The corner-reflector of order n: $\mathbb{R}^2/D_n$ where D_n is the dihedral group generated by reflections about two lines meeting at an angle π/n.

(The actions here are isometries on $\mathbb{R}^2$.)

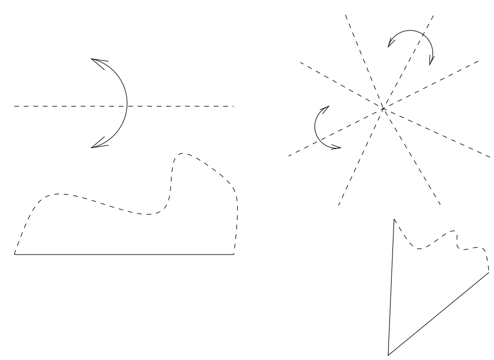

FIGURE 1. The singular points in two-dimensional orbifolds. Mirror points and corner-reflectors are drawn.

Definition 1.2. A map $f : Q \to Q'$ between two orbifolds Q and Q' is an *orbifold map* or simply a *map*, if it induces a continuous function $X_Q \to X_{Q'}$, and for each point of form $f(x)$ for $x \in X_Q$ with (U, Γ) with a homeomorphism ϕ_U from U/Γ to a neighborhood of $f(x)$ there is a pair (V, Γ') with homeomorphism ϕ_V from V/Γ' to a neighborhood of x so that there exists a differentiable map $\tilde{f} : V \to U$ equivariant with respect to a homomorphism $\psi : \Gamma' \to \Gamma$:

$$
\begin{array}{ccc}
V & \xrightarrow{\tilde{f}} & U \\
\downarrow & & \downarrow \\
V/\Gamma' & \xrightarrow{f} & U/\Gamma.
\end{array} \tag{3}
$$

That is, we need to record $(\tilde{f}, \psi)$ but $\tilde{f}$ is determined only up to the actions of Γ and Γ' and ψ changed correspondingly.

Let I be the unit interval with an obvious smooth structure seen as an orbifold structure. Given an orbifold Q, $X_Q \times I$ has an obvious orbifold structure with boundary equal to the union of two orbifolds $Q \times \{0\}$ and $Q \times \{1\}$, orbifold-diffeomorphic to Q itself. Let $Q \times I$ denote the orbifold. A *homotopy* between two orbifold maps from Q to another orbifold Q' is an orbifold map $Q \times I$ to Q' which restricts to the two maps at 0 and 1.

An *isotopy* of an orbifold Σ is a self-diffeomorphism f so that there exists an orbifold map $F : \Sigma \times I \to \Sigma$ where $F_t : \Sigma \to \Sigma$ given by $F_t(x) = F(x, t)$ is a diffeomorphism for each t and F_0 is the identity and $F_1 = f$.

Definition 1.3. A *covering orbifold* of an orbifold Q is an orbifold $\tilde{Q}$ with a surjection $p : X_{\tilde{Q}} \to X_Q$ such that each point $x \in X_Q$ has a neighborhood U with a homeomorphism $\phi : \tilde{U}/\Gamma \to U$ for an open subset of $\tilde{U}$ in $\mathbb{R}^n$ or $\mathbb{R}^{n,+}$ with a group Γ acting on it so that each component V_i of $p^{-1}(U)$ has a diffeomorphism $\tilde{\phi}_i : \tilde{U}/\Gamma_i \to V_i$ (in the orbifold structure) where Γ_i is a subgroup of Γ. We require the quotient map $\tilde{U} \to V_i$ induced by $\tilde{\phi}_i$ composed with p is the quotient map $\tilde{U} \to V$ induced by ϕ.

Clearly, if $f : Q' \to Q$ is an orbifold covering and Q has an (X, G)-structure, then so does Q'.

Given a smooth manifold M and a group Γ acting properly discontinuously, M/Γ has a unique orbifold structure for which the quotient projection $M \longrightarrow M/\Gamma$ is an orbifold covering map.

Definition 1.4. A *good* orbifold is an orbifold which has a covering that is a manifold.

The inverse image of a suborbifold under the covering map of an orbifold is a suborbifold again. If the covering orbifold is a manifold, then the inverse image is a submanifold.

An orbifold always has a so-called universal covering orbifold:

Proposition 1.5. *An orbifold Q has a covering orbifold $p : \tilde{Q} \to Q$ with the following property. If x is a nonsingular point, $p(\tilde{x}) = x$ for $\tilde{x} \in \tilde{Q}$, and $p' : Q' \to Q$ is a covering map with $p'(x') = x$, then there is a lifting orbifold map $q : \tilde{Q} \to Q'$ with $q(\tilde{x}) = x'$.*

Proof. See [9] or §5 of Thurston [37] or Chapter 13 of Ratcliffe [34]. □

A universal covering orbifold is unique up to isomorphisms of covering spaces; that is, given two universal coverings $p_1 : \tilde{Q}_1 \to Q$ and $p_2 : \tilde{Q}_2 \to Q$, there is a diffeomorphism $f : \tilde{Q}_1 \to \tilde{Q}_2$ so that $p_1 \circ f = p_2$.

For good orbifolds, the universal covering orbifolds are simply connected manifolds. For two-dimensional good orbifolds, they are diffeomorphic to either a disk or a sphere.

Let $p : Q' \to Q$ be an orbifold covering map. A *deck transformation* of a covering orbifold Q' of Q is an orbifold self-diffeomorphism of Q' which composed with p is equal to p. When Q' is the universal cover of Q, then the group of deck transformations are said to be the (*orbifold*) *fundamental group* of Q and denoted by $\pi_1(Q)$. (See Chapter 13 of Ratcliffe [34].) Thus, a good orbifold is a quotient orbifold of a simply connected manifold by the fundamental group. We denote by $\pi_1(Q)$ the group of deck transformations.

Let M and M' be two orbifolds, and let $\tilde{M}$ and $\tilde{M}'$ be their universal covers with deck transformations $\pi_1(M)$ and $\pi_1(M')$ respectively. Given a map $f : \tilde{M} \to \tilde{M}'$ lifting a diffeomorphism $f' : M \to M'$, define a homomorphism

$$\begin{aligned} \tilde{f}_* : \pi_1(M) &\to \pi_1(M') \\ \gamma &\mapsto \tilde{f} \circ \gamma \circ \tilde{f}^{-1}. \end{aligned}$$

1.2. **The Euler characteristics of orbifolds.** In dimension 1 or 2, the underlying space X_Q of an orbifold Q has a cellular decomposition such that each point of an open cell has the same model open set and the same finite group action. We define the Euler characteristic to be

$$\chi(Q) = \sum_{c_i} (-1)^{\dim(c_i)} (1/|\Gamma(c_i)|),$$

where c_i ranges over the open cells and $|\Gamma(c_i)|$ is the order of the group Γ_i associated with c_i.

For example, a full 1-orbifold has Euler characteristic zero.

We recall that the cardinality of inverse image under a covering map $p : Q' \to Q$ is constant over nonsingular points. If $p : Q' \to Q$ is k-sheeted, then $\chi(Q') = k\chi(Q)$. (This follows since over the regular points, the map is an ordinary covering map.)

Suppose that a 2-orbifold Σ with or without boundary has the underlying space X_Σ and m cone-points of order q_i and n corner-reflectors of order r_j and n_Σ boundary full 1-orbifolds. Then the following generalized Riemann-Hurwitz formula is very useful:

$$\chi(\Sigma) = \chi(X_\Sigma) - \sum_{i=1}^{m} \left(1 - \frac{1}{q_i}\right) - \frac{1}{2} \sum_{j=1}^{n} \left(1 - \frac{1}{r_j}\right) - \frac{1}{2} n_\Sigma, \tag{4}$$

which is proved by a doubling argument. (See Thurston [37] or Scott [36] for details.)

For 2-orbifolds Σ_1, Σ_2 meeting in a compact 1-orbifold Y forming a 2-orbifold Σ as a union, we have the following additivity formula:

$$\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_2) - \chi(Y), \tag{5}$$

which can be verified by counting cells with weights since the orders of singular points in the boundary orbifold equal the ambient orders.

Thurston showed that compact 2-orbifolds of nonpositive Euler characteristic with or without boundary admit euclidean or hyperbolic structures (with geodesic boundary when there is a nonempty boundary). See Theorem 6.9. They are good orbifolds; that is, they are covered by surfaces, since orbifolds admitting geometric structures are good again by Thurston [37] (see [9] also). We also claim that these orbifolds admit finite regular-covers by surfaces; that is, they are very good. In other words, there is a finite group F acting on a surface S such that our orbifold is of form S/F, a quotient orbifold: Since such an orbifold is of form the hyperbolic plane H^2 quotient by an infinite discrete group Γ. There is a finite-index torsion-free normal subgroup Γ' by Selberg's lemma. Thus H^2/Γ' is a closed surface and our orbifold H^2/Γ is a finite quotient orbifold of it.

Since $\pi_1(S) = \Gamma$ is finitely presented, we obtain:

Theorem 1.6. *Let Σ be a compact orbifold of negative Euler characteristic. Then the group $\pi_1(\Sigma)$ of deck-transformations is finitely presented.*

1.3. **Splitting and sewing on topological 2-orbifolds.** We now describe the process of splitting and sewing: Let S be a very good orbifold. Suppose that its underlying space X_S is the interior of a compact surface-with-boundary (For example, S could be a suborbifold obtained by removing from an orbifold S' boundary components, embedded 1-orbifolds, or circles. The completion is in general different from $X_{S'}$.)

Let $\hat{S}$ be a very good covering space, that is, a finite regular covering space, of S, so that S is orbifold-diffeomorphic to $\hat{S}/F$ where F is a finite group acting on $\hat{S}$. Since $X_{\hat{S}} = \hat{S}$ is also pre-compact and has a path-metric, complete it to obtain a compact surface $X'_{\hat{S}}$. Since F acts isometric with respect to some metric of $\hat{S}$, it acts on $X'_{\hat{S}}$. We easily see that X_S can be identified naturally as a subspace of the 2-nd countable Hausdorff space $X'_{\hat{S}}/F$ which has a quotient orbifold structure induced from $X'_{\hat{S}}$ by the quotient-orbifold process described briefly.

Definition 1.7. $X'_{\hat{S}}/F$ with the quotient orbifold structure is said to be the *orbifold-completion* of S.

Let S be a 2-orbifold with an embedded circle or a full 1-orbifold l in the interior of S. The completion S' of $S - l$ is said to be obtained from *splitting* S along l. Since $S - l$ has an embedded copy in S', we see that there exists a map $S' \to S$ sending the copy to $S - l$. Let l' denote the boundary component of S corresponding to l under the map. Conversely, S is said to be obtained from *sewing S' along l'*.

If the interior of the underlying space of l lies in the interior of the underlying space of S, then the components of S' are said to be *decomposed components of S along l*, and we also say that S *decomposes* into S' along l. Of course, if l is a union of disjoint embedded circles or full 1-orbifolds, the same definition holds.

There are two distinguished classes of splitting and sewing operations: A nonsingular compact boundary component can be replaced by a mirror, and in a unique way: a boundary point has a neighborhood which is realized as a quotient of an open ball by a $\mathbf{Z}_2$-action generated by a reflection about a line. Such a system of model neighborhoods can be chosen consistently to produce an orbifold structure. A boundary full 1-orbifold can be replaced by 1-orbifold of mirror points and two corner-reflectors of order two; this construction is also unique. The interior points of the 1-orbifold have neighborhoods as above, and a boundary point has a neighborhood which is a quotient space of a dihedral group of order four acting on the open ball generated by two reflections. Again, such a system produces an orbifold structure. We call the forward process *silvering* and the reverse process *clarifying*.

1.4. **Euler-characteristic-zero 2-orbifolds.** An *edge* is a segment in the singular locus of a 2-orbifold which ends in corner-reflectors or in the boundary. An edge is a 1-orbifold only if its endpoints are corner-reflectors of order two or boundary points.

Let A be a compact annulus with boundary. The quotient orbifold of an annulus has Euler characteristic zero. From equation (4), we can determine all of the Euler characteristic zero 2-orbifolds with nonempty boundary. We call them the *annular* orbifolds as they are quotients of annuli. Each of them is diffeomorphic to one of the following orbifolds:

(1) an annulus,
(2) a Möbius band,
(3) an annulus with one boundary component silvered (a *silvered annulus*),
(4) a disk with two cone-points of order two with no mirror points (a $(;2,2)$-*disk* from Thurston's notation),
(5) a disk with two boundary 1-orbifolds, two edges (*a silvered strip*),
(6) a disk with one cone-point and one boundary full 1-orbifold (a *bigon with a cone-point of order two*), that is, it has only one edge, and
(7) a disk with two corner-reflectors of order two and one boundary full 1-orbifold (a *half-square*). (It has three edges.)

To prove this simply notice that the underlying space must have a nonnegative Euler characteristic. When the Euler characteristic of the space is zero, there are no cone-points, corner-reflectors, and the boundary full 1-orbifolds, since they will make the Euler characteristic negative. Hence the first three occur. Suppose that the underlying space is a disk. If there are no singular points in the boundary, then we obtain the fourth case as there has to be exactly two cone-points of order two for the Euler characteristic to be zero. If there are two boundary full 1-orbifolds, then there are no singular points in the interior and no corner-reflector can exist; thus, we have the fifth case. Assume that there exists exactly one boundary full 1-orbifold. If there is a cone-point, then it must be the unique one of order two. Thus, we have the sixth case. If there are no cone-points, but corner-reflectors, then there are exactly two corner-reflectors of order two and no more. We have the seventh case.

1.5. **Regular neighborhoods of 1-orbifolds.** Suppose that there exists a circle or a 1-orbifold l embedded in the interior of a 2-orbifold S, and assumed not to be homotopic to a point. l has Euler characteristic zero. Thus in a good cover $\hat{S}$ of S, the inverse image of certain of its neighborhoods is a disjoint union of annuli or Möbius bands. Thus, l has a neighborhood of zero Euler characteristic: Since the inverse image of l consists of closed curves which represent generators of the fundamental group of the neighborhoods, it follows that in the first two cases (1) and (2), l is the closed curve representing the generator of the fundamental group; in case (3), l is the mirror set that is a boundary component; in case (4), l is the arc connecting the two cone-points unique up to homotopy; in case (5), l is an arc connecting two interior points of two edges respectively; in case (6), l is an arc connecting an interior point of an edge and the

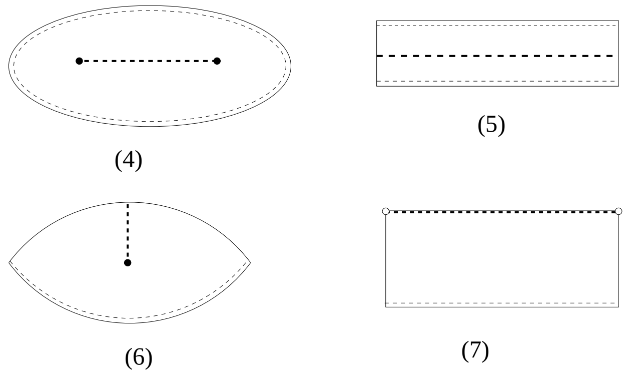

FIGURE 2. Orbifolds of zero-Euler-characteristic in (4)-(7). A thin dashed arc indicates boundary and a thick dashed arc a suborbifold l. A black dot indicates a cone-point of order two and a white dot a corner-reflector of order two.

cone-point of order two; and in the final case (7), the edge in the topological boundary connecting the two corner-reflectors of order two.

Definition 1.8. Given a 1-orbifold l and a neighborhood N of it in some ambient 2-orbifold, N is said to be a *regular neighborhood* if the pair (N, l) is diffeomorphic to one of the above.

Proposition 1.9. *A* 1*-orbifold in a good* 2*-orbifold has a regular neighborhood which is unique up to isotopy.*

Proof. The existence is proved above. The uniqueness up to isotopy is proved as follows: Each regular neighborhood fibers over a 1-orbifold with fibers connected 1-orbifolds in the orbifold sense. A regular neighborhood can be isotoped into any other regular neighborhood by contracting in the fiber directions. To see this, we can modify the proof of Theorem 5.3 in Chapter 4 of [21] to be adopted to an annulus with a finite group acting on it and an imbedded circle. □

1.6. **Splitting and sewing on 2-orbifolds reinterpreted.** An orbifold is said to be an *open orbifold* if the underlying space is noncompact and the boundary is empty. If one removes l from the interior of these orbifolds, we obtain either a union of one or two open annuli, or a union of one or two open silvered strip. In (2)-(4), an open annulus results. For (1), a union of two open annuli results. For (6)-(7), an open silvered strip results. For (5), we obtain a union of two open silvered strips. These can be easily completed to be a union of one or two compact annuli or a union of one or two silvered strips respectively. To see this simply identify them to be dense open suborbifolds of the unions of compact annuli or silvered strips.

Let l be a 1-orbifold embedded in the interior of an orbifold S. We can complete $S - l$ in this manner: We take a closed regular neighborhood N of l in S. We remove $N - l$ to obtain the above types and complete it and re-identify with $S - l$ to obtain a compactified orbifold. This process is the splitting of S along l.

Conversely, we can describe sewing: Take an open annular 2-orbifold N which is a regular neighborhood of a 1-orbifold l. Suppose that l is a circle. We obtain $U = N - l$

which is a union of one or two annuli. Take an orbifold S' with a union l' of one (resp. two) boundary components which are circles. Take an open regular neighborhood of l' and remove l' to obtain V. U and V are the same orbifold. We identify $S'-l'$ and $N-l$ along U and V. This gives us an orbifold S, and it is easy to see that S is obtained from S' by sewing along l'. l corresponds to a 1-orbifold l'' in S in a one-to-one manner. We can obtain (1),(2),(3)-type neighborhoods of l'' in this way. The operation in case (1) is said to be *pasting*, in case (2) *cross-capping*, and in case (3) *silvering* along simple closed curves.

Suppose that l is a full 1-orbifold. $U = N - l$ is either an open annulus or a union of one (resp. two) silvered strips. The former happens if N is of type (4) and the latter if N is of type (5)-(7). In case (4), take an orbifold S' with a boundary component l' a circle. Then we can identify U with a regular neighborhood of l' removed with l' to obtain an orbifold S. Then l corresponds a full 1-orbifold l'' in S in a one-to-one manner. l'' has a type-(4) regular neighborhood. The operation is said to be *folding* along a simple closed curve.

In the remaining cases, take an orbifold S' with a union l' of one (resp. two) boundary full 1-orbifolds. Take a regular neighborhood N of l' and remove them to obtain V. Identify U with V for $S' - l'$ and $N - l$ to obtain S. Then S is obtained from S' by sewing along l'. Again l corresponds to a full 1-orbifold l'' in S in a one-to-one manner. We obtain (5),(6), and (7)-type neighborhoods of l'' in this way, where the operations are said to be *pasting*, *folding*, and *silvering* along full 1-orbifolds respectively.

In other words, silvering is the operation of removing a regular neighborhood and replacing by a silvered annulus or a half square. Clarifying is an operation of removing the regular neighborhood and replacing an annulus or a silvered strip.

Proposition 1.10. *The Euler characteristic of an orbifold before and after splitting or sewing remains unchanged.*

Proof. Form regular neighborhoods of the involved boundary components of the split orbifold and those of the original orbifold. They have zero Euler characteristic. Since their boundary 1-orbifolds have zero Euler characteristic, the lemma follows by the additivity formula (5). □

1.7. **Identification interpretations of splitting and sewing.** In the following we describe the topological identification process of the underlying space involved in these six types of sewings. The orbifold structure on the sewed orbifold should be clear.

Let an orbifold Σ have a boundary component b. (Σ is not necessarily connected.) b is either a simple closed curve or a full 1-orbifold. We find a 2-orbifold Σ'' constructed from Σ by sewing along b or another component of Σ.

(A) Suppose that b is diffeomorphic to a circle; that is, b is a closed curve. Let Σ' be a component of the 2-orbifold Σ with boundary component b'. Note that Σ' may be the same component as the component containing b, and b' may equal b. Suppose that there is a diffeomorphism $f : b \to b'$. Then we obtain a bigger orbifold Σ'' glued along b and b' topologically.

(I) The construction so that Σ'' does not create any more singular point results in an orbifold Σ'' so that
$$\Sigma'' - (\Sigma - b \cup b')$$
is a circle with neighborhood either diffeomorphic to an annulus or a Möbius band.
  (1) In the first case, $b \neq b'$ (pasting).
  (2) In the second case, $b = b'$ and $\langle f \rangle$ is of order two without fixed points (cross-capping).

(II) When $b = b'$, the construction so that Σ'' does introduce more singular points to occur in an orbifold Σ'' so that
$$\Sigma'' - (\Sigma - b)$$
is a circle of mirror points or is a full 1-orbifold with endpoints in cone-points of order two depending on whether $f : b \to b$
  (1) is the identity map (silvering), or
  (2) is of order two and has exactly two fixed points (folding).

(B) Consider when b is a full 1-orbifold with endpoints mirror points.
  (I) Let Σ' be a component orbifold (possibly the same as one containing b) with boundary full 1-orbifold b' with endpoints mirror points where $b \neq b'$. We obtain a bigger orbifold Σ'' by gluing b and b' by a diffeomorphism $f : b \to b'$. This does not create new singular points (pasting).
  (II) Suppose that $b = b'$. Let $f : b \to b$ be the attaching map. Then
    (1) if f is the identity, then b is silvered and the end points are changed into corner-reflectors of order two (silvering).
    (2) If f is of order two, then Σ'' has a new cone-point of order two and has one-boundary component orbifold removed away. b corresponds to a mixed type 1-orbifold in Σ' (folding).

It is obvious how to put the orbifold structure on Σ'' using the previous descriptions using regular neighborhoods above.

2. Projective orbifolds and the Hitchin-Teichmüller components

By an $\mathbb{RP}^2$-*structure* or *projectively flat structure* on a 2-orbifold Σ we mean an $(\mathbb{RP}^2, \mathrm{PGL}(3, \mathbb{R}))$-structure on Σ. From now on, we look at two-dimensional $\mathbb{RP}^2$-orbifolds, that is, orbifolds with $\mathbb{RP}^2$-structures.

In this section, we first go over some needed materials in projective plane geometry.

Next, we define the deformation spaces of $\mathbb{RP}^2$-structures on orbifolds, describe local properties, and define convex $\mathbb{RP}^2$-structures (when the orbifolds are boundaryless).

We discuss the relationship between the $\mathbb{RP}^2$-structures and holonomy representations. First, we deduce that the deformation space is locally Hausdorff from the corresponding property of the holonomy representation variety. Next, we discuss convex $\mathbb{RP}^2$-structures. We show that the deformation space of convex $\mathbb{RP}^2$-structures on an orbifold is an open subset of the full deformation space. We identify the deformation space of convex $\mathbb{RP}^2$-structures on orbifolds with a subset of the space of conjugacy classes of representations of its fundamental group using the above relationship.

2.1. **Projective geometry.** We go over basic definitions and facts, which can be found in Coxeter [14]: Recall that the plane projective geometry is a geometry based on the pair consisting of the projective plane $\mathbb{RP}^2$, the space of lines passing through the origin in $\mathbb{R}^3$ with the group $\mathrm{PGL}(3, \mathbb{R})$, the projectivized general linear group acting on it. $\mathbb{RP}^2$ is considered as the quotient space of $\mathbb{R}^3 - \{O\}$ by the equivalence relation $v \sim w$ iff $v = sw$ for a scalar s.

A *point* is an element of $\mathbb{RP}^2$ and a *line* is a codimension-one subspace of $\mathbb{RP}^2$, i.e., the image of a two-dimensional vector subspace of $\mathbb{R}^3$ with the origin removed under the quotient map. Two points are contained in a unique line and two lines meet at a unique point. Points are *collinear* if they lie on a common line. Lines are *concurrent* if they pass through a common point. A pair of points or lines are *incident* if they meet each other.

The dual projective plane $\mathbb{RP}^{2\dagger}$ is given as the space of lines in $\mathbb{RP}^2$. We can identify it as the quotient of the dual vector space of $\mathbb{R}^3$ with the origin removed by the scalar equivalence relations as above.

An element of $\mathrm{PGL}(3, \mathbb{R})$ acting on $\mathbb{RP}^2$ is said to be a *collineation*. The elements are uniquely represented by matrices of determinant equal to 1. Their conjugacy classes are in one-to-one correspondence with the topological conjugacy classes of their actions on $\mathbb{RP}^2$. (Sometimes, we will use matrices of determinant -1 for convenience.)

Among collineations, an order-two element is said to be a *reflection*. It has a unique line of fixed points and an isolated fixed point. Actually, any pair of reflections are conjugate to each other, and given a line and a point not on the line, we can find a unique reflection with these fixed point sets. A reflection will often be represented by a matrix of determinant equal to -1 and the isolated fixed point corresponds to the eigenvector of -1 eigenvalue.

Given two lines a map between the points in one line l_1 to the other l_2 is a *projectivity* if the map is induced from a rank-two linear map from the vector subspace corresponding to l_1 to that corresponding to l_2.

By *duality*, we mean the one-to-one correspondence between the lines in $\mathbb{RP}^2$ with the points in $\mathbb{RP}^{2\dagger}$ and one between the points in $\mathbb{RP}^2$ with the lines in $\mathbb{RP}^{2\dagger}$. The correspondence preserves incidence relations. (There are many dualities between $\mathbb{RP}^2$ and $\mathbb{RP}^{2\dagger}$.)

Under duality, a line in $\mathbb{RP}^{2\dagger}$ correspond to the set of all lines through a point in $\mathbb{RP}^2$, so-called a pencil of lines, and vice-versa.

By duality, given the pencil of lines through a point p and the pencil of lines through another point q, a *projectivity* between the two pencils is a one-to-one correspondence that is the projectivity from the dual line to p to that of q.

A pair of four points in $\mathbb{RP}^2$ are always equivalent by a collineation. Let l_1 and l_2 be two lines and let p_1^1, p_2^1, p_3^1 be three distinct points of l_1 and let p_1^2, p_2^2, p_3^2 be three distinct points in l_2. Then there is a projectivity sending p_i^1 to p_i^2 for $i = 1, 2, 3$.

A nonzero vector v in $\mathbb{R}^3$ *represents* a point p of $\mathbb{RP}^2$ if v is in the equivalence class of p or in the ray p. We often label a point of $\mathbb{RP}^2$ by a vector representing it and vice versa by an abuse of notation.

A homogeneous coordinate of a point p is given by $[x_1, x_2, x_3]$ if p is represented by a vector (x_1, x_2, x_3). Note that $[x_1, x_2, x_3] = [\lambda x_1, \lambda x_2, \lambda x_3]$ for a nonzero real number λ.

We recall:

Definition 2.1. Let y, z, u, v be four distinct collinear points with $u = \lambda_1 y + \lambda_2 z$ and $v = \mu_1 y + \mu_2 z$. The cross-ratio $[y, z; u, v]$ is defined to be $\lambda_2\mu_1/\lambda_1\mu_2$.

Given a set of four mutually distinct points $p^1_1, p^1_2, p^1_3, p^1_4$ on a line l_1 and another such set $p^2_1, p^2_2, p^2_3, p^2_4$ on a line l_2, there is a projectivity $l_1 \to l_2$ sending p^1_i to p^2_i iff

$$[p^1_1, p^1_2; p^1_3, p^1_4] = [p^2_1, p^2_2; p^2_3, p^2_4].$$

Another convenient formula is given by

$$[y, z; u, v] = \frac{(\bar{u} - \bar{y})(\bar{v} - \bar{z})}{(\bar{u} - \bar{z})(\bar{v} - \bar{y})}$$

where $\bar{x}$ is the coordinate of an affine coordinate system on the line containing y, z, u, v. (See Berger [2] for definitions of affine coordinates.) Even by a further abuse of notation, we will identify points with their affine coordinates sometimes. In particular, the cross-ratio $[0, \infty; 1, z]$ equals z.

For example, if $y = 1, z = 0$, and $1 > u > v > 0$, then the cross ratio $[1, 0, u, v]$ equals

$$\frac{1-u}{u}\frac{v}{1-v}$$

which is positive and can realize any values in the open interval $(0, 1)$.

The cross-ratio of four concurrent lines is also defined similarly (see Busemann-Kelly [4]) using the dual projective plane where they become four collinear points.

Given a notation $[y, z; u, v]$ with four points y, z, u, v, they are to be on an image of a segment under a projective map where y, z the endpoints and y, v separates u from z. This is the standard position of the four points in this paper.

2.2. **Developing orbifolds.** Thurston shows that all orbifolds admitting (X, G)-structures are good [37]. It also follows from his work that for a 2-orbifold Σ, the existence of $(\mathbb{RP}^2, \mathrm{PGL}(3, \mathbb{R}))$-structure is equivalent to giving a developing map $\mathbf{dev} : \tilde{\Sigma} \to \mathbb{RP}^2$ from the universal covering space $\tilde{\Sigma}$ of Σ equivariant with respect to the holonomy homomorphism $h : \pi(\Sigma) \to G$ where $\pi(\Sigma)$ is the deck transformation group of $\tilde{\Sigma}$. In other words h satisfies:

$$h(\vartheta) \circ \mathbf{dev} = \mathbf{dev} \circ \vartheta, \vartheta \in \pi(\Sigma). \tag{6}$$

(See Proposition 5.4.2 of Thurston [37] or [9] or Bridson-Haefliger [3].)

Two pairs $(\mathbf{dev}, h)$ and $(\mathbf{dev}', h')$ are *equivalent* if and only if there exists $\psi \in G$ satisfying

$$\mathbf{dev}' = \psi \circ \mathbf{dev}, h'(\cdot) = \psi \circ h(\cdot) \circ \psi^{-1} \text{ for some } \psi \in G. \tag{7}$$

The development pair is determined by the structure and a germ of a structure at the basepoint; changing the germ of the structure changes the pair in its equivalence class. Thus the *equivalence class* of the development pair is uniquely determined by the structure.

The image of **dev** is a *developing image* and that of h is a *holonomy group*.

2.3. **Types of Singularities.** An automorphism of $\mathbb{RP}^2$ is said to be a *reflection* if its matrix is conjugate to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

A reflection has a line of fixed points and an isolated fixed point, which is said to be the *reflection point*. An automorphism of $\mathbb{RP}^2$ is said to be a *rotation of order* n, $n = 2, 3, \dots$, if its matrix is conjugate to

$$\begin{bmatrix} \cos 2\pi/n & -\sin 2\pi/n & 0 \\ \sin 2\pi/n & \cos 2\pi/n & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A rotation has a unique isolated fixed point, called a *rotation point*, and an invariant line. A one-parameter family of invariant conics fills the complement in $\mathbb{RP}^2$ of the rotation point and the invariant line. A rotation of order two is a reflection also and conversely.

For $\mathbb{RP}^2$-orbifolds, the singular points have neighborhoods modelled on quotients of domains by finite groups corresponding to one of the following types:

(i) A mirror point: An open disk in $\mathbb{RP}^2$ meeting a line of fixed points of a reflection.
(ii) A cone-point of order n: An open disk in $\mathbb{RP}^2$ containing a rotation point of the rotation of order n.
(iii) A corner-reflector of order n: An open disk in $\mathbb{RP}^2$ containing the intersection point of the lines of fixed points of two reflections g_1 and g_2 so that $g_1 \circ g_2$ is a rotation of order n.

For example the quotient of $\mathbb{R}^2$ by reflections about horizontal and vertical lines through integer points is a square with boundary mirror points and corner-reflectors of order two.

2.4. **Example: Elementary annuli.** Let ϑ be a collineation represented by a diagonal matrix with distinct positive eigenvalues. Then it has three fixed points in $\mathbb{RP}^2$: an attracting fixed point of the action of $\langle\vartheta\rangle$, a repelling fixed point, and a saddle type fixed point. Three lines passing through two of them are ϑ-invariant, as are four open triangles bounded by them. Choosing two open sides of an open triangle ending at an attracting fixed point or a repelling fixed point simultaneously, their union is acted properly and freely upon by $\langle\vartheta\rangle$. The quotient space is diffeomorphic to an annulus. The $\mathbb{RP}^2$-surface projectively diffeomorphic to the quotient space is said to be an *elementary annulus*.

2.5. **Example: π-Annuli.** Take two adjacent triangles, and three open sides of them all ending in an attracting fixed point or a repelling fixed point. Then the quotient of the union by $\langle\vartheta\rangle$ is diffeomorphic to an annulus. The projectively diffeomorphic surfaces are said to be *π-annuli*. (See [5] and [6] for more details.)

A reflection sending one triangle to the other induces an order-two group. The quotient map is an orbifold map, and the quotient space carries an orbifold structure so that one boundary component is made of mirror points. (See Page 31 for more details).

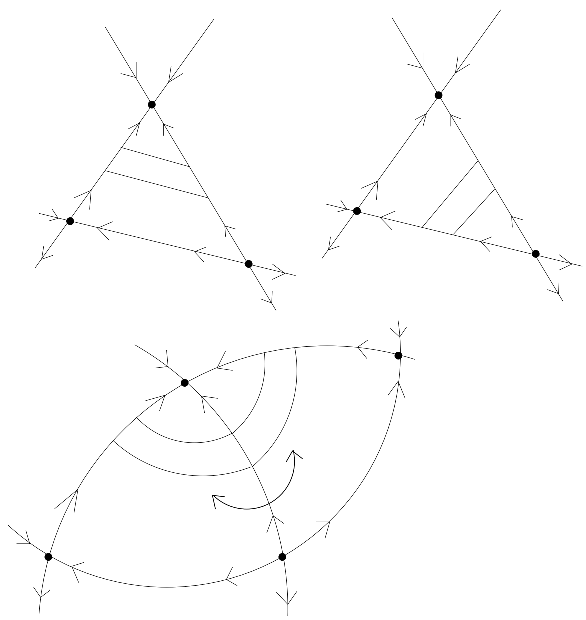

FIGURE 3. Elementary annuli and a π-annulus and an action on it.

2.6. **The deformation spaces and holonomy.** We define the deformation space $\mathbb{RP}^2(\Sigma)$ of $\mathbb{RP}^2$-structures on a connected 2-orbifold Σ as follows (assuming Σ is connected and has empty boundary): Give the C^1-topology to the set $\mathcal{S}(\Sigma)$ of all pairs $(\mathbf{dev}, h)$ satisfying equation (6) on $\tilde{\Sigma}$. Two pairs $(\mathbf{dev}, h)$ and $(\mathbf{dev}', h')$ are *equivalent under isotopy* if there exists a self-diffeomorphism f of the universal cover $\tilde{\Sigma}$ of Σ commuting with the deck transformations so that $\mathbf{dev}' = \mathbf{dev} \circ f$ and $h' = h$. We denote by $\mathbb{RP}^{2*}(\Sigma)$ the space of equivalence classes with the quotient topology.

The pairs $(\mathbf{dev}, h)$ and $(\mathbf{dev}', h')$ are equivalent under $\mathrm{PGL}(3, \mathbb{R})$-action, if there exists an element g of $\mathrm{PGL}(3, \mathbb{R})$ so that $\mathbf{dev}' = g \circ \mathbf{dev}$ and $h'(\cdot) = gh(\cdot)g^{-1}$. The quotient space of $\mathbb{RP}^{2*}(\Sigma)$ under the $\mathrm{PGL}(3, \mathbb{R})$-equivalence relation is denoted by $\mathbb{RP}^2(\Sigma)$.

The deformation space may also be intepreted as the space of isotopy classes of $\mathbb{RP}^2$-structures on Σ.

Non-isotopic $\mathbb{RP}^2$-structures represent different points in the deformation spaces. An example is a pair of $\mathbb{RP}^2$-orbifolds with non-conjugate holonomy homomorphisms (see [9] for details).

By forgetting $\mathbf{dev}$ from the pair $(\mathbf{dev}, h)$, we obtain an induced map

$$\mathcal{H}' : \mathbb{RP}^{2*}(\Sigma) \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbb{R}))$$

to the space of homomorphisms of $\pi_1(\Sigma)$ since the isotopy does not change the holonomy homomorphism.

By Theorem 1.6, we see that $\pi_1(\Sigma)$ is a finitely presented group. From now on, we denote

$$H(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PGL}(3,\mathbb{R}))$$

for the $\mathbb{R}$-algebraic subset of $\mathrm{PGL}(3,\mathbb{R})^n$ satisfying the relations corresponding to the relations of the presentation of $\pi_1(\Sigma)$ where n is the number of the generators of $\pi_1(\Sigma)$.

The main result of [9] is that the map $\mathcal{H}'$ is a local homeomorphism since $\pi_1(\Sigma)$ is finitely presented. The proof of this is not much different from the manifold case. The idea of proof is based on [18] and Morgan and Lok [31] (from lectures of Morgan) generalizing Weil's work [39].

Let U^n denote the open subset of $\mathrm{PGL}(3,\mathbb{R})^n$ consisting of $(X_1,\dots,X_n)$ such that no line in $\mathbb{R}^3$ is simultaneously invariant under $X_1,\dots,X_n$. The $\mathrm{PGL}(3,\mathbb{R})$-action is proper and free on the set

$$U(\Sigma) := H(\Sigma) \cap U^n.$$

(see [19]).

Theorem 2.2. *Let Σ be a connected closed 2-orbifold with $\chi(\Sigma) < 0$. Then $\mathbb{RP}^2(\Sigma)$ has the structure of Hausdorff real analytic variety modeled on $U(\Sigma)/\mathrm{PGL}(3,\mathbb{R})$, and the induced map*

$$\mathcal{H} : \mathbb{RP}^2(\Sigma) \to U(\Sigma)/\mathrm{PGL}(3,\mathbb{R})$$

is a local homeomorphism.

Proof. First, we show that the image of $\mathcal{H}'$ is in $U(\Sigma)$. By Lemma 2.5 of [19],the holonomy group of a 2-orbifold of negative Euler characteristic fixes no line, since a finite-index subgroup is a fundamental group of a closed surface of negative Euler characteristic. Since $\pi_1(\Sigma)$ is finitely presented by Theorem 1.6, the holonomy map is a local homeomorphism by [9]. □

Suppose now that Σ has more than one component. Let $\Sigma_1,\dots,\Sigma_n$ denote the connected components of Σ. Define $\mathbb{RP}^2(\Sigma)$ to be the product

$$\mathbb{RP}^2(\Sigma_1) \times \cdots \times \mathbb{RP}^2(\Sigma_n).$$

Similarly define $\mathbb{RP}^{2*}(\Sigma)$ and $\mathcal{S}(\Sigma)$ and $H(\Sigma)$. If the Euler characteristic of each component of Σ is negative, the product map

$$\mathcal{H}' : \mathbb{RP}^{2*}(\Sigma) \to H(\Sigma) := \prod_{i=1}^{n} H(\Sigma_i)$$

is a local homeomorphism, and so is the product map

$$\mathcal{H} : \mathbb{RP}^2(\Sigma) \to U(\Sigma)/\mathrm{PGL}(3,\mathbb{R}) := \prod_{i=1}^{n} (U(\Sigma_i)/\mathrm{PGL}(3,\mathbb{R})).$$

2.7. **Hitchin-Teichmüller components.** Let Σ be a closed connected 2-orbifold with $\chi(\Sigma) < 0$. An $\mathbb{RP}^2$-structure on Σ is *convex* if **dev** is a diffeomorphism to a convex subset of an affine patch in $\mathbb{RP}^2$. Then Σ is projectively diffeomorphic to Ω/Γ where Γ is a discrete subgroup of $\mathrm{PGL}(3,\mathbb{R})$ acting on a convex domain Ω. By Theorem 3.2 of [19], Ω is a strictly convex precompact subset of an affine patch of $\mathbb{RP}^2$. Its boundary $\partial\Omega$ is C^1 and is *strictly convex:* that is, it contains no line segment.

A conic in $\mathbb{RP}^2$ is the zero set of a quadratic form of signature $(2,1)$ on $\mathbb{RP}^2$ with homogeneous coordinates x_0, x_1, x_2. The stabilizer of a conic is conjugate to the projectivized orthogonal group $\mathrm{PO}(1,2)$. Applying a collineation, we may assume that the conic is defined by the standard diagonal quadratic form of signature $(2,1)$ and that the holonomy group lies in $\mathrm{PO}(1,2)$.

Hyperbolic structures on Σ form a distinguished class of convex $\mathbb{RP}^2$-structures: Klein's model of hyperbolic geometry identifies a hyperbolic plane and its isometry group with the convex domain Ω bounded by a conic and its stabilizer $\mathrm{PO}(1,2)$, respectively. A hyperbolic structure on a 2-orbifold has a chart into Ω with transition functions in $\mathrm{PO}(1,2)$. Since Ω is a subset of $\mathbb{RP}^2$ and $\mathrm{PO}(1,2)$ is a subgroup of $\mathrm{PGL}(3,\mathbb{R})$, such a 2-orbifold has an $\mathbb{RP}^2$-structure, which is said to be a *hyperbolic* $\mathbb{RP}^2$*-structure*. A hyperbolic structures with geodesic boundary on a 2-orbifold determines an $\mathbb{RP}^2$-structure with geodesic boundary, a *hyperbolic* $\mathbb{RP}^2$*-structure*.

The subspace of $\mathbb{RP}^2(\Sigma)$ of elements represented by convex $\mathbb{RP}^2$-structures will be denoted by $\mathcal{C}(\Sigma)$. The subspace of $\mathcal{C}(\Sigma)$ corresponding to hyperbolic $\mathbb{RP}^2$-structures is denoted by $\mathcal{T}(\Sigma)$ and identifies with the Teichmüller space of hyperbolic structures on Σ as determined by Thurston [37]. This follows since a projective diffeomorphism of two hyperbolic $\mathbb{RP}^2$-orbifolds is an isometry of them. This is also a topology preserving identification since the topologies of the deformation spaces and the Teichmüller spaces are both defined by C^1-topology of developing maps.

For later purposes, we define $\mathcal{C}'(\Sigma)$ be the subset of $\mathbb{RP}^{2*}(\Sigma)$ consisting of isotopy classes of convex structures on Σ.

The *pre-Hitchin-Teichmüller component* of $H(\Sigma)$ is a component $C_{\mathcal{T}}$ of it which contains representations

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PO}(1,2))$$

corresponding to holonomy homomorphisms of hyperbolic structures on Σ (discrete embeddings $\pi_1(\Sigma) \longrightarrow \mathrm{PO}(1,2)$).

The group $\mathrm{PGL}(3,\mathbb{R})$ acts on $H(\Sigma)$ by conjugation, that is,

$$h(\cdot) \mapsto \vartheta h(\cdot)\vartheta^{-1}, \vartheta \in \mathrm{PGL}(3,\mathbb{R})$$

for $h \in H(\Sigma)$. Let $H(\Sigma)^{st}$ be the subspace of representations r acting freely on $\mathbb{RP}^2$. By Lemma 1.12 of [19], $\mathrm{PGL}(3,\mathbb{R})$ acts properly on this subset.

We call a component $C_{\mathcal{T}}/\mathrm{PGL}(3,\mathbb{R})$ of

$$H(\Sigma)^{st}/\mathrm{PGL}(3,\mathbb{R})$$

a *Hitchin-Teichmüller component* of Σ (following Hitchin's theory [22] for closed surfaces). $C_{\mathcal{T}}$ may be defined to be a component of $H(\Sigma)$ as we defined in the introduction. Theorem B states that this component is identical with the deformation space $\mathcal{C}(\Sigma)$ of convex $\mathbb{RP}^2$-structures on Σ.

2.8. **Openness of convex $\mathbb{RP}^2$-structures.** We will need Propositions 2.3 and 2.5 to prove Theorem B. They will be modified to Propositions 3.8 and 3.9 in the next section.

Proposition 2.3. *Let S be a closed orbifold with $\chi(S) < 0$. Then $\mathcal{C}(S)$ is an open subset of $\mathbb{RP}^2(S)$. (So is $\mathcal{C}'(S)$ of $\mathbb{RP}^{2*}(S)$.)*

Proof. It is shown in [19] using J. L. Koszul's result [28] that $\mathcal{C}(S')$ is an open subset of $\mathbb{RP}^2(S')$ if S' is a closed surface. This is done by taking product $S' \times \mathbf{S}^1$ and finding an affine structure on it corresponding to an $\mathbb{RP}^2$-structure on S'. An affine structure on $S' \times \mathbf{S}^1$ is convex if and only if the $\mathbb{RP}^2$-structure on S' is convex. Since all discussions in [28] apply to differentiable orbifolds as well, we can replace S' by S. ☐

2.9. **Closedness of $\mathbb{RP}^2$-structures.** The following proposition shows that 2-orbifolds of negative Euler characteristic with isomorphic fundamental groups are diffeomorphic. The harmonic map theory of Schoen-Yau [35] and the Nielsen realization theorem proved by Kerckhoff [27] are essential for the proof:

Proposition 2.4. *Let Σ_1 and Σ_2 be closed 2-orbifolds of negative Euler characteristic where $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are homeomorphic to disks. If $k : \pi_1(\Sigma_1) \to \pi_1(\Sigma_2)$ is an isomorphism, then there is an orbifold diffeomorphism $f : \Sigma_1 \to \Sigma_2$ so that $\tilde{f}_* = k$ for a lift $\tilde{f} : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$.*

Proof. Since Σ_1 admits a hyperbolic structure, $\pi_1(\Sigma_1)$ is isomorphic to a discrete cocompact subgroup of $\mathrm{PSL}(2, \mathbb{R})$. There is a torsion-free finite-index normal subgroup Γ_1 of $\pi_1(\Sigma_1)$ by Selberg's lemma [34]. Let Γ_2 be $k(\Gamma)$ in $\pi_1(\Sigma_2)$. There is a finite covering surface Σ_1' of Σ_1 corresponding to Γ_1 and Σ_2' of Σ_2 corresponding to Γ_2.

The finite group $G_1 = \pi_1(\Sigma_1)/\Gamma_1$ maps injectively into $\mathrm{Out}(\Gamma_1) = \mathrm{Aut}(\Gamma_1)/\mathrm{Inn}(\Gamma_1)$. Similarly, $G_2 = \pi_1(\Sigma_2)/\Gamma_2$ maps into $\mathrm{Out}(\Gamma_2)/\mathrm{Inn}(\Gamma_2)$. Thus, the commutative diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{k} & G_2 \\ \downarrow & & \downarrow \\ \mathrm{Out}(\Gamma_1)/\mathrm{Inn}(\Gamma_1) & \xrightarrow{k_*} & \mathrm{Out}(\Gamma_2)/\mathrm{Inn}(\Gamma_2) \end{array} \tag{8}$$

holds where k_* is an induced isomorphism.

By the Nielsen realization theorem of Kerckhoff [27], G_1 acts on Σ_1 and G_2 on Σ_2. A homeomorphism $f' : \Sigma_1' \to \Sigma_2'$ realizes $k|\Gamma$ and for each $g \in G_1$, $f' \circ g$ is homotopic to $k(g) \circ f'$ for $k(g) \in G_2$.

Give Σ_1' and Σ_2' arbitrary hyperbolic metrics which are G_1- and G_2-invariant respectively (that is, using Thurston's orbifold hyperbolization of Σ_1 and Σ_2). Then choose a unique harmonic diffeomorphism $\hat{f} : \Sigma_1' \to \Sigma_2'$ in the homotopy class of f' as obtained by Schoen-Yau [35]. Since $\hat{f} \circ g = k(g) \circ \hat{f}$ by uniqueness, $\hat{f}$ induces a desired orbifold diffeomorphism $f : \Sigma_1 \to \Sigma_2$. ☐

Proposition 2.5. *Let S be a closed orbifold with $\chi(S) < 0$. The image of*

$$\mathcal{H}' : \mathcal{C}'(S) \to H(S)$$

is closed.

Proof. Choose a sequence of representations in $\mathcal{H}'$

$$h_i : \pi_1(S) \to \mathrm{PGL}(3, \mathbb{R})$$

so that

$$(h_i(g_1), \dots, h_i(g_m)) \to (h(g_1), \dots, h(g_m))$$

for a representation $h : \pi_1(S) \to \mathrm{PGL}(3, \mathbb{R})$; that is, h_i converges to h algebraically. We show that h is in the image proving the closedness.

Let S_i be the orbifold S with an $\mathbb{RP}^2$-structure corresponding to h_i, and let $\tilde{S}$ be the universal cover of S. Let $\tilde{S}_i$ be the universal cover $\tilde{S}$ with the induced $\mathbb{RP}^2$-structure from S_i, and D_i a developing map of S_i associated with h_i. Then $D_i : \tilde{S}_i \to \mathbb{RP}^2$ maps onto a strictly convex domain Ω_i in an affine patch of $\mathbb{RP}^2$. (This follows since $\tilde{S}_i$ is a universal cover of a convex $\mathbb{RP}^2$-surface finitely covering S_i. See [19].) $D_i : \tilde{S}_i \to \Omega_i$ induces an $\mathbb{RP}^2$-diffeomorphism $S_i \to \Omega_i/h_i(\pi_1(S))$.

There is a unique $\mathbb{RP}^2$-structure on the sphere $\mathbf{S}^2$ such that the covering projection $p_{\mathbb{RP}^2} : \mathbf{S}^2 \to \mathbb{RP}^2$ is a projective map. The nontrivial deck transformation is represented by the antipodal map of the sphere. Its collineation group $\mathrm{Aut}(\mathbf{S}^2)$ is isomorphic to $\mathrm{SL}_{\pm}(3, \mathbb{R})$, generated by $\mathrm{PGL}(3, \mathbb{R})$ and the antipodal map,

We can show that $D_i : \tilde{S}_i \to \mathbb{RP}^2$ always lifts to an embedding $D'_i : \tilde{S}_i \to \mathbf{S}^2$ and h_i lifts to a homomorphism $h'_i : \pi_1(S) \to \mathrm{Aut}(\mathbf{S}^2)$ (see [5]): we can lift first, and for a deck-transformation ϑ of $\pi_1(S)$, $D'_i \circ \vartheta$ is another developing map, and hence it must equal $\varphi \circ D'_i$ for $\varphi \in \mathrm{Aut}(\mathbf{S}^2)$. Defining $h'_i(\vartheta) = \varphi$, we see that h'_i is a lift of h_i.

The image Ω'_i of D'_i is a convex open subset of an open hemisphere in $\mathbf{S}^2$ with a standard geodesic structure.

By choosing a subsequence if necessary, the sequence of the closures $\mathrm{Cl}(\Omega'_i)$ converges to a compact convex subset of $\mathbf{S}^2$ in a closed hemisphere in the Hausdorff topology (Choi-Goldman [10]). Therefore, Ω'_∞ is a compact convex subset of an open hemisphere in $\mathbf{S}^2$: The limit Ω'_∞ is neither a point, a line segment, a lune, nor a closed hemisphere since by taking a finite subcover S' of S if necessary all $D'_i(\tilde{S}_i)$ are images of a sequence of developing images of convex $\mathbb{RP}^2$-structures on a closed surface S'.

By choosing a subsequence, h'_i converges to a representation $h' : \pi_1(S) \to \mathrm{Aut}(\mathbf{S}^2)$ lifting h. As in [11], $h'(\pi_1(S))$ acts on Ω'_∞. Since h' is a map to $\mathrm{SL}_{\pm}(3, \mathbb{R})$, h' is discrete and faithful by Lemma 1.1 of Goldman-Millson [20]. ($\pi_1(S)$ has a finite index subgroup which is torsion-free. Apply Lemma 1.1 of [20] here and the finite index extension argument is trivial.)

Therefore, $h'(\pi_1(S))$ acts on an open disk Ω'^o_∞ with quotient orbifold S''. By Proposition 2.4, there is a diffeomorphism $S'' \longrightarrow S$ inducing h'. Since $p_{\mathbb{RP}^2}|\Omega'^o_\infty$ is an embedding onto an $h(\pi_1(S))$-invariant convex open domain $\Omega \subset \mathbb{RP}^2$, we see that S'' is realized also as the quotient space $\Omega/h(\pi_1(S))$. Thus, h is realized as a holonomy homomorphism of a convex $\mathbb{RP}^2$-structure on S and lies in the image of $\mathcal{H}'(\mathcal{C}(S))$. □

2.10. **The proof of Theorem B.** For the proof, we need Theorem A, which will be proved in §6.

Proof of Theorem B. Let Σ be a closed 2-orbifold with $\chi(\Sigma) < 0$. Let $\mathcal{T}'(\Sigma)$ the connected subset of $\mathcal{C}'(\Sigma)$ consisting of hyperbolic $\mathbb{RP}^2$-structures on Σ. Since by

Theorem A, $\mathcal{C}'(\Sigma)$ is connected, and $\mathcal{H}'$ sends $\mathcal{T}'(\Sigma)$ into $C_{\mathcal{T}}$, it follows that $\mathcal{H}'$ sends $\mathcal{C}'(\Sigma)$ into $C_{\mathcal{T}}$.

By Proposition 2.3, $\mathcal{C}'(\Sigma)$ is an open subset of $\mathbb{RP}^{2*}(\Sigma)$. Since $\mathcal{H}'$ is an open map, $\mathcal{H}'(\mathcal{C}'(\Sigma))$ is an open subset of $C_{\mathcal{T}}$. By Proposition 2.5, the image is a closed subset of $C_{\mathcal{T}}$. Hence, the image equals $C_{\mathcal{T}}$.

The holonomy group of a convex $\mathbb{RP}^2$-orbifold is discrete since it acts on an open domain discontinuously. Thus, $C_{\mathcal{T}}$ consists of discrete embeddings.

Recall also that since $\mathrm{PGL}(3,\mathbb{R})$ acts properly on $U(\Sigma)$, $C_{\mathcal{T}}$ is a subset of $H(\Sigma)^{st}$.

To complete the proof of Theorem B, we show that $\mathcal{H}'|\mathcal{C}'$ is injective: That is, given two holonomy representations h and h' for $(D, \tilde{f} : \tilde{\Sigma} \to \tilde{M})$ and $(D', \tilde{f}' : \tilde{\Sigma} \to \tilde{M}')$ for convex $\mathbb{RP}^2$-orbifolds M and M'. Then,

$$h = h_1 \circ \tilde{f}_*, \qquad h' = h_1' \circ \tilde{f}_*'$$

for holonomy homomorphisms of h_1 and h_1' of M and M' respectively. We show that if $h = h'$, then $(D, \tilde{f})$ and $(D', \tilde{f}')$ are isotopic equivariantly with respect to $h' \circ h^{-1}$. (According to the definition of deformation spaces in [9], this will prove the injectivity of $\mathcal{H}'$.)

Let Σ' be a closed surface finitely covering Σ. Let Ω be the image of D composed with $\tilde{f}$ and Ω' that of D' composed with $\tilde{f}'$. Since h and h' restricted to $\pi_1(\Sigma')$ are the same, Proposition 3.4 of [19] shows $\Omega = \Omega'$. The images of $D \circ \tilde{f}$ and $D' \circ \tilde{f}'$ are the same, and they are both equivariant under the homomorphism $h = h' : \pi_1(\Sigma) \to \mathrm{PGL}(3,\mathbb{R})$. The map

$$g = (D')^{-1} \circ D : \tilde{M} \to \tilde{M}'$$

satisfies $D' \circ g = D$ and is equivariant under the homomorphism

$$g_* = i_{M,M'} = \tilde{f}_*' \circ \tilde{f}_*^{-1} : \pi_1(M) \to \pi_1(M').$$

Give Σ a hyperbolic metric μ. Then $\tilde{f}$ and $\tilde{f}'$ induce metrics on $\tilde{M}$ and $\tilde{M}'$ respectively. We now show that $g \circ \tilde{f}$ and $\tilde{f}'$ are $i_{M,M'}$-equivariantly isotopic. $\tilde{M}'$ has induced hyperbolic metrics μ_0 and μ_1 induced from $g \circ \tilde{f}$ and $\tilde{f}'$ respectively. There is a path of Riemannian metrics

$$\mu_t = t\mu_1 + (1-t)\mu_0$$

for $t \in [0,1]$ from μ_0 to μ_1.

Equivariance implies they induce metrics on M', denoted by same letters. Recall that Σ' is the closed surface covering Σ. Let M_s' denote the corresponding closed surface covering M'. Let μ_t' denote the Riemannian metrics of M_s' corresponding to μ_t. By Theorem B.26 of Tromba [38], there exists a smooth one-parameter family of harmonic diffeomorphisms $S'(\mu_t) : (M_s', \mu_t') \to (\Sigma', \mu)$. Since these harmonic diffeomorphisms are unique in their homotopy classes, they are equivariant under automorphisms of M_s' and Σ'. The maps $S'(\mu_t)$ descend to a one-parameter family of diffeomorphisms

$$S(\mu_t) : (M', \mu_t) \to (\Sigma, \mu).$$

One can lift the inverse map $S(\mu_t)^{-1}$ of $S(\mu_t)$ to a smooth one-parameter family of diffeomorphisms $\tilde{S}(\mu_t)^{-1} : \tilde{\Sigma} \to \tilde{M}'$ by analytic continuation. Uniqueness of harmonic diffeomorphisms implies the inverse map $S(\mu_0)^{-1}$ lifts to $g \circ \tilde{f} : \tilde{\Sigma}_1 \to \tilde{M}'$, and by analytic continuation $S(\mu_1)^{-1}$ lifts to $\gamma \circ \tilde{f}' : \tilde{\Sigma} \to \tilde{M}'$ for some deck transformation γ of $\tilde{M}'$, where $g \circ \tilde{f}$ is isotopic to $\gamma \circ \tilde{f}$ equivariant with respect to $i_{M,M'}$. $\tilde{S}(\mu_1)_*^{-1}(\cdot)$ must equal $\gamma \circ \tilde{f}'_*(\cdot) \circ \gamma$. Since $g_* \circ \tilde{f}_* = \tilde{f}'_*$, Proposition 8 of [9] implies that γ equals the identity since the center of $\pi_1(\Sigma)$ is trivial. Therefore $\tilde{S}(\mu_1)^{-1}$ equals $\tilde{f}'$. Thus $g \circ \tilde{f}$ and $\tilde{f}'$ are $i_{M,M'}$-equivariantly isotopic.

Applying D' to $g \circ \tilde{f}$ and $\tilde{f}'$ and D'_* to $i_{M,M'}$, implies that $(D, \tilde{f})$ and $(D', \tilde{f}')$ are equivalent. Therefore, $\mathcal{H}' : \mathcal{C}'(\Sigma) \to C_{\mathcal{T}}$ is a homeomorphism, inducing a map

$$\mathcal{H} : \mathcal{C}(\Sigma) \to C_{\mathcal{T}}/\mathrm{PGL}(3, \mathbb{R}).$$

□

3. Splitting and sewing $\mathbb{RP}^2$-orbifolds

We describe geometric operations on convex $\mathbb{RP}^2$-orbifolds corresponding to the topological operations in §1. (We recommend [19], [5], [6], and [7] for background knowledge of $\mathbb{RP}^2$-structures on surfaces.) The boundary invariants of a convex $\mathbb{RP}^2$-orbifold are used to build bigger convex orbifolds from by constructions along 1-dimensional suborbifolds. Finally, we discuss how the deformation space of $\mathbb{RP}^2$-orbifolds with boundary relates to the space of conjugacy classes of representations, following §2. We prove the openness and closedness of convex $\mathbb{RP}^2$-structures, i.e., Propositions 3.8 and 3.9, which we need later, generalizing Propositions 2.3 and 2.5. The geometric operations discussed here induce fibrations of orbifold deformation spaces.

3.1. **The deformation spaces of boundary closed curves.** Let Σ be a compact convex $\mathbb{RP}^2$-orbifold with nonempty boundary. Let $(\mathbf{dev}, h)$ be its development pair and $\tilde{\Sigma}$ the universal cover. Let b be a closed curve in Σ and $\tilde{b}$ a lift to $\tilde{\Sigma}$, which is an embedded arc. Let γ be the corresponding deck transformation. Then $h(\gamma)$ must be conjugate to one of the following matrices:

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{9}$$

where $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$, and $\lambda_1\lambda_2\lambda_3 = 1$ or

$$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \tag{10}$$

where $\lambda_1^2\lambda_1 = 1$. In the former case, if all eigenvalues are positive, b or the holonomy of b is said to be *hyperbolic*, and in the second case, if all eigenvalues are positive again *quasi-hyperbolic*. A curve with quasi-hyperbolic holonomy is homotopic to the boundary. (See Theorem 3.2 of [19] and Proposition 4.5 in [6].)

Remark 3.1. Let Σ be an orbifold with $\chi(\Sigma) < 0$ and a closed geodesic boundary component γ. Then the holonomy of γ is either hyperbolic or quasi-hyperbolic. This follows from the analogous property for closed surfaces; see [5, 6].

In the hyperbolic case, if an eigenvalue is negative, we say that b or the holonomy of b is said to be a *hyperbolic glide-reflection*.

Remark 3.2. The conjugacy classes of hyperbolic automorphisms are classified by two real numbers $\lambda = \lambda_3$, and $\tau = \lambda_1 + \lambda_2$. They are in the space

$$\mathcal{R} = \{(\lambda, \tau) | 0 < \lambda < 1, 2/\sqrt{\lambda} < \tau < \lambda + \lambda^{-2}\}.$$

Also, for $A \in \mathrm{SL}(3, \mathbb{R})$, A is hyperbolic if and only if $A \in (\lambda, \tau)^{-1}(\mathcal{R})$. If $\tau = 1 + \lambda^{-1}$, then $\lambda_2 = 1$, and the hyperbolic element is called *purely hyperbolic*. Since hyperbolic elements of $\mathrm{PO}(1, 2)$ are purely hyperbolic, the holonomy of a closed essential curve in a hyperbolic $\mathbb{RP}^2$-orbifold is purely hyperbolic.

A hyperbolic collineation of $\mathbb{RP}^2$ has three or two fixed points, and $\mathbf{dev} \circ \tilde{b}$ is a line connecting two of the fixed points. In the first case, there are three fixed points, which are attracting, repelling, or saddle type ones in the dynamics of infinite cyclic action of the powers of the automorphism. If $\mathbf{dev} \circ \tilde{b}$ connects the attractor to the repelling fixed point, then b is said to be *principal*.

If Σ has no boundary and is convex, then any closed geodesic b is principal (see [19] and [6]). When Σ has boundary diffeomorphic to a circle, we will require each boundary component to be a principal closed geodesic in this paper for convenience. As a consequence, all closed curves are homotopic to a unique principal closed geodesic (see [6]).

For a principal closed geodesic b, we define the *space of invariants* as the subspace:

$$\mathcal{R}(b) = \{(\lambda, \tau) | 0 < \lambda < 1, 2/\sqrt{\lambda} < \tau < \lambda + \lambda^{-2}\} \subset \mathbb{R}^2$$

which is diffeomorphic to $\mathbb{R}^2$.

An *open* 2-orbifold is an orbifold with empty boundary whose underlying space is noncompact. An $\mathbb{RP}^2$-orbifold with principal closed geodesic boundary component b' is contained in an ambient 2-orbifold so that b' has an annulus neighborhood.

The space of invariants is the deformation space of germs of convex $\mathbb{RP}^2$-structures on b: a point of this space determines an $\mathbb{RP}^2$-structure on a thin neighborhood. A principal simple closed geodesic is characterized by its holonomy along it; two principal simple closed geodesics have projectively isomorphic neighborhoods (in some open ambient 2-orbifolds) if and only if they have conjugate holonomy: There exists a neighborhood of a principal simple closed geodesic which is projectively diffeomorphic by a map induced by $\mathbf{dev}$ to a quotient under $\langle h(\vartheta) \rangle$ of a domain which is a sufficiently thin-neighborhood of the line connecting the attracting fixed point and the repelling one.

3.2. **A classification of geodesic 1-orbifolds.** A *geodesic* 1-suborbifold in Σ is a suborbifold in Σ so that it is locally modeled on a subspace $\mathbb{RP}^1$ in $\mathbb{RP}^2$ with projective group actions on $\mathbb{RP}^2$ preserving $\mathbb{RP}^1$. Since a geodesic full 1-suborbifold has two points which are mirror points, the universal cover of a full 1-orbifold is diffeomorphic to an open interval. (A full 1-orbifold is two-fold covered by a circle with one-dimensional projective structure.) $\mathbb{RP}^1$-structures on a circle are easily classified: the universal cover of a full 1-orbifold is isomorphic to one of the following $\mathbb{RP}^1$-manifolds:

- $(0,1)$ in $\mathbb{R}$, considered as an affine patch of $\mathbb{RP}^1$.
- $\mathbb{R}$ itself.
- An infinite cyclic cover of $\mathbb{RP}^1$.

In the first case, l is said to be *principal*. A geodesic 1-dimensional suborbifold of a convex two-dimensional orbifold S of negative Euler characteristic is always principal as a component of the inverse image of it in the universal cover of S must be isomorphic to the first item.

Let Σ be an $\mathbb{RP}^2$-orbifold. A singular point of a principal geodesic full 1-orbifold in Σ is modeled on an open set with an order-two group acting on it fixing a point. The group must be generated by an element with a matrix conjugate to:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, either it is an isolated fixed point of a reflection or in a line of fixed point. A principal geodesic full 1-suborbifold in Σ

cone-type: either connects two points which are both cone-points of order two and lies in the interior of Σ;

mirror-type: connects two points which are both mirror points and lies in the interior or the boundary of Q entirely and

- **boundary-mirror-type:** it furthermore is in the boundary,
- **singular-mirror-type:** it furthermore lies in the singular locus of Σ and connects two corner-reflectors of order two (it lies in the interior);

mixed-type: connects a cone-point of order two with a mirror point and lies in the interior.

A geodesic segment is of *singular-type* if it lies in the singular locus of Σ. We say that such a segment is a *singular segment*.

Notice from these that a boundary component of a compact convex 2-orbifold always is a closed curve or a full mirror-type 1-orbifold.

3.3. **The deformation spaces of full 1-orbifolds.** A principal geodesic full 1-orbifold is covered by a component l of its inverse image in the universal cover $\tilde{\Sigma}$ of Σ. Since l is projectively diffeomorphic to $(0,1)$, $\mathbf{dev}|l$ is an embedding onto a line, a precompact subset, in an affine patch of $\mathbb{RP}^2$. The holonomy group of the 1-orbifold is generated by two reflections r_1 and r_2.

Let us discuss mirror-type 1-orbifolds first. The lines of fixed points of two reflections r_1 and r_2 meet $\mathbf{dev}(l)$ at two points p_1 and p_2 respectively. Since $\mathbf{dev}(l)$ is invariant under r_1 and r_2, the respective isolated fixed points f_1 and f_2 of r_1 and r_2 lie in the one-dimensional subspace containing $\mathbf{dev}(l)$. The points f_1 and f_2 may not coincide with any of p_1 or p_2, as $\mathbf{dev}(l)/\langle r_1, r_2\rangle$ is a 1-orbifold isomorphic to l itself and so $\langle r_1, r_2\rangle$ acts properly discontinuously on $\mathbf{dev}(l)$. Also f_1 may not coincide with f_2 since then $\mathbf{dev}(l)$ is projectively diffeomorphic to an entire affine line $\mathbb{R}$ (In this case, l is said to be an affine 1-orbifold).

Since l is projectively diffeomorphic to $(0,1)$, the four points f_1, f_2, p_1, and p_2 are distinct, and these points are located on a segment in an affine patch with endpoints

f_1 and f_2 so that p_1 separates f_1 and p_2. Otherwise, we obtain a noninjective developing map, a holonomy element with non-real eigenvalues, or affine 1-orbifolds, which contradicts principality. The cross ratio $[f_1, f_2; p_1, p_2]$, which lies in the interval $(0, 1)$, of these points is invariant under choices of **dev** or the conjugation of holonomy.

There is always an affine coordinate so that $(f_1, f_2, p_1, p_2) = (1, 0, y, x)$ where $0 < x < y < 1$. The cross ratio equals

$$\frac{1-y}{y}\frac{x}{1-x}$$

and hence it is positive and may assume any value in $(0, 1)$.

Conversely, if two 1-orbifolds l_1 and l_2 of mirror type have the same invariants, then there exist isomorphic neighborhoods (in some ambient $\mathbb{RP}^2$-orbifolds). If the interiors of the underlying spaces of l_1 and l_2 lie in the interiors of projective 2-orbifolds, then the neighborhoods can be chosen to be open 2-orbifolds. If l_1 and l_2 lie in the boundary, then the 2-orbifolds can be enlarged so that the neighborhoods become open. The same can be said for each of the boundary-mirror-orbifold case or the singular-mirror-orbifold case.

For orbifolds of cone-type, the isolated fixed points f_1 and f_2 lie on $\mathbf{dev}(l)$. Let p_1 and p_2 be the points of intersection of the lines of fixed points of r_1 and r_2 meet the one-dimensional subspace containing $\mathbf{dev}(l)$. The points f_1, f_2, p_1, p_2 are distinct, and lie on a segment with endpoints p_1 and p_2. We assume that f_1 separates p_1 and f_2. The cross-ratio $[f_1, f_2; p_1, p_2] \in (0, 1)$ is independent of the choice of of **dev**. Conversely, if two 1-orbifolds of cone-type have the same invariants, then there exists isomorphic open neighborhoods (possibly after extending the collar neighborhoods).

The mixed-type case is entirely similar with invariant defined by $[f_1, f_2; p_1, p_2]$ again for f_1, f_2, p_1, p_2 defined as above. (Here, f_1 and f_2 are not the endpoints of a segment.)

We remark that a principal 1-orbifold has a double-covering circle s, and the generator of whose fundamental group has holonomy which is hyperbolic with eigenvalues $\lambda, 1, \lambda^{-1}$; i.e., it is purely hyperbolic. This can be easily seen since the intersection point of the lines of fixed points of r_1 and r_2 correspond to an eigenvector with eigenvalue 1 for the generator.

Given a full principal geodesic 1-orbifold b in Σ, the *space of invariants* is defined as $\mathcal{C}(b) = (0, 1)$.

Definition 3.3. Given a convex 2-orbifold Σ, let $\partial\Sigma$ denote the union of boundary 1-orbifolds. Let $\mathcal{C}(\partial\Sigma)$ denote the product of the spaces of invariants of all components of $\partial\Sigma$.

As with closed case, $\mathcal{C}(\Sigma)$ is a subspace of $\mathbb{RP}^2(\Sigma)$ and $\mathcal{T}(\Sigma)$ is a subspace of $\mathcal{C}(\Sigma)$.

3.4. **Geometric constructions of $\mathbb{RP}^2$-orbifolds.** Now, let Σ' be a compact convex $\mathbb{RP}^2$-orbifold with principal boundary. Given that certain boundary conditions are met, we will describe how to obtain a convex $\mathbb{RP}^2$-orbifold Σ'' obtained from Σ' by the above topological operations in §1.7 and construct all convex structures on Σ'' so that Σ' with its original convex structure is obtained back when we split. We of course obtain principal boundary for the resulting 2-orbifolds.

We follow the notation of §1.7.

3.4.1. *Pasting or crosscapping* (A)(I). In the former case, if two boundary component curves b and b' have a conjugate holonomy, then Σ'' is also a projective 2-orbifold. This construction is given in §3.6 of [19] for surface cases.

We start with pasting: Let b and b' be distinct with equal invariants. Suppose also that the component Σ' of Σ containing b' is distinct from the component S of Σ containing b. Without loss of generality, assume that Σ' has the two components only. Let $\tilde{S}$ be the universal cover of S and $(\mathbf{dev}_S, h_S)$ the development pair. Let $\tilde{\Sigma}'$ and $(\mathbf{dev}_{\Sigma'}, h_{\Sigma'})$ be the universal cover and the pair for Σ'. Let l and l' denote components of inverse images of b and b' in $\tilde{S}$ and $\tilde{\Sigma}'$ respectively. Let ϑ and ϑ' denote the deck transformations corresponding under f. Then $h_S(\vartheta)$ and $h_{\Sigma'}(\vartheta')$ act on $\mathbf{dev}_S(l)$ and $\mathbf{dev}_{\Sigma'}(l')$ respectively. Since $h_S(\vartheta)$ and $h_{\Sigma'}(\vartheta')$ are conjugate,

$$f' h_S(\vartheta) f'^{-1} = h_{\Sigma'}(\vartheta') \tag{11}$$

for some collineation f'. Let $\Omega = \mathbf{dev}_S(\tilde{\Sigma})$ and $\Omega' = \mathbf{dev}_{\Sigma'}(\tilde{\Sigma}')$. Then by post-composing f' with a reflection if necessary, we may assume without loss of generality that $f'(\Omega)$ and Ω' meet exactly in $f'(\mathbf{dev}(l)) = \mathbf{dev}(l')$.

Their union is convex: $f'(\Omega)$ is a subset of $f'h_S(\vartheta)f'^{-1}$-invariant triangle with an open side $f'(\mathbf{dev}(l))$ and Ω is a subset of $h_{\Sigma'}(\vartheta')$-invariant triangle with side $\mathbf{dev}(l')$. The second triangle must be adjacent to the first one. Since a supporting line of $f'(\Omega)$ at a vertex of $\mathbf{dev}(l')$ coincide with a side of the first triangle and that of Ω' coincide with the second triangle and the sides extend each other being the sides of the invariant triangles, it follows that the support lines coincide. The same holds at the other vertex of $\mathbf{dev}(l')$. Elementary geometry shows that $f'(\Omega) \cup \Omega'$ is convex.

Let Γ be the image of the homomorphism $f'h_S(\cdot)f'^{-1}$ and Γ' that of $h_{\Sigma'}$. Let Γ'' be the group generated by Γ and Γ', which is isomorphic to an amalgamated product of Γ and Γ' actually. Let Ω'' be the union of images of Ω and Ω' under Γ''. We claim that Ω'' is a convex domain: any two points lie in a finite connected union of images of Ω. A finite connected union is always convex. We can order the images and keep adding domains one by one. Let Ω_n be the n-th union. At each step, the union of Ω_n and a new domain Ω' to be added meet at a line acted upon by a hyperbolic transformation, which is a conjugate of $h_{\Sigma'}(\vartheta')$. Moreover, Ω_n and Ω' are subsets of adjacent triangles acted upon by the same hyperbolic transformation. The above supporting line argument applies.

Since Γ'' acts properly on Ω'', we see that Ω''/Γ'' is a compact convex 2-orbifold Σ'' obtained from S and Σ' by pasting along b and b' and leaving other components untouched.

Lemma 3.4. *A subsurface S' of a convex surface S bounded by closed geodesics is convex.*

Proof. A path in the subsurface S' is homotopic to a geodesic in S. Since there are no bigons, the geodesic itself is in S'. A closed geodesic in S is always principal. Thus, S' is convex (see [6]). □

Conversely, given a convex $\mathbb{RP}^2$-structure on Σ'' with a principal geodesic b'' as above, we see that the completions of $\Sigma'' - b''$ is a convex 2-orbifold by taking a finite cover and Lemma 3.4. Therefore, such a structure can always be constructed in this manner.

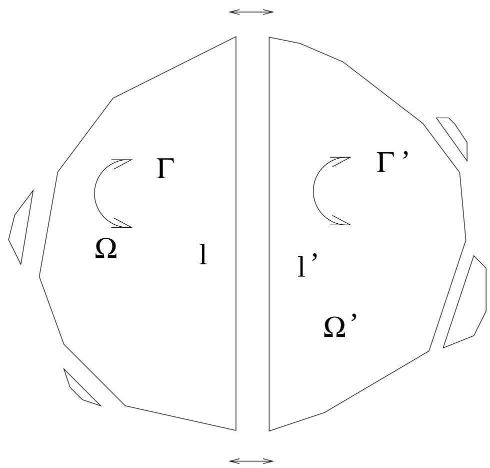

FIGURE 4. A convex orbifold obtained by pasting two smaller convex orbifolds.

In fact, the choice of f' is not unique. f' can be replaced by

$$\hat{f} = g \circ f' \tag{12}$$

where g is in the identity component of $\mathrm{PGL}(3, \mathbb{R})$ commuting with $h_{\Sigma'}(\vartheta')$. Thus, g is diagonalizable with positive eigenvalues, and hence the space of such g is $\mathbb{R}^2$.

Let $h''_{\hat{f}} : \pi_1(\Sigma'') \to \mathrm{PGL}(3, \mathbb{R})$ be given by amalgamating the fundamental groups and extending homomorphisms h_Σ and $h_{\Sigma'}$ in the obvious manner with image group Γ''. There is an $\mathbb{R}^2$-parameter space of possible $\hat{f}$. For different choices of $\hat{f}$, $h''_{\hat{f}}$ yields non-conjugate actions. Hence the resulting $\mathbb{RP}^2$-structures are non-isotopic (Lemma 3.5). For given fixed invariants for b and b' and fixed Σ', this construction gives an $\mathbb{R}^2$-parametrized family of non-isomorphic Σ''.

Lemma 3.5. *Let h be a holonomy homomorphism of a convex 2-orbifold of negative Euler characteristic with possibly nonempty but principal geodesic boundary. Let f_t be a one-parameter family of collineations. Then $f_t h(\cdot) f_t^{-1}$ equals $f_0 h(\cdot) f_0^{-1}$ if and only if f_t is constant.*

Proof. Pass to a finite covering surface S with $\chi(S) < 0$. Let $(\mathbf{dev}, h)$ be a development pair and $\tilde{S}$ the universal cover of S. Since S is convex, its holonomy representation is faithful. Let g_1 and g_2 be two deck-transformations corresponding to simple closed curves not homotopic to each other. Then these curves are homotopic to principal closed geodesics and g_1 and g_2 act on geodesic lines l_1 and l_2 in the universal cover $\tilde{S}$ corresponding to the closed geodesics respectively. The endpoints of $\mathbf{dev}(l_1)$ are the attracting and repelling fixed points of $h(g_1)$ by the principal conditions; and the endpoints of $\mathbf{dev}(l_2)$ those of $h(g_2)$. No two of these points coincide. Otherwise, $\mathbf{dev}(l_1)$ can be sent arbitrarily close to $\mathbf{dev}(l_2)$ by $h(g_2^n)$. Since l_1 maps to a simple closed curve and $\mathbf{dev}$ is an embedding, this cannot happen. No three of these points are collinear since $h(g_i)$ acts on $\mathbf{dev}(l_i)$ freely, and by convexity.

Suppose that $f' h(\cdot) f'^{-1} = h(\cdot)$ for a collineation f'. Then $f' h(g_i) f'^{-1} = h(g_i)$ for $i = 1, 2$, and f' acts on each of the two pairs of four noncollinear points. Since the images of four points, no three of which are collinear, determine the collineations, f' is the identity map or a unique reflection determined by the four points. Therefore $f_t = f'$ is constant. □

We describe this construction in a different language following [19]: First find a slightly bigger $\mathbb{RP}^2$-orbifold T containing Σ so that $T - \Sigma$ is a union of two annuli parallel to b and b' respectively. Then there are open tubular neighborhoods of b and b' which are isomorphic as b and b' are conjugate elements of $\pi_1(\Sigma)$. Remove from T what are outside these annuli to obtain T'. Now identify these two annuli by a projective diffeomorphism f'. As b and b' are principal geodesics, f' sends b to b'. Then Σ'' is independent of the choice of annuli but depends on the germ of f' near b.

There is a two-parameter family of projective diffeomorphisms with corresponding annular neighborhoods of b and b'. First consider the annuli as quotients of domains D_1 and D_2 in $\mathbb{RP}^2$ and a collineation $\tilde{f}$ sending D_1 to D_2 lifting f'. Such $\tilde{f}$ satisfies

$$\tilde{f} \circ \vartheta_1 \circ \tilde{f}^{-1} = \vartheta_2$$

where ϑ_1 and ϑ_2 are generators of the infinite cyclic groups acting on D_1 and D_2 respectively. Therefore, there are choices of maps f parametrized by $\mathbb{R}^2$. Essentially by Lemma 3.5, different choices of $\tilde{f}$ yield non-conjugate holonomy groups, and hence non-isotopic $\mathbb{RP}^2$-structures. (These are the projective versions of Fenchel-Nielsen twists for hyperbolic surfaces.)

To summarize: a family of distinct $\mathbb{RP}^2$-structures on Σ'' is parametrized by $\mathbb{R}^2$ when the common conjugacy class of the holonomy of b and b' is fixed and Σ' is fixed. The group $\mathbb{R}^2$ acts on the deformation space of Σ'' by changing the gluing map as above. Thus, we obtain a principal $\mathbb{R}^2$-fibration description as in Proposition 3.11. See §5 of [19] for details,

We may also assume that $S = \Sigma'$ but b not equal to b'. In this case, the discussions are similar with Ω and Ω' being equal and Γ becoming an HNN-extension.

We now go over the geometric crosscapping construction. Suppose now that $b = b'$. In this case, $S = \Sigma'$, and b corresponds to a simple closed curve b'' in Σ'' with a Möbius band neighborhood.

Let $(\mathbf{dev}, h)$ denote the development pair of S, and $\tilde{S}$ the universal cover of S. Then let $\Gamma = h(\pi_1(S))$. Let $\tilde{b}$ denote a component of the inverse image of b and ϑ be the corresponding deck transformation acting on $\tilde{b}$. Let ϑ' be the unique projective automorphism acting on $\tilde{b}$ preserving an orientation of $\tilde{b}$ but reversing the orientation of $\mathbb{RP}^2$ so that $\vartheta'^2 = \vartheta$; that is, we want ϑ' to be a hyperbolic glide-reflection. (This can obviously solved by conjugating the hyperbolic ϑ to a diagonal form.) Let $\Omega = \mathbf{dev}(\tilde{S})$. $\vartheta'(\Omega)$ and Ω meet exactly at $\mathbf{dev}(\tilde{b})$. Since Ω and $\vartheta'(\Omega)$ are ϑ-invariant, $\Omega \cup \vartheta'(\Omega)$ is a convex domain similarly to the pasting case. Let Γ'' denote the group generated by $\Gamma = h(\pi_1(\Sigma))$ and ϑ'. Let Ω'' be the union of images of Ω under Γ''. Then Ω'' is a convex domain as in the pasting case, and Ω''/Γ'' is an orbifold diffeomorphic to a component orbifold of Σ''.

Any convex $\mathbb{RP}^2$-orbifold diffeomorphic to Σ'' can be constructed in this manner. Finally, we remark that given a fixed invariant on b, there is a unique Σ'' that can be constructed.

3.4.2. *Silvering and folding* (A)(II). In this case, f either (1) is the identity map or (2) has exactly two fixed points reversing the orientation of b and is of order two. Let $\tilde{S}$ be the universal cover of a component S containing b and $(\mathbf{dev}, h)$ the development pair of S. Let $\tilde{b}$ be a component of the inverse image of b in $\tilde{S}$ and ϑ the corresponding deck transformation acting on $\tilde{b}$.

(1) When f is the identity, there is a unique reflection $F : \mathbb{RP}^2 \to \mathbb{RP}^2$ so that the line of fixed points contain $\mathbf{dev}(\tilde{b})$ and $F \circ h(\vartheta) \circ F^{-1} = h(\vartheta)$. Thus the isolated fixed point of F coincides with the fixed point of the hyperbolic automorphism $h(\vartheta)$ not on the closure of $\mathbf{dev}(\tilde{b})$. As above, consider the group Γ'' generated by Γ and F, and the union Ω'' of images of $\mathbf{dev}(\tilde{S})$ under the action of this group. Ω'' is a convex domain. Then Ω''/Γ'' as a component of Σ''. As F is unique, Σ determines Σ''.

When the holonomy of b is hyperbolic and b is geodesic, then b can be always silvered since such an element F exists. A boundary component with quasi-hyperbolic holonomy cannot be silvered.

(2) When f has exactly two fixed points, there is a reflection F so that the isolated fixed point of F lies on $\mathbf{dev}(\tilde{b})$ and

$$F \circ h(\vartheta) \circ F^{-1} = h(\vartheta)^{-1}. \tag{13}$$

This forces the hyperbolic $h(\vartheta)$ to have eigenvalues $\lambda, 1, \lambda^{-1}$, for $\lambda > 1$, and F exchanges the endpoints of $\mathbf{dev}(\tilde{b})$ and fixes the third fixed point of $h(\vartheta)$, which is not an endpoint of $\mathbf{dev}(\tilde{b})$. The choice of isolated fixed point of F on $\mathbf{dev}(\tilde{b})$ determines the intersection of the fixed line of F with the line containing $\mathbf{dev}(\tilde{b})$. Let Γ'' be generated by Γ and F, and Ω'' be the union of images of $\mathbf{dev}(\tilde{\Sigma})$ under Γ'', which is again convex. (This construction produces two cone-points of order two, one corresponding to F and the other to $h(\vartheta) \circ F$, which by (13) is a reflection.)

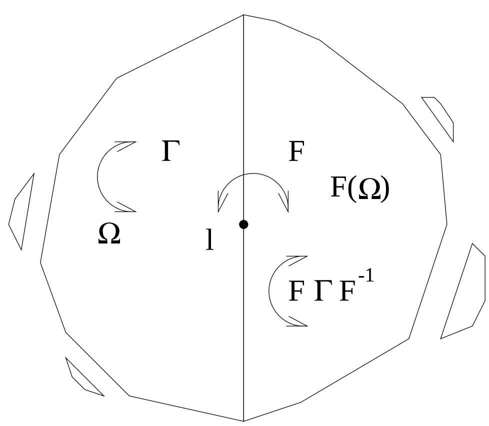

FIGURE 5. How to fold an orbifold.

The choice of the respective isolated fixed points in $\mathbf{dev}(\tilde{b})$ itself produces non-isotopic $\mathbb{RP}^2$-structures on Σ''. Thus, there is an $\mathbb{R}$-family of non-isotopic $\mathbb{RP}^2$-structures on Σ'' for $h(\vartheta)$ in a fixed conjugacy class of a purely hyperbolic transformation. Any other choice of F equals $g \circ F \circ g^{-1}$ for unique g commuting with $h(\vartheta)$ and with positive eigenvalues including 1. Such g is diagonalizable with positive eigenvalues. Thus $\mathbb{R}$ acts on the deformation space of convex $\mathbb{RP}^2$-structures on Σ''.

3.4.3. *Pasting* (B)(I). Now suppose that b is a full 1-orbifold. Consider a diffeomorphism $f : b \to b'$ of b with another 1-orbifold b'.

The holonomy group of b is generated by two reflections r_1 and r_2 acting on the line l to which a lift of b to $\tilde{\Sigma}$ develops. The fixed lines of r_1 and r_2 are transversal to l. The respective fixed points p_1 and p_2 of r_1 and r_2 lie on l. Letting q_1 and q_2 denote the respective intersection points of fixed lines of r_1 and r_2 with l, the cross ratio $[p_1, p_2; q_1, q_2]$ determines the respective conjugacy classes of r_1 and r_2. If b and b' have equal invariants, we can find a projective automorphism conjugating the reflections r_1 and r_2 to that corresponding to b'. The pasted orbifold Σ'' carries a convex $\mathbb{RP}^2$-structure by a similar construction.

The $\mathbb{R}$-family of conjugating elements determines an $\mathbb{R}$-family of non-isotopic $\mathbb{RP}^2$-structures on Σ'' for fixed invariants of b and b'.

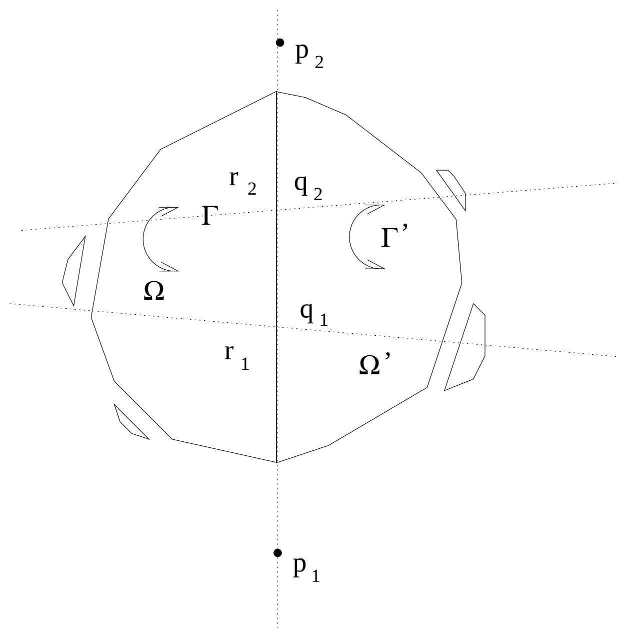

FIGURE 6. Pasting along a 1-orbifold.

3.4.4. *Silvering and folding* (B)(II). Now suppose $b' = b$. Then f is the identity map or fixes a unique point. If f is the identity, b contains mirror points. Then Σ determines Σ'' uniquely.

If f fixes a unique point, $\Sigma'' - (\Sigma - b)$ is a 1-orbifold with one endpoint a cone-point of order 2 and the other endpoint a mirror point. Then a unique convex $\mathbb{RP}^2$-structure on Σ'' exists, given one in Σ as above in (A)(II)(2). This follows since the gluing automorphism has to switch two reflections by conjugations. A fixed invariant for b determines a unique convex structure on Σ''.

These operations (A)(I), (A)(II), (B)(I), and (B)(II) can be performed for $\mathbb{RP}^2$-structures on possibly nonconvex orbifolds since these operations are supported on components of the boundary.

(However, all these operations preserve convexity.)

3.5. **Deformation spaces for bounded orbifolds.** When $\partial\Sigma \neq \emptyset$, we consider $\mathbb{RP}^2$-structures on Σ so that its boundary components are principal (see Remark 3.1). (Clearly convex structures belong here.) Denote by $\mathbb{RP}^2(\Sigma)$ the subspace of the deformation space of all $\mathbb{RP}^2$-structures with principal boundary components. $\mathbb{RP}^2(\Sigma)$ is the quotient space by the action of $\mathrm{PGL}(3,\mathbb{R})$ as before. Given an element of $\mathbb{RP}^2(\Sigma)$ for an orbifold Σ, one can silver them producing an $\mathbb{RP}^2$-structure on Σ' with silvered boundary since there exists a unique involution centralizing the holonomy of the boundary component. For any $\mathbb{RP}^2$-orbifold Σ', clarifying the boundary components produces $\mathbb{RP}^2$-structures on Σ. Thus, $\mathcal{S}(\Sigma)$ bijectively corresponds to $\mathcal{S}(\Sigma')$ using the silvering and clarifying operations. Thus $\mathbb{RP}^2(\Sigma)$ bijectively corresponds to $\mathbb{RP}^2(\Sigma')$. (A bit of subtlety lies with the fact that isotopies on Σ' need to have some symmetry for new mirror-point sets while isotopies on Σ need not. However, the charts of Σ' need to have the symmetries also. The net effect is the homomorphism.)

We define $\mathcal{C}(\Sigma)$ as a subspace of $\mathbb{RP}^2(\Sigma)$ of elements represented by convex structures on Σ with principal boundary.

A similar bijection exists between a subset of $\mathcal{C}(\Sigma)$ and $\mathcal{C}(\Sigma')$.

Theorem 3.6. *The process of silvering and clarifying induces a one-to-one correspondence, i.e., homeomorphism, between* $\mathbb{RP}^2(\Sigma)$ *and* $\mathbb{RP}^2(\Sigma')$ *where* Σ' *is obtained from silvering* Σ *in the topological sense. The same can be said for* $\mathcal{C}(\Sigma)$ *and* $\mathcal{C}(\Sigma')$.

When $\partial\Sigma \neq \emptyset$, denote by $H(\Sigma)_p$ the open subset of $H(\Sigma)$ consisting of structures for which the holonomy of each component of $\partial\Sigma$ is hyperbolic.

Define as above

$$U(\Sigma)_p = H(\Sigma)_p \cap U^g$$

for appropriate U^g. We can form a one-to-one correspondence between $U(\Sigma)_p$ and $U(\Sigma')$ by adding or removing reflections corresponding to the boundary components of Σ. The correspondence is obviously a non-proper homeomorphism.

Theorem 3.7. *Let* Σ *be a connected compact* 2*-orbifold with* $\chi(\Sigma) < 0$ *and* $\partial\Sigma \neq \emptyset$. *Then* $\mathbb{RP}^2(\Sigma)$ *has the structure of a Hausdorff real analytic variety modeled on* $U(\Sigma)_p/\mathrm{PGL}(3,\mathbb{R})$*, for which the induced map*

$$\mathcal{H} : \mathbb{RP}^2(\Sigma) \to U(\Sigma)_p/\mathrm{PGL}(3,\mathbb{R})$$

is a local homeomorphism onto its image.

Proof. Since the silvered Σ' has empty boundary,

$$\mathcal{H}' : \mathbb{RP}^2(\Sigma') \to U(\Sigma')/\mathrm{PGL}(3,\mathbb{R})$$

is a local homeomorphism. The one-to-one correspondences satisfy a commutative diagram:

$$\begin{array}{ccc} \mathbb{RP}^2(\Sigma) & \xrightarrow{\mathcal{H}} & U(\Sigma)_p/\mathrm{PGL}(3,\mathbb{R}) \\ \updownarrow & & \updownarrow \\ \mathbb{RP}^2(\Sigma') & \xrightarrow{\mathcal{H}'} & U(\Sigma')/\mathrm{PGL}(3,\mathbb{R}). \end{array}$$

The desired conclusion follows. □

Proposition 3.8. *Let Σ be a compact connected orbifold with $\chi(\Sigma) < 0$ and $\partial\Sigma \neq \emptyset$. Then $\mathcal{C}(\Sigma)$ is an open subset of $\mathbb{RP}^2(\Sigma)$.*

Proof. By Proposition 2.3, $\mathcal{C}(\Sigma')$ is an open subset of $\mathbb{RP}^2(\Sigma')$ where Σ' is obtained from Σ by silvering boundary components. By the above one-to-one correspondence between $\mathcal{C}(\Sigma)$ and $\mathcal{C}(\Sigma')$ and that $\mathbb{RP}^2(\Sigma)$ between $\mathbb{RP}^2(\Sigma')$, we obtain the result. □

Proposition 3.9. *The image of $\mathcal{C}'(\Sigma)$ under $\mathcal{H}'$ to $U(\Sigma)_p$ is a (relatively) closed subset.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}(\Sigma) & \xrightarrow{\mathcal{H}} & U(\Sigma)_p/\mathrm{PGL}(3,\mathbb{R}) \\ \updownarrow & & \updownarrow \\ \mathcal{C}(\Sigma') & \xrightarrow{\mathcal{H}'} & U(\Sigma')/\mathrm{PGL}(3,\mathbb{R}). \end{array}$$

Since the image of $\mathcal{H}'$ is closed, our result follows. □

3.5.1. *A question of Fock.* We will not need the following theorem in this paper, but to answer a question (for bounded surfaces) of Vladimir Fock, we state:

Theorem 3.10. *Let Σ be a compact connected orbifold of negative Euler characteristic and with nonempty boundary. Then*

$$\mathcal{H} : \mathcal{C}(\Sigma) \to U(\Sigma)_p$$

is a homeomorphism onto a component of $U(\Sigma)_p$ containing the image of $\mathcal{T}(\Sigma)$ said to be the Hitchin-Teichmüller component.

Proof. Theorem 2.2 and Proposition 3.8 show that the image under $\mathcal{H}$ is an open set. The proof follows from Proposition 3.9 and the fact that $\mathcal{C}(\Sigma)$ is connected by Theorem 6.1. □

3.6. **Fibrations of deformation spaces.** We summarize the result of this section with the following proposition with an obvious proof:

Proposition 3.11. *Consider only compact 2-orbifolds of negative Euler characteristic.*

- **(A)(I)(1):** *Let the 2-orbifold Σ'' be obtained from pasting along two closed curves b, b' in a 2-orbifold Σ'. The map resulting from splitting*

  $$\mathcal{SP} : \mathcal{C}(\Sigma'') \to \Delta \subset \mathcal{C}(\Sigma')$$

  is a principal $\mathbb{R}^2$-fibration, where Δ is the subset of $\mathcal{C}(\Sigma')$ where b and b' have equal invariants.

(A)(I)(2): *Let Σ'' be obtained from Σ' by cross-capping. The resulting map*

$$\mathcal{SP} : \mathcal{C}(\Sigma'') \to \mathcal{C}(\Sigma')$$

is a diffeomorphism.

(A)(II)(1): *Let Σ'' be obtained from Σ' by silvering. The clarifying map*

$$\mathcal{SP} : \mathcal{C}(\Sigma'') \to \mathcal{C}(\Sigma')$$

is a diffeomorphism.

(A)(II)(2): *Let Σ'' be obtained from Σ' by folding a boundary closed curve l'. The unfolding map*

$$\mathcal{SP} : \mathcal{C}(\Sigma'') \to \Delta \subset \mathcal{C}(\Sigma')$$

is a principal $\mathbb{R}$-fibration, where Δ is a subspace of $\mathcal{C}(\Sigma')$ consisting of $\mathbb{RP}^2$-structures with hyperbolic holonomy for l'.

(B)(I): *Let Σ'' be obtained by pasting along two full 1-orbifolds b and b' in Σ'. The splitting map*

$$\mathcal{SP} : \mathcal{C}(\Sigma'') \to \Delta \subset \mathcal{C}(\Sigma')$$

is a principal $\mathbb{R}$-fibration where Δ is a subset of $\mathcal{C}(\Sigma')$ where the invariants of b and b' are equal.

(B)(II): *Let Σ'' be obtained by silvering or folding a full 1-orbifold. The clarifying or unfolding map*

$$\mathcal{SP} : \mathcal{C}(\Sigma'') \to \mathcal{C}(\Sigma')$$

is a diffeomorphism.

4. Decomposition of convex orbifolds into elementary orbifolds

This section describes how to decompose a compact convex $\mathbb{RP}^2$-orbifold with negative Euler characteristic and principal boundary into elementary orbifolds along disjoint simple closed geodesics and geodesic full 1-orbifolds. Crucial is Lemma 4.1, whereby simple closed curves and 1-orbifolds are realized by simple closed geodesics and geodesic 1-orbifolds. Next, we define elementary orbifolds and prove that we can decompose convex orbifolds into elementary ones.

4.1. **The existence of 1-suborbifolds.**

The $\mathbb{RP}^2$-orbifolds are decomposed using the following lemma:

Lemma 4.1. *Suppose that Σ is a compact convex $\mathbb{RP}^2$-orbifold with $\chi(\Sigma) < 0$ and principal geodesic boundary. Let $c_1, \dots, c_n$ be a mutually disjoint collection of simple closed curves or 1-orbifolds so that the orbifold Euler characteristic of the completion of each component of $\Sigma - c_1 - \dots - c_n$ is negative. Then $c_1, \dots, c_n$ are isotopic to principal simple closed geodesics or principal geodesic 1-orbifolds $d_1, \dots, d_n$ respectively where $d_1, \dots, d_n$ are mutually disjoint.*

Proof. Consider first the case $n = 1$. The completions of components of $\Sigma - c_1$ have negative Euler characteristic by Proposition 1.10 since they are obtained by sewing the components of $\Sigma - c_1 - c_2 - \dots - c_n$. Let Σ' be a finite cover of Σ which is an orientable surface with principal closed geodesic boundary. Let G be the group of automorphisms of Σ' so that Σ'/G is projectively diffeomorphic to Σ in the orbifold sense. (We showed the existence of such a cover in §1.)

Let c_{1i}, $i = 1, \dots, k$, be the components of the inverse image of c_1. Since Σ' is convex, orientable, and $\chi(\Sigma') < 0$, each c_{1i} is homotopic to a simple closed geodesic (see [19]). Let d_{1j}, $j = 1, \dots, l$, be the union of all simple closed geodesics in Σ' homotopic to some c_{1i}. Then clearly G acts on

$$\bigcup_{j=1,\dots,l} d_{1j}.$$

Let us take a component d_{1j}, and let G_j be the subgroup stabilizing d_{1j}. Let $c_{1,j_1}, \dots, c_{1,j_m}$ be the curves in $\{c_{11}, \dots, c_{1k}\}$ homotopic to d_{1j} in Σ', on the union of which G_j acts on. G_j must act transitively on the set

$$C_j = \{c_{1,j_1}, \dots, c_{1,j_m}\}$$

as c_1 is the unique image in Σ. If C_j has a unique element, then G_j also acts on the unique element, and the image of d_{1j} is the unique principal closed geodesic or principal 1-orbifold d_1 isotopic to c_1. (A G-equivariant isotopy in Σ' can be constructed by perturbations of c_1 and d_{1j}s and using innermost-bigons which occur G-equivariantly.)

If C_j has more than one element, then let A_j be the unique maximal annulus bounded by elements in C_j. Then it is easy to see that G_j acts on A_j and G permutes A_js by projective diffeomorphisms. A_j covers A_j/G_j embedded in Σ, and A_j/G_j is a suborbifold of Σ bounded by c_1. The Euler characteristic $\chi(A_j/G_j) = 0$ since $\chi(A_j) = 0$. Since the completions of components of $\Sigma - c_1$ have negative Euler characteristic and the embedded suborbifolds A_j/G_j are such completions, we obtain a contradiction. Hence, the element of C_j is unique.

We obtain d_2 in a similar manner. If d_1 and d_2 are principal closed geodesics or geodesic full 1-orbifolds, subsegments of d_1 and d_2 do not bound a bigon. (To prove this, simply lift to the universal cover which may be considered a convex domain in an affine patch.) Thus,

- d_1 and d_2 are disjoint,
- d_1 and d_2 meet transversally at least once in the least possible number of points that closed curves in their respective free homotopy classes can meet
- or $d_1 = d_2$.

If $d_1 = d_2$, then a power of $c_{1,i}$ is homotopic to that of $c_{2,j}$ for some i, j. Since they are disjoint simple closed curves, they bound an annulus A in the closed surface Σ'. If there are $c_{1,k}$ or $c_{2,l}$ for some k and l in the interior of A, they are essential in A. We find an annulus $A' \subset A$ bounded by them and with interior disjoint from such curves. (Possibly $A' = A$.) Now, A' covers a suborbifold in Σ with zero Euler characteristic, which is not possible. We conclude that d_1 and d_2 are distinct.

Suppose d_1 and d_2 meet transversally at least once, then $d_{1,i}$ and $d_{2,j}$ meet transversally for some pair i, j at least once. Since $d_{1,i}$ and $d_{2,j}$ do not bound a bigon also, they meet in the minimal number of points in their respective free homotopy classes. Since $d_{1,i}$ and $d_{2,j}$ for every pair i, j are homotopic to disjoint closed curves on Σ', this means that $d_{1,i}$ and $d_{2,j}$ are disjoint, a contradiction. Therefore, d_1 and d_2 are disjoint.

By induction, there exist principal closed geodesics or principal 1-orbifolds $d_1, \ldots, d_k$ disjoint from each other so that c_i is isotopic to d_i for each i. $\square$

4.2. **Elementary Orbifolds.** The following orbifolds are said to be *elementary.* They are required to be convex and have principal geodesic boundary components and have negative Euler characteristic. We require that no closed geodesic is in their singular locus. We give nicknames and Thurston's notation in parentheses:

(P1) A pair-of-pants.
(P2) An annulus with one cone-point of order n. $(A(;n))$
(P3) A disk with two cone-points of order p, q, one of which is greater than 2. $(D(;p,q))$
(P4) A sphere with three cone-points of order p, q, r where $1/p + 1/q + 1/r < 1$. $(\mathbf{S}^2(;p,q,r))$
(A1) An annulus with one boundary component a union of a singular segment and one boundary-orbifold. The other boundary component is a principal closed geodesic. (We call it a 2-*pronged crown* and denote it $A(2,2;\,)$.) It has two corner-reflectors of order 2 if the boundary components are silvered.
(A2) An annulus with one boundary component of the underlying space in a singular locus with one corner-reflector of order n, $n \geq 2$. (The other boundary component is a principal closed geodesic which is the boundary of the orbifold.) (We call it a *one-pronged crown* and denote it $A(n;\,)$.)
(A3) A disk with one singular segment and one boundary 1-orbifold and a cone-point of order greater than or equal to three $(D^2(2,2;n))$.
(A4) A disk with one corner-reflector of order m and one cone-point of order n so that $1/2m + 1/n < 1/2$ (with no boundary orbifold). ($n \geq 3$ necessarily.) $(D^2(m;n)$.)
(D1) A disk with three edges and three boundary 1-orbifolds. No two boundary 1-orbifolds are adjacent. (We call it a *hexagon* or $D^2(2,2,2,2,2,2;\,)$.)
(D2) A disk with three edges and two boundary 1-orbifolds on the boundary of the underlying space. Two boundary 1-orbifolds are not adjacent, and two edges meet in a corner-reflector of order n, and the remaining one a segment. (We called it a *pentagon* and denote it by $D^2(2,2,2,2,n;\,)$.)
(D3) A disk with two corner-reflectors of order p, q, one of which is greater than or equal to 3, and one boundary 1-orbifold. The singular locus of the disk is a union of three edges and two corner-reflectors. (We call it a *quadrilateral* or $D^2(2,2,p,q;\,)$.)
(D4) A disk with three corner-reflectors of order p, q, r where $1/p+1/q+1/r < 1$ and three edges (with no boundary orbifold). (We call it a *triangle* or $D^2(p,q,r;\,)$.)

We justify our notation: P1, P2, P3, and P4 type 2-orbifolds are all obtained from a pair-of-pants by "degenerating" one, two, and three boundary components to cone-points respectively. (Such processes are realizable as deformations of hyperbolic structures first making the boundary components into cusps and from cusps to cone-points as can be accomplished by Kerckhoff's paper [26]. See also Cooper-Hodgson-Kerchkoff [12] in three-dimensional cases.) A1, A2, A3, and A4 type 2-orbifolds are doubly-covered by P1, P2, P3, and P4 type 2-orbifolds respectively where for each of them the

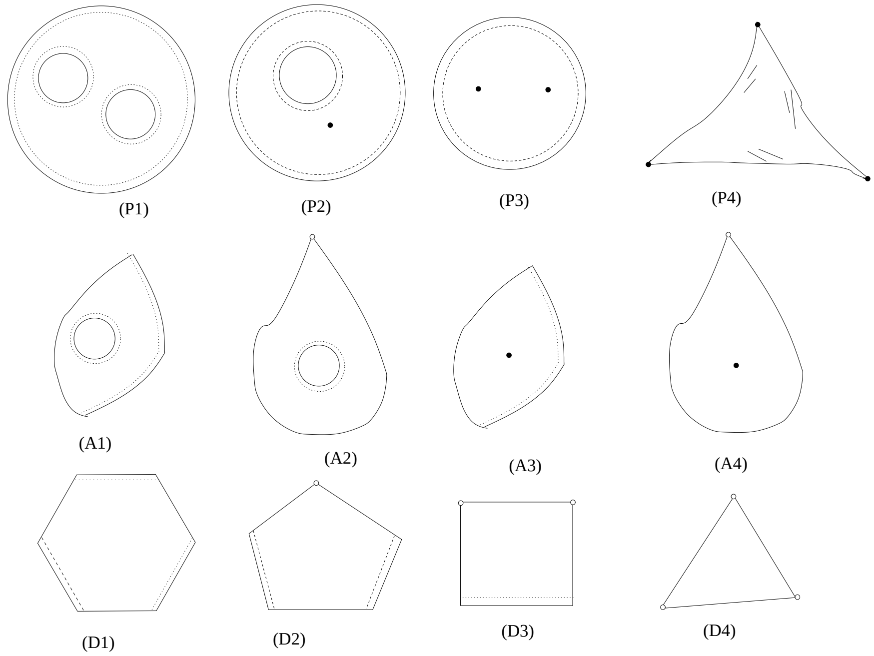

FIGURE 7. The elementary orbifolds. Arcs with dotted arcs next to them indicate boundary components. Black points indicate cone-points and white points the corner-reflectors.

order-two deck-transformation acts with the set of fixed points equal to an arc from a single boundary component or a single cone-points to itself. D1, D2, D3, and D4 are also doubly-covered by P1, P2, P3, and P4 respectively where for each of them the order-two deck-transformation acts with the set of fixed points the union of mutually disjoint three arcs obtained by connected any pair of cone-points or boundary components. (Again obtaining an A2 orbifold from an A1 orbifold can be realized as a deformation of hyperbolic structures. The same can be said for other types here.)

Since a unique complete hyperbolic structure exists on a 3-punctured sphere, an orientation-preserving self-homeomorphism of a pair-of-pants fixing each of the ends is isotopic to the identity. Automorphism groups of P1, P2, P3, or P4 type orbifolds are thus determined by the action on boundary components and cone-points. They are realized as permutation groups of oriented boundary components and cone-points. It follows that the above elementary orbifolds are all of the quotient orbifolds orbifolds of type P1, P2, P3, and P4.

Remark 4.2. None of these have 1-orbifolds decomposing them into unions of negative Euler characteristic orbifolds.

If Σ has a convex $\mathbb{RP}^2$-structure, and each boundary component of Σ is geodesic and principal, then the decomposed orbifolds have $\mathbb{RP}^2$-structures with principal geodesic boundary.

4.3. **Decomposition into elementary orbifolds.**

Theorem 4.3. *Let Σ be a compact convex $\mathbb{RP}^2$-orbifold with $\chi(\Sigma) < 0$ and principal geodesic boundary. Then there exists a mutually disjoint collection of simple closed geodesics and mirror- or cone- or mixed-type geodesic 1-orbifolds so that Σ decomposes along their union to a union of elementary 2-orbifolds or such elementary 2-orbifolds with some boundary 1-orbifolds silvered additionally.*

Proof. We essentially follow the sketch of the proof of Chapter 5 of Thurston [37]. We remark that we don't really need to have silvered elementary 2-orbifolds in the conclusion since we can define clarifying as "decomposing" also; however, we don't need this fact in this paper. In this proof, we won't distinguish between silvered elementary annuli and elementary annuli mainly in indicating their types.

Suppose first that Σ has no corner-reflectors or boundary full 1-orbifolds; that is, singular points of Σ are cone-points or in a closed geodesic of mirror points in the boundary of X_Σ, and the boundary components of Σ are principal closed geodesics.

Let $p_1, \dots, p_m$ be the cone-points of Σ. Assume for the moment that the number of cone-points m is greater than or equal to 3, and $\chi(X_\Sigma) \leq 0$. Let D be a disk in Σ containing all $p_1, \dots, p_m$. Then D has a structure of a 2-orbifold of negative Euler characteristic (by equation (4)). If at least one p_i has an order greater than two, then we find a simple closed curve bounding a disk D_1 containing p_i and another cone-point, say p_j. We find a simple closed curve bounding a disk D_2 including D_1 and a cone-point and a next simple closed curve bounding a disk D_3 including D_2 and a cone-point and so on. Therefore, D is divided into a disk D_1 and annuli A_i containing unique cone-points. Each of D_1 and A_is has negative Euler characteristic. $X_\Sigma - D$ contains no singular points and has negative Euler characteristic. Decompose $X_\Sigma - D^o$ into pairs-of-pants. By Lemma 4.1, we can find a collection of principal closed geodesics in Σ decomposing Σ into convex $\mathbb{RP}^2$-suborbifolds of negative Euler characteristic of type (P1), (P2), and (P3).

If all cone-points have order two, then we choose a disk D_1 in D containing three cone-points. Split along a cone-type 1-orbifold connecting the two cone-points in D_1. Now find $D_2, D_3, \dots,$ as above to obtain the decomposition into elementary 2-orbifolds of type (P1) or (P2).

If $\chi(X_\Sigma) = 1$, then X_Σ is homeomorphic to a disk or $\mathbb{RP}^2$. In the former case, X_Σ decomposes as above. In the latter case, the decomposition along a one-sided simple closed curve produces a disk with cone-points. In particular the resulting orbifold has negative Euler characteristic. Now apply Lemma 4.1.

Suppose that $\chi(X_\Sigma) = 2$. When $m = 3$, Σ is an elementary 2-orbifold of type (P4).

Next suppose $m \geq 4$ and at least two cone-points of order greater than 2. Then there exists a simple closed curve in X_Σ bounding two disks containing at least two cone-points, each containing a cone-point of order greater than 2, and with negative Euler characteristic. Divide the complement of the union of these disks to annuli as above.

Next suppose $m \geq 4$ and a single cone-point of order greater than two. Then there exists a cone-type 1-orbifold connecting two cone-points of order two. Cutting along the 1-orbifold produces a disk with principal geodesic boundary. Decompose the disk into union of a disk containing the single cone-point of order greater than two, and annuli with single cone-points. If all cone-points are of order two, then $m \geq 5$. Then decompose along two disjoint cone-type 1-orbifolds, obtaining an annulus with cone-points of order two. Now proceed as above.

Suppose that $m = 1$ or 2. If $\chi(X_\Sigma) < 0$, then we can introduce one or two annuli or Möbius bands containing the one singular point each and we obtain a decomposition into (P1), (P2), (P3) type orbifolds. If $\chi(X_\Sigma) = 0$, then an essential simple closed curve cuts X_Σ into an annulus. The decomposed orbifold is connected and of negative Euler characteristic. If $m = 1$ and X_Σ is homeomorphic to $\mathbb{RP}^2$ or $\mathbf{S}^2$, then $\chi(\Sigma) \geq 0$. If $m = 2$ and $\chi(X_\Sigma) \geq 0$, then X_Σ is homeomorphic to $\mathbb{RP}^2$. Split Σ along a simple closed curve into a disk with two cone-points, leading to the next case. Finally if X_Σ is homeomorphic to a disk, then Σ is an elementary 2-orbifold of type (P3) and $m = 2$. (This settles case (d) below.)

We now suppose that Σ has corner-reflectors and/or boundary full 1-orbifolds. Let $b_1, b_2, \ldots, b_l$ be the boundary components of X_Σ containing corner-reflectors or boundary 1-orbifolds, and $b_{l+1}, \ldots, b_k$ denote the remaining components.

Let c_i be a simple closed curve in X^o_Σ homotopic to b_i for $i = 1, \ldots, l$. The component of $\Sigma - c_i$ containing b_i has negative Euler characteristic by equation (4).

Suppose that Σ split along the union of c_is has a component C with nonnegative Euler characteristic. Then the underlying space of this component is homeomorphic to either a disk, an annulus, or a Möbius band. If C is homeomorphic to an annulus or Möbius band, no singular point of Σ lies on the component. When C is an annulus, cut along the center closed curve obtaining C decomposed into two annuli without singular points. They can be pasted with other components of Σ without changing their Euler characteristics, obtaining a different decomposition of Σ. When C is a Möbius band, cut along a one-sided closed curve obtaining an annulus with singular points. The annulus is then pasted with the adjacent component of Σ without changing the Euler characteristic, obtaining a new decomposition. If C is homeomorphic to a disk, then there are either two cone-points of order two or a single cone-point of arbitrary order by the Euler characteristic condition or no cone-points, and X_Σ is homeomorphic to a disk. If there are two cone-points of order two, we can split Σ along a cone-type 1-orbifold in C to obtain an annulus with corner-reflectors. If there is a single cone-point, then Σ must have been a disk with one cone-point. If there is no cone-point in C, then Σ is a disk.

These facts and Lemma 4.1 imply that Σ decomposes into convex 2-suborbifolds listed below:

(a) A disk with corner-reflectors and boundary full 1-orbifolds and no cone-points.
(b) An annulus with corner-reflectors and boundary full 1-orbifolds in one boundary component of its underlying space and no cone-points.
(c) A disk with corner-reflectors, boundary 1-orbifolds, and one cone-point of arbitrary order.
(d) A 2-orbifold without corner-reflectors or boundary full 1-orbifolds.

When there are boundary full 1-orbifolds, we silver them. This does not change the Euler characteristic of any 2-suborbifold containing it by equation (4). We mark the full 1-orbifolds to be of *boundary type*, and *we will never cut them apart*. After our decomposition, we will clarify them.

Let Σ' be one of the above (a)-(c). Let m denote the number of corner-reflectors. In case (a), assume $m \geq 3$. If $m = 3$, the orbifold Σ' is an elementary one of type (D4). Suppose that $m \geq 4$. From a side A, we choose one 1-orbifold, say $a_1, \ldots, a_{m-2}$, ending at A and each of the other sides except the sides adjacent to A. We choose these to be disjoint. If a_i ends at a side was a boundary full 1-orbifold, we delete a_i. If any component of Σ' split along the remaining 1-orbifolds a_i has Euler characteristic greater than or equal to zero, then we delete one of the 1-orbifolds adjacent to the component. Continuing in this manner, we obtain a collection a_i so that if Σ' is split along them, each of the component is an orbifold of type (D1), (D2), or (D3). Finally, clarifying the boundary types yields the desired decomposition.

In case (b), the underlying space of Σ' is a disk minus an open disk in its interior. As before let $p_1, p_2, \ldots, p_m$ denote the corner-reflectors and $e_1, e_2, \ldots, e_m$ edges. If $m = 1$, then Σ' is an elementary 2-orbifold of type (A2).

Suppose $m \geq 2$. From an edge κ which is not boundary type, we choose 1-orbifolds $a_1, \ldots, a_m$ ending at the sides $e_1, \ldots, e_m$ so that the two endpoints of a_m are in κ and other a_i has endpoints in distinct edges. We eliminate a_i's that end in the edges adjacent to κ and ones of boundary type. If we split along the rest of a_i's and obtain a component of Euler characteristic greater than or equal to zero, then we eliminate one of the adjacent a_is. As in case (a), by continuing in this manner, we decompose Σ' into an elementary orbifold of type (A2) and ones of type (D1), (D2), or (D3).

In case (c), suppose that the cone-point has order greater than two. If the number of corner-reflectors is one, then Σ' is an elementary 2-orbifold of type (A4). As above, we find 1-orbifolds $a_1, \ldots, a_m$ and do similar constructions.

Suppose that the cone-point has order two. Draw a mixed-type 1-orbifold τ from the cone-point to a mirror point in the nonboundary type full 1-orbifold. The decomposition results in a disk. Silver the 1-orbifold which folds to τ. Decompose the disk and clarify it back. □

5. The Teichmüller spaces of elementary orbifolds

This section contains a proof of part of Thurston's Theorem 6.9 for elementary 2-orbifolds .

We will need in the next section that the space of convex $\mathbb{RP}^2$-structures with principal boundary on each elementary 2-orbifold is nonempty. Since hyperbolic projective structures on elementary 2-orbifolds are certainly convex with principal boundary, the following proposition implies this nonemptyness. Recall that the Hilbert metric of a hyperbolic $\mathbb{RP}^2$-structure is the hyperbolic metric.

Proposition 5.1. *Let Σ be a compact 2-orbifold with empty boundary and negative Euler characteristic diffeomorphic to an elementary 2-orbifold. Then the deformation space $\mathcal{T}(\Sigma)$ of hyperbolic $\mathbb{RP}^2$-structures on Σ is homeomorphic to a cell of dimension $-3\chi(X_\Sigma) + 2k + l + 2n$ where X_Σ is the underlying space and k is the number of cone-points, l is the number of corner-reflectors, and n is the number of boundary full 1-orbifolds.*

The strategy of the proof is to show that for each elementary 2-orbifold S, $\mathcal{T}(S)$ is homeomorphic to $\mathcal{T}(\partial S)$, where $\mathcal{T}(\partial S)$ is the product of $\mathbb{R}^+$ for each component of ∂S corresponding to the hyperbolic-metric lengths of components of ∂S. Then for hyperbolic structures, to obtain a bigger orbifold in the sense of this paper, we simply need to match lengths of boundary components.

A generalized triangle in the hyperbolic plane is one of following:

(a) A hexagon: a disk bounded by six geodesic sides meeting in right angles labeled $A, \beta, C, \alpha, B, \gamma$.
(b) A pentagon: a disk bounded by five geodesic sides labeled A, B, α, C, β where A and B meet in an angle γ, and the rest of the angles are right angles.
(c) A quadrilateral: a disk bounded by four geodesic sides labeled A, C, B, γ where A and C meet in an angle β, C and B meet in an angle α and the two remaining angles are right angles.
(d) A triangle: a disk bounded by three geodesic sides labeled A, B, C where A and B meet in an angle γ and B and C meet in an angle α and C and A meet in angle β.

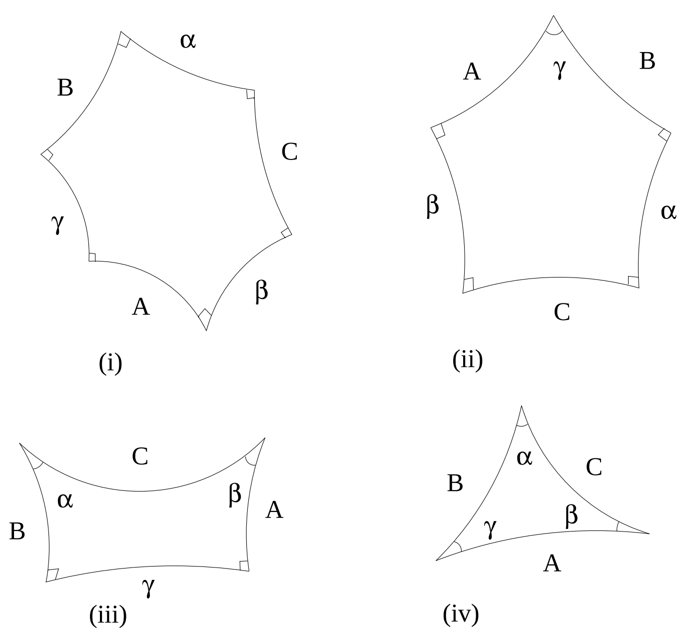

FIGURE 8. Generalized triangles and their labels.

Lemma 5.2. *For generalized triangles in the hyperbolic plane, the following equations hold.*

$$\begin{aligned}
&\text{(a)} \quad \cosh C &= \frac{\cosh\alpha\cosh\beta + \cosh\gamma}{\sinh\alpha\sinh\beta}\\
&\text{(b)} \quad \cosh C &= \frac{\cosh\alpha\cosh\beta + \cos\gamma}{\sinh\alpha\sinh\beta}\\
&\text{(c)} \quad \sinh A &= \frac{\cosh\gamma\cos\beta + \cos\alpha}{\sinh\beta\sin\gamma}\\
&\text{(d)} \quad \cosh C &= \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}
\end{aligned} \tag{14}$$

In (a), (α, β, γ) *can be any triple of positive real numbers. In* (b), (α, β) *can be any pair of positive numbers and* γ *a real number in* $(0, \pi)$ *In* (c), (α, β) *can be any pair of positive real numbers in* $(0, \pi)$ *satisfying* $\alpha + \beta < \pi$. *In* (d), (α, β, γ) *can be any triple of real numbers in* $(0, \pi)$ *satisfying* $\alpha + \beta + \gamma < \pi$.

Proof. For a proof, see Coxeter [13] or Ratcliffe [34]. □

The following lemma implies Proposition 5.1 for elementary 2-orbifolds of type (D1), (D2), (D3), and (D4).

Lemma 5.3. *Let* P *be an orbifold whose underlying space is a disk constructed as follows. Silver edges labeled by the capital letters* A, B, C. *Assign to each vertex an angle of the form* π/n *(where* ($n > 1$ *is an integer*), *for which it is a corner-reflector of that angle. Each edge labeled by Greek letters* α, β, γ *is a boundary full* 1*-orbifold. Then in cases* (a), (b), (c), (d) $\mathcal{F} : \mathcal{T}(P) \to \mathcal{T}(\partial P)$ *for each of the above orbifolds* P *is a homeomorphism*; *that is,* $\mathcal{T}(P)$ *is homeomorphic to a cell of dimension* 3*,* 2*,* 1*, or* 0 *respectively.*

Proof. We see that the edge lengths or angle measures of α, β, γ completely determine the unique disks. The edges labeled by Greek letters can be made arbitrarily large or small after the angles labeled by Greek letters are assigned as the reader can easily verify. By the above formulas, $\mathcal{F}$ is a homeomorphisms. □

Lemma 5.4. *Let* S *be an elementary* 2*-orbifold of type* (A1), (A2), (A3), *or* (A4). *Then* $\mathcal{F} : \mathcal{T}(S) \to \mathcal{T}(\partial S)$ *is a homeomorphism. Thus,* $\mathcal{T}(S)$ *is a cell of dimension* $2, 1, 1,$ *or* 0 *when* S *is of type* (A1), (A2), (A3) *or* (A4) *respectively.*

Proof. In case (A1), find the shortest segment s from one boundary component to the other, which is a full 1-orbifold. Cutting along it, we obtain a hexagon, where the boundary is cut into three alternating sides of the hexagon α, β, γ. Let α and β be adjacent to the boundary full 1-orbifold of S and γ from the boundary component circle. Let the length of α and β be equal. By symmetry, the lengths of two sides corresponding to s become equal. Hence, we can glue back such a hexagon to obtain an elementary 2-orbifold of type (A1) always. Thus, we see that given 2α and γ, we can obtain an elementary 2-orbifold. Since 2α and β are lengths of the boundary components, we have shown that $\mathcal{F}$ is a homeomorphism.

In case (A2), we silver the boundary component temporarily, find a shortest segment s from the mirror edge to the boundary component. Then cutting along s, we obtain a pentagon. Let α and β be the boundary full 1-orbifolds of the pentagon. Letting the lengths of α and β equal, we can always glue back to obtain an elementary 2-orbifold of type (A2). Since we can change the length of C arbitrarily by changing the common length of α and β, we see that $\mathcal{F}$ is a homeomorphism.

In case (A3), we draw a shortest segment s from the cone-point to the boundary 1-orbifold. We obtain a pentagon where s corresponds to edges labeled A and B above. If the lengths of A and B are equal, we can glue back to obtain an elementary 2-orbifold of type (A3). Here, α and β are from the boundary 1-orbifold by cutting, and their lengths are the same. If the lengths of α and β are the same, then those of A and B are equal, and we can glue back. Since 2α is the length of the boundary 1-orbifold, we see that $\mathcal{F}$ is a homeomorphism.

In (A4), we draw a shortest segment s from the cone-point to the mirror edge. Then we obtain a quadrilateral with angles π/p, $\pi/2$, $2\pi/q$, and $\pi/2$ where p is the order of the corner-reflector and q is that of the cone-point. Such a quadrilateral is unique, and we can glue back to obtain an elementary 2-orbifold of type (A4) always. Thus the Teichmüller space is a single point. □

Finally, if S is an elementary 2-orbifold of type (P1),(P2),(P3), or (P4), then there exists an order-two self-isometry so that the quotient 2-orbifolds are of type (D1), (D2), (D3), or (D4) respectively. This can be seen by choosing shortest segments connecting any pair of the boundary components or cone-points. Proposition 5.1 in these cases follows from Lemma 5.3. To conclude, Lemmas 5.3 and 5.4 imply Proposition 5.1. (In the final part of §6, we will prove Theorem 6.9.)

6. The deformation spaces of elementary 2-orbifolds and the proof of Theorem A.

This section concludes the proof of Theorem A, that the deformation spaces of convex $\mathbb{RP}^2$-structures on 2-orbifolds are homeomorphic to cells. The inductive proof begins from elementary orbifolds: We show that the map from the deformation space to the deformation space of the boundary is a fibration with base and fiber homeomorphic to cells. After this, when we build bigger orbifolds, this fact automatically holds.

The method we use for elementary orbifolds of type (P1)-(P4) is from [19] for a pair-of-pants. In fact, we need very small changes for our purposes. That is, we first show that the 2-orbifold of type (P1)-(P4) is built up from two triangles. We study the quadruples of triangles in order to understand how they are glued. By converting the geometric conditions that they assemble to convex elementary orbifolds into algebraic relations and solving the relations, we show that the deformation spaces are described by cells. For some annular elementary 2-orbifolds (A1)-(A4), we need Steiner's theorem defining conics as the set of intersection points of pencils of lines through two given points related by a projectivity (see Chapters 6 and 7 of Coxeter [14]). To study elementary 2-orbifolds with corner-reflectors (D1)-(D4), we generalize the methods in [17]. Finally, Theorem A is proved by noticing that the above fibration property can be inductively proved as we build up 2-orbifolds from elementary 2-orbifolds. Finally, we prove Theorem 6.9 (originally due to Thurston) for the sake of completeness of the paper.

The result proving Theorem A is as follows:

Theorem 6.1. *Let Σ be a compact 2-orbifold with negative Euler characteristic. Then the deformation space of convex $\mathbb{RP}^2$-structures $\mathcal{C}(\Sigma)$ is homeomorphic to a cell of dimension*

$$d(\Sigma) := -8\chi(X_\Sigma) + (6k_c - 2b_c) + (3k_r - b_r) + 4n_0 \tag{15}$$

where k_c is the number of cone-points, k_r the number of corner-reflectors, b_c the number of cone-points of order two, b_r the number of corner-reflectors of order two, and n_o is the number of boundary full 1-orbifolds.

6.1. The Fibration Property.

Definition 6.2. We say that the deformation space of Σ satisfies the *fibration property* if:

(i) the deformation space of convex $\mathbb{RP}^2$-structures $\mathcal{C}(\Sigma)$ is homeomorphic to a cell of dimension $d(\Sigma)$ as in equation (15), and

(ii) there exists a principal fibration $\mathcal{F} : \mathcal{C}(\Sigma) \to \mathcal{C}(\partial\Sigma)$ with the action of a cell of dimension $\dim \mathcal{C}(\Sigma) - \dim \mathcal{C}(\partial\Sigma)$, equal to

$$-8\chi(X_\Sigma) + (6k_c - 2b_c) + (3k_r - b_r) + 3n_0 - 2b_\Sigma = d(\Sigma) - n_0 - 2b_\Sigma$$

where b_Σ is the number of boundary components of Σ homeomorphic to S^1.

We will prove first the following:

Proposition 6.3. *Let Σ be an elementary 2-orbifold of negative Euler characteristic. Then the deformation space of Σ has the fibration property.*

6.2. Simple geodesics on 2-orbifolds.

Proposition 6.4. *Let l be an geodesic arc in an orientable convex 2-orbifold of negative Euler characteristic with boundary points in the boundary of the good orbifold or an interior point. Suppose that l is homotopic to a simple arc by a homotopy fixing the boundary points in the boundary of the orbifold or fixing the point if it is an interior point. Then l is simple.*

Proof. The universal cover of the orbifold is a convex domain and contains no bigons or monogons with geodesic boundary. Any nonsimple arc which is homotopic to a simple arc must have some bigon or monogon in the universal cover. Thus l is simple. □

We say that a sequence l_i of segments or line in a convex domain *converges to a line* l_∞ in Ω if $l_i \cap K$ converges to $l_\infty \cap K$ in the Hausdorff sense for every compact subset K of Ω.

A *lamination* in a convex 2-orbifold S is a subset of the regular set of S such that for each coordinate neighborhood U not meeting the singular points meets the set in a closed subset that is a union of disjoint lines that pass through the open set completely. (We allow half-open lines as long as they are complete in the neighborhood.) A *leaf* of a lamination is an arcwise connected subset.

A lamination is *finite* if each open set meets finitely many leaves. We say that an end of an infinite lamination *winds around* a simple closed curve, if a half-infinite arc corresponding to the end is so that all of its accumulation points lie in the curve.

Lemma 6.5. *Suppose that S is a convex 2-orbifold which either is diffeomorphic to a pair-of-pants, an annulus with a cone-point, a disk with two cone-points, or a sphere with three cone-points. We choose orientations on the boundary components. Then there exists a geodesic lamination l with three leaves so that S with the boundary components and the cone-points and the lamination l removed is a union of two convex triangles. Each leaf of l has two ends, each of which either ends in a cone-point or winds around a boundary component of S infinitely often. We can choose l so that the direction of the winding follows the arbitrarily chosen orientation.*

Proof. The universal covering of S is a convex domain with a projectively invariant Hilbert metric. This metric induces a Finsler metric on S with a corresponding notion of arclength.

Define S_ϵ for each $\epsilon > 0$ to be the compact subset of S obtained by removing from S the union of convex open neighborhoods of the boundary components and the cone-points so that S_ϵ is homeomorphic to a pair-of-pants. We assume that as $\epsilon \to 0$, S_ϵ is strictly increasing and contains any interior nonsingular point of S eventually.

First, there exists a topological lamination λ with this property. We can approximate it by a finite-length three-leaf topological lamination λ_t with six (distinct) endpoints either in the boundary components of S or the cone-points. We assume that for each $\epsilon > 0$, as $t \to \infty$, $S_\epsilon \cap \lambda_t$ approximates $S_\epsilon \cap \lambda$ very closely in C^1-sense. Moreover, we assume that the endpoints of λ_t winds around ∂S along the chosen orientation infinitely as $t \to \infty$.

Since S is convex, there exists a geodesic lamination $\hat{\lambda}_t$ homotopic to λ_t by a homotopy fixing the endpoints. We may choose S_ϵ for each $\epsilon > 0$ so that $\hat{\lambda}_t$ meets it always in a union of six connected arcs near the six end points of $\hat{\lambda}_t$. Thus $\hat{\lambda}_t \cap S_\epsilon$ is a finite-length finite lamination with endpoints in ∂S_ϵ by Lemma 6.6. Since $\hat{\lambda}_t \cap S_\epsilon$ has six endpoints, it has three leaves.

The length of $\hat{\lambda}_t \cap S_\epsilon$ is bounded for fixed ϵ as $t \to \infty$. Otherwise the Hausdorff limit set is a lamination in S_ϵ containing an infinite leaf. This contradicts Lemma 6.6.

Thus for each ϵ, we can choose a subsequence so that $\hat{\lambda}_{t_i} \cap S_\epsilon$ converges to a three-leaf finite-length finite lamination l_ϵ. By using a diagonal subsequence argument, we obtain a sequence $\hat{\lambda}_{t_j}$ so that $\hat{\lambda}_{t_j} \cap S_\epsilon$ converges to such a lamination for each $\epsilon > 0$.

The direction of the winding is the same as the topological lamination by Lemma 6.7.

The limit is now a geodesic lamination with three leaves. The intersection of the geodesic lamination with the compact set S_ϵ has fixed topological type. Therefore the complement in S of the union of the boundary, the lamination, and the cone-points is precisely the union of two disjoint convex open disks, each of whose boundary are leaves of the lamination.

Lifting to the universal covering convex domain, such disks develop to convex open triangles. □

Lemma 6.6. *Suppose that S is as above. Then every leaf L of a geodesic lamination in S ends in a cone of point of S or winds around a boundary component, and meets S_ϵ in a finite-length finite lamination.*

Proof. An end of any embedded infinite arc in S which lifts to a properly embedded arc in a universal cover winds around a simple closed curve parallel to a boundary component or a simple closed curve around a cone-point or end at a cone-point. (That is, its limit points comprise one of these.) Thus, L winds around a closed geodesic parallel to a boundary component. Since S is convex and with principal boundary and S can be covered by a convex surface with principal boundary, the closed curve is a boundary component (see [6]). □

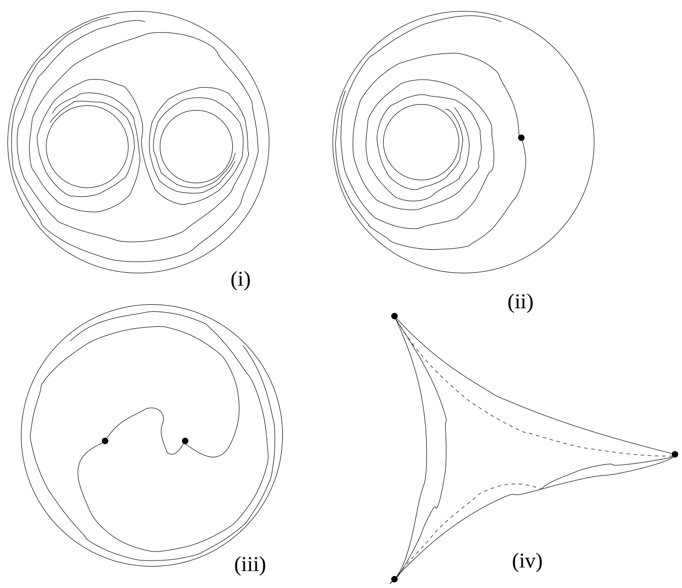

FIGURE 9. The finite laminations in elementary 2-orbifolds of type (P1)-(P4).

Lemma 6.7. *A leaf l of a geodesic lamination in S winds around a boundary component γ in a given direction if and only if l lifts to a curve in $\tilde{S} = \Omega$ ending in an attracting fixed point of the holonomy of γ.*

6.3. **Proof of Proposition 6.3.** We begin the case by case proof of Proposition 6.3. Case (P1) was already treated in [19].

6.4. **Annuli with one cone-point (P2).** We will decompose the elementary orbifold of type (P2) into two triangles following [19]. We make a one-to-one correspondence of $\mathcal{C}'(P)$ with configuration spaces of four adjacent triangles in the projective plane. Since $\mathcal{C}(P)$ is the quotient of $\mathcal{C}'(P)$ by $\mathrm{PGL}(3, \mathbb{R})$, we show that the configuration space quotient by $\mathrm{PGL}(3, \mathbb{R})$ is the configuration space with a triangle with standard vertices quotient by a group of diagonal matrices. Finally, we show that the latter configuration space fibers over the boundary invariants by solving algebraic equations.

First, we will reduce an elementary orbifold of type (P2) to a configuration: Let P be an annulus with a cone-point c of order n, $n \geq 2$ and two boundary components a, b. The fundamental group of P has presentation

$$\pi = \langle A, B, C | C^n = 1, ABC = 1 \rangle$$

where A and B are loops around two boundary components with boundary orientation and C loops around the cone-point. We find simple arcs e_b from c to a and e_b from c to a. We also find a disjoint simple arc e_c from a to b avoiding c. We will spiral e_a, e_b, and e_c positively in the boundary components a and b, with respect to the boundary orientation, to create geodesic laminations with leaves l_a, l_b, l_c by using above results on realizations.

We denote the geodesics by same notation. The complement of the lamination in P with boundary and the cone-point removed is a union of two open triangles. Let us denote them by T_0 and T_1. Choose a base point $p \in T_0$, and a point $\tilde{p}$ in the inverse image in the universal $\tilde{P}$ of P. Then the components of inverse images of T_0 and T_1 develop in $\mathbb{RP}^2$ to disjoint triangles. Let $\tilde{T}_0$ be the triangle containing $\tilde{p}$ and let T_a, T_b, and T_c be the components of the inverse images of T_1 adjacent to $\tilde{T}_0$ along the lifts of l_a, l_b, and l_c respectively. Therefore developing these triangles yields four triangles $\triangle_0, \triangle_a, \triangle_b, \triangle_c$ in $\mathbb{RP}^2$ and collineations A, B, C satisfying:

(i) $\mathrm{Cl}(\triangle_a)$, $\mathrm{Cl}(\triangle_b)$, $\mathrm{Cl}(\triangle_c)$ meet $\mathrm{Cl}(\triangle_0)$ in three edges of $\mathrm{Cl}(\triangle_0)$.

(ii) The union

$$\mathrm{Cl}(\triangle_a) \cup \mathrm{Cl}(\triangle_b) \cup \mathrm{Cl}(\triangle_c) \cup \mathrm{Cl}(\triangle_0)$$

is an embedded polygon with six or five vertices. (It is a convex hexagon if $n \geq 4$, and a pentagon may occurs when $n = 3$ but only rarely.) (More precisely, any two of $\mathrm{Cl}(\triangle_a), \mathrm{Cl}(\triangle_b)$, and $\mathrm{Cl}(\triangle_c)$ meet exactly in a singleton as shown in Figure 10.)

(iii) $ABC = \mathrm{Id}$, and $A(\mathrm{Cl}(\triangle_b)) = \mathrm{Cl}(\triangle_c)$, $B(\mathrm{Cl}(\triangle_c)) = \mathrm{Cl}(\triangle_a)$, and $C(\mathrm{Cl}(\triangle_a)) = \mathrm{Cl}(\triangle_b)$.

(iv) A and B are hyperbolic and the vertex where $\mathrm{Cl}(\triangle_b)$ and $\mathrm{Cl}(\triangle_c)$ meet is the repelling fixed point of A, and the vertex where $\mathrm{Cl}(\triangle_c)$ and $\mathrm{Cl}(\triangle_a)$ meet is one of B. C is a rotation of order n with the isolated fixed point the vertex at $\mathrm{Cl}(\triangle_a) \cap \mathrm{Cl}(\triangle_b)$.

Only the proof of (ii) is not immediate.

The complement of the union of lines containing the three sides of $\mathrm{Cl}(\triangle_0)$ is a union of three disjoint open triangles. Denote their closures by S_0, S_a, S_b, S_c where $S_0 = \mathrm{Cl}(\triangle_0)$. The existence of the actions by A, B, C implies the union of $\mathrm{Cl}(\triangle_0)$ with any one of $\mathrm{Cl}(\triangle_a), \mathrm{Cl}(\triangle_b), \mathrm{Cl}(\triangle_c)$ is a convex quadrilateral-(*). Thus $\mathrm{Cl}(\triangle_a), \mathrm{Cl}(\triangle_b)$, and $\mathrm{Cl}(\triangle_c)$ lie in S_a, S_b, S_c respectively.

By a patch of these triangles, we mean the disk obtained by identifying the sides as specified by the universal cover, and it is a disk with corners with an $\mathbb{RP}^2$-structure. If $n \geq 4$, then the interior angles of the patch of $\mathrm{Cl}(\triangle_0), \mathrm{Cl}(\triangle_a), \mathrm{Cl}(\triangle_b)$, and $\mathrm{Cl}(\triangle_c)$ are always less than π. Hence the patch is an embedded hexagon.

Let p_1, p_2, and p_3 denote the vertices of $\triangle_0$ so that the triangles $\mathrm{Cl}(\triangle_a)$ and $\mathrm{Cl}(\triangle_0)$ meet in a segment with vertices p_2 and p_3, the triangles $\mathrm{Cl}(\triangle_b)$ and $\mathrm{Cl}(\triangle_0)$ in that with p_1 and p_3, and the triangles $\mathrm{Cl}(\triangle_c)$ and $\mathrm{Cl}(\triangle_0)$ in that with p_1 and p_2. (See figure 10.)

Suppose that $n = 2$ or 3. If $n = 3$, then the vertices of $\mathrm{Cl}(\triangle_a), \mathrm{Cl}(\triangle_b)$, and $\mathrm{Cl}(\triangle_c)$ other than the vertices of $\mathrm{Cl}(\triangle_0)$ are in the open triangles S^o_a, S^o_b, S^o_c as C has order three. Thus, the embeddedness of the patch follows from this fact. (Notice that the angle at p_3 could be π when the vertex of $\triangle_a$, p_3, and the vertex of $\triangle_b$ are collinear, which could happen.)

If $n = 2$, then C is of order two and this forces that the vertex of $\mathrm{Cl}(\triangle_a)$ lies on the line containing p_1 and p_3, and the vertex of $\mathrm{Cl}(\triangle_b)$ lies on the line containing p_2 and p_3. Since A and B are hyperbolic, we see that the angle of the patch is less than π at p_1 and p_2. This implies that the patch corresponds to an embedded hexagon with a concave vertex at p_3. (See figure 11.)

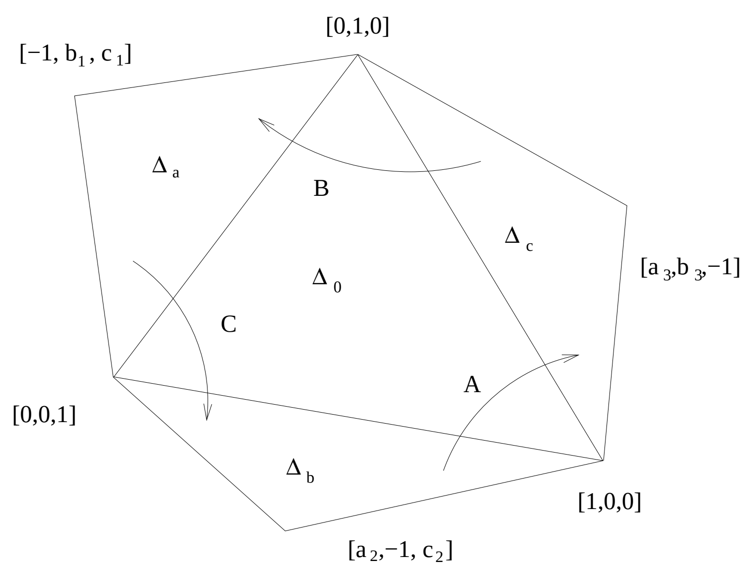

FIGURE 10. The four triangles needed to understand the deformation space of an annulus with one cone-point.

Let $\mathcal{C}'$ denote the space of all 7-tuples $(\triangle_0, \triangle_a, \triangle_b, \triangle_c, A, B, C)$ satisfying conditions (i)-(iv) with topology induced from the product

$$\left(\mathbb{RP}^2\right)^{12} \times \left(\mathrm{PGL}(3, \mathbb{R})\right)^3.$$

We showed that each element of $\mathcal{C}'(P)$ gives rise to one of the configurations in $\mathcal{C}'$. Our aim is to identify them now. The process is similar to [19]:

(i) We show that a configuration corresponds to an $\mathbb{RP}^2$-structure.
(ii) We embed the configuration space to the deformation space as an open subset.
(iii) We show that the deformation space of convex $\mathbb{RP}^2$-structures is a subset of the image, i.e., our construction above.
(iv) We show that the deformation space is an open and closed subset of the image using the Koszul openness and the closedness in their representation space.

Let $\hat{H}$ be the space of triples (A, B, C) that occur in $\mathcal{C}'$. Give $\hat{H}$ the subspace topology of the triple product $\mathrm{PGL}(3, \mathbb{R})^3$ of $\mathrm{PGL}(3, \mathbb{R})$. Then $\hat{H}$ is open: For A', B', C' sufficiently close to A, B, C in $\hat{H}$, the set of attracting fixed points of A', B' and the isolated fixed point of C' form vertices of a convex triangle. Any other collection of the triple points also form vertices of convex triangles. Thus, the seven tuple is well-defined.

We claim that an element (A, B, C) of $\hat{H}$ determines an element of $\mathcal{C}'$, and the projection $p_1 : \mathcal{C}' \to \mathrm{PGL}(3, \mathbb{R})^3$ defined by

$$(\triangle_0, \triangle_a, \triangle_b, \triangle_c, A, B, C) \mapsto (A, B, C)$$

is a homeomorphism onto $\hat{H}$: The eigenvectors of A, B, C and their images under A, B, C determine the triangles $\triangle_0, \triangle_a, \triangle_b, \triangle_c$ and hence we can define a section of this fibration from their image. The continuity of the section follows from that of the map from the subspace of hyperbolic matrices in $\mathrm{PGL}(3, \mathbb{R})$ to the space of fixed points of their largest eigenvalue is a continuous one and that of the map from the subspace of rotations in $\mathrm{PGL}(3, \mathbb{R})$ to their unique isolated fixed points.

We can identify the subspace $\mathrm{PGL}(3\mathbb{R})^3$ comprising triples (A, B, C) satisfying $ABC = \mathrm{Id}$ as the space of representations $U(P)_p$. Therefore, p_1 embeds $\mathcal{C}'$ to the subspace $\hat{H}$ of $U(P)_p$.

The group $\mathrm{PGL}(3, \mathbb{R})$ acts properly on the subset of $(\mathbb{RP}^2)^4$ consisting of 4-tuples of points, no three of which are collinear. Therefore the $\mathrm{PGL}(3, \mathbb{R})$-action on $\mathcal{C}'$, defined by

$$\big(\triangle_0, \triangle_a, \triangle_b, \triangle_c, A, B, C\big) \overset{\vartheta}{\mapsto} \big(\vartheta(\triangle_0), \vartheta(\triangle_a), \vartheta(\triangle_b), \vartheta(\triangle_c), \vartheta A \vartheta^{-1}, \vartheta B \vartheta^{-1}, \vartheta C \vartheta^{-1}\big),$$

for $\vartheta \in \mathrm{PGL}(3, \mathbb{R})$, is proper and free. Let $\mathcal{C}$ denote the quotient space.

Now, we demonstrate that $\mathcal{C}(P)$ is diffeomorphic to $\mathcal{C}$ as in Proposition 4.4 of [19]: Given an element of $\mathcal{C}'$, consider the $\mathbb{RP}^2$-orbifold with the same underlying space as P, but with the cone-point, boundary points, and mirror points removed. The open 2-orbifold P' extends to a homeomorph P'' of P, whose development pair is already determined by the 7-tuple: Let E be an end of P' corresponding to a boundary component of P'' and say A. The universal cover $\tilde{P}'$ of P' develops to a convex domain tessellated by the images of the triangles under the holonomy group action. Then a component of the inverse image of an open neighborhood of E in $\tilde{P}'$ is filled with triangles developing into triangles converging to a segment connecting the attracting and repelling fixed points of A. Adding the interior of the segment corresponds to the completion of E. If E is an end of P' corresponding to a cone-point. Then triangles meeting E develops periodically around the fixed point of a conjugate of C. Adding the fixed point corresponds to the completion of E. Therefore the homeomorph P'' carries a real projective structure since the completion is a geometric operation of attaching a principal geodesic boundary component and a cone-point. Therefore, such a 7-tuple determines an element of $\mathbb{RP}^{2*}(P)$, that is, an isotopy class of $\mathbb{RP}^2$-structures on P.

The embedding $\iota : \mathcal{C}' \longrightarrow \mathbb{RP}^{2*}(P)$ defined in this way is an imbedding onto an open subset of $\mathbb{RP}^{2*}(P)$: The composition $\mathcal{H}' \circ \iota$ equals projection p_1 from $\mathcal{C}'$ to the open subspace $\hat{H}$ in the space of representations $U(P)_p$. Because $\mathcal{H}'$ is a local homeomorphism by Theorem 3.7 and p_1 is continuous, ι is continuous. Because $\mathcal{H}'$ is a local homeomorphism and p_1 is open, ι is open and its image is an open subset of $\mathbb{RP}^{2*}(P)$. Because p_1 is injective, ι is injective. Therefore, ι is an embedding onto a subset of $\mathbb{RP}^{2*}(P)$.

The set $\mathcal{C}'(P)$ corresponding to convex structures on P is an open subset of $\mathbb{RP}^{2*}(P)$ by Proposition 3.8. Since a convex $\mathbb{RP}^2$-structure on P determines a 7-tuple by our construction, $\mathcal{C}'(P)$ is open in the image of ι.

By Proposition 3.9, the image of $\mathcal{C}'(P)$ under $\mathcal{H}'$ is a closed subset of $U(P)_p$. Since $\mathcal{H}'$ is a local homeomorphism, $\mathcal{C}'(P)$ is locally closed in $\mathbb{RP}^{2*}(P)$. Therefore $\mathcal{C}'(P)$ is open and locally closed in the image of ι.

The Teichmüller space $\mathcal{T}(P)$ of hyperbolic $\mathbb{RP}^2$-structures is a subset of $\mathcal{C}(P)$. $\mathcal{C}(P)$ is not empty since it contains $\mathcal{T}(P) \neq \emptyset$. Therefore $\mathcal{C}'(P) \neq \emptyset$. Since the image of ι is connected and open and relatively closed, $\mathcal{C}'(P)$ is the entire image of ι. This completes our identification of $\mathcal{C}(P)$ with $\mathcal{C}'$.

Now, we act by $\mathrm{PGL}(3,\mathbb{R})$ on $\mathcal{C}'(P)$ to obtain a quotient space $\mathcal{C}(P)$ and act by the same group on $\mathcal{C}'$ to obtain $\mathcal{C}$. Since the above correspondence is $\mathrm{PGL}(3,\mathbb{R})$-equivariant, we see that $\mathcal{C}$ and $\mathcal{C}(P)$ are diffeomorphic by the map induced by ι.

The following proposition proves Proposition 6.3 when P is an elementary 2-orbifold of type (P2):

Proposition 6.8. *$\mathcal{C}$ is an open cell of dimension 6 or $6-2=4$ depending on whether or not the order of the cone-point is 2. The map*

$$\mathcal{C} \rightarrow \mathcal{R}_A \times \mathcal{R}_B \tag{16}$$

$$(\triangle_0, \triangle_a, \triangle_b, \triangle_c, A, B, C) \mapsto ((\lambda,\tau)_A, (\lambda,\tau)_B) \tag{17}$$

is a principal fibration with fiber an open two-cell over the four-cell $\mathcal{R}^2$ for $n \geq 3$. If $n = 2$, the map is a diffeomorphism.

We begin the proof of Proposition 6.8: We may put $\triangle_0$ to a standard triangle with vertices $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$ in the homogeneous coordinates of $\mathbb{RP}^2$. The remaining vertices of $\triangle_a, \triangle_b$, and $\triangle_c$ are $[-1,b_1,c_1]$, $[a_2,-1,c_2]$, and $[a_3,b_3,-1]$ respectively. If $n > 2$, then $b_1, c_1, a_2, c_2, a_3, b_3$ are positive. If $n = 2$, C sends lines through its isolated fixed point to the same lines with orientation reversed. Thus,

$$a_2 = 0, b_1 = 0, \tag{18}$$

and the rest are positive.

Denoting by $\mathcal{C}''$ the configurations of four triangles with the above vertices. We see that $\mathcal{C}(P)$ is a quotient of $\mathcal{C}''$ by a group of collineations conjugate to a group of diagonal matrices with positive eigenvalues.

We will now analyze $\mathcal{C}''$ below using the above notation:

6.5. **(P2) The cone-point order $\neq 2$.** Now assume $n \geq 3$. If $n \geq 4$, then the angles of the union hexagon are less than π. Then

$$b_1, c_1, a_2, c_2, a_3, b_3 > 0.$$

At each vertex of $\triangle_0$, the cross-ratios of the four lines containing the edges of the incident triangles, determine invariants

$$\rho_1 = b_3c_2, \rho_2 = a_3c_1, \rho_3 = a_2b_1,$$

satisfying

$$\rho_1, \rho_2, \rho_3 > 1$$

since $\triangle_0 \cup \triangle_a \cup \triangle_b \cup \triangle_c$ is convex. (See Figure 10.) If $n = 3$, then

$$b_1, c_1, a_2, c_2, a_3, b_3 > 0, \rho_1, \rho_2 > 1$$

but ρ_3 may assume any positive real value. (When $\rho_3 = 1$, we have an angle π at $[0, 0, 1]$.)

The group of diagonal matrices

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}, \lambda\mu\nu = 1, \lambda, \mu, \nu > 0,$$

acts on $\triangle_0, \triangle_a, \triangle_b, \triangle_c$, taking

$$\begin{pmatrix} b_1 \\ c_1 \\ a_2 \\ c_2 \\ a_3 \\ b_3 \end{pmatrix} \mapsto \begin{pmatrix} (\mu/\lambda)b_1 \\ (\nu/\lambda)c_1 \\ (\lambda/\mu)a_2 \\ (\nu/\mu)c_2 \\ (\lambda/\nu)a_3 \\ (\mu/\nu)b_3 \end{pmatrix}.$$

Then $\sigma_1 = a_2b_3c_1$ and $\sigma_2 = a_3b_2c_2$ are invariants under the diagonal group action and hence are invariants of the 7-tuple under the action of collineations. $\sigma_1, \sigma_2 > 0$ and $\sigma_1\sigma_2 = \rho_1\rho_2\rho_3$.

By applying such a diagonal matrix, we may assume $a_3 = 2$ and $b_3 = 2$, obtaining a slice for the $\mathrm{PGL}(3, \mathbb{R})$-action on 7-tuples:

$$a_2 = t, a_3 = 2, b_1 = \rho_3/t, b_3 = 2, c_1 = \rho_2/2, c_2 = \rho_1/2 \tag{19}$$

where $t = \sigma_1/\rho_2 > 0$ is an arbitrary positive number. Elements of $\mathrm{PGL}(3, \mathbb{R})$ may be uniquely represented by elements of $\mathrm{SL}(3, \mathbb{R})$. The most general collineation sending $\triangle_b$ to $\triangle_c$ is given by the matrix

$$\begin{aligned} A &= \alpha_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot [1, a_2, 0] + \beta_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot [0, -1, 0] \\ &+ \gamma_1 \cdot \begin{bmatrix} a_3 \\ b_3 \\ -1 \end{bmatrix} \cdot [0, c_2, 1] \\ &= \begin{bmatrix} \alpha_1 & \alpha_1 a_2 + \gamma_1 a_3 c_2 & \gamma_1 a_3 \\ 0 & -\beta_1 + \gamma_1 b_3 c_2 & \gamma_1 b_3 \\ 0 & -\gamma_1 c_2 & -\gamma_1 \end{bmatrix} \end{aligned} \tag{20}$$

for $\alpha_1, \beta_1, \gamma_1 > 0$. We obtain for B and C the following

$$\begin{aligned} B &= \alpha_2 \cdot \begin{bmatrix} -1 \\ b_1 \\ c_2 \end{bmatrix} \cdot [1, 0, a_3] + \beta_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot [0, 1, b_3] \\ &+ \gamma_2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [0, 0, -1] \\ &= \begin{bmatrix} -\alpha_2 & 0 & -\alpha_2 a_3 \\ \alpha_2 b_1 & \beta_2 & \beta_2 b_3 \alpha_2 a_3 b_1 \\ \alpha_2 c_1 & 0 & -\gamma_2 + \alpha_2 a_3 c_1 \end{bmatrix} \qquad (21) \end{aligned}$$

and

$$\begin{aligned} C &= \alpha_3 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot [-1, 0, 0] + \beta_3 \cdot \begin{bmatrix} a_2 \\ -1 \\ c_2 \end{bmatrix} \cdot [b_1, 1, 0] \\ &+ \gamma_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [c_1, 0, 1] \\ &= \begin{bmatrix} -\alpha_3 + \beta_3 a_2 b_1 & \beta_3 a_2 & 0 \\ -\beta_3 b_1 & -\beta_3 & 0 \\ \gamma_3 c_1 + \beta_3 b_1 c_2 & \beta_3 c_2 & \gamma_3 \end{bmatrix} \qquad (22) \end{aligned}$$

where

$$\alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3 > 0.$$

$ABC = I$ and $\det(A) = \det(B) = \det(C) = 1$ imply

$$\alpha_1 \alpha_2 \alpha_3 = \beta_1 \beta_2 \beta_3 = \gamma_1 \gamma_2 \gamma_3 = 1 \qquad (23)$$

and

$$\alpha_1 \beta_1 \gamma_1 = \alpha_2 \beta_2 \gamma_2 = \alpha_3 \beta_3 \gamma_3 = 1. \qquad (24)$$

The invariants of A and B are given by:

$$\lambda(A) = \lambda_1 = \alpha_1, \qquad (25)$$

$$\tau(A) = \tau_1 = -\beta_1 + \gamma_1(\rho_1 - 1), \qquad (26)$$

$$\lambda(B) = \lambda_2 = \beta_2, \qquad (27)$$

$$\tau(B) = \tau_2 = -\gamma_2 + \alpha_2(\rho_2 - 1). \qquad (28)$$

For C to be of order n (for any $n \geq 2$),

$$\gamma_3 = 1, \text{ and} \qquad (29)$$

$$-\alpha_3 + \beta_3(\rho_3 - 1) = 2\cos(2\pi/n). \qquad (30)$$

We now restrict our attention purely to the system of equations from (23) to (30). We wish to solve for $\alpha, \beta, \gamma, \rho$s given the invariant $(\lambda_1, \tau_1, \lambda_2, \tau_2) \in \mathcal{R}^2$ and the order $n \geq 3$.

From equation (19), we obtain the coordinates of vertices. It is clear that the set of such solutions correspond in one to one manner to $\mathcal{R}$.

This system of equations appear in [19] where λ_3 is replaced by γ_3 and τ_3 by $2\cos(2\pi/n)$ where (λ_3, τ_3) is the invariant of C if C were hyperbolic.

The way to solve the system is to realize that with α_1 and β_2 and γ_3 fixed by λ_1, λ_2, and 1 from equations (25), (27), and (29), equations (23) and (24) can be made into a system of linear equations of rank five of six variables

$$\log\alpha_2, \log\alpha_3, \log\beta_1, \log\beta_3, \log\gamma_1, \log\gamma_2.$$

There is a space of solutions diffeomorphic to $\mathbb{R}^+$, say parametrized by a variable s. For each of the solutions of these equations, we plug into equations (26), (28), and (30) to solve for ρ_1, ρ_2, ρ_3. Then we plug these to equation (19) adding $\mathbb{R}^+$-parameter from the variable t, obtaining a solution space diffeomorphic to $\mathbb{R}^2$. Thus, we obtain for every fixed

$$(\lambda_1, \tau_1, \lambda_2, \tau_2) \in \mathcal{R}^2, (s, t) \in \mathbb{R}^{+2},$$

a unique solution. (Actually, equations 4-21 and 4-23 of [19] with setting $\lambda_3 = 1$ and $\tau_3 = 2\cos(2\pi/n)$ are the solutions for any choice of s and t, $s, t > 0$, there.) This completes the proof of Proposition 6.8 when $n \geq 3$.

6.6. **(P2) the cone-point order 2.** Now consider $n = 2$. Apply a unique collineation (see Figure 11) so that $\mathrm{Cl}(\triangle_0)$ has vertices

$$[1, 0, 0], [0, 1, 0], [0, 0, 1],$$

$\mathrm{Cl}(\triangle_a)$ is an adjacent triangle with vertices

$$[0, 1, 0], [-1, 0, 1], [0, 0, 1],$$

$\mathrm{Cl}(\triangle_b)$ is an adjacent triangle with vertices

$$[0, -1, 1], [1, 0, 0], [0, 0, 1],$$

and $\mathrm{Cl}(\triangle_c)$ is an adjacent triangle with vertices

$$[0, 1, 0], [1, 0, 0], [a_3, b_3, -1].$$

We can define cross-ratios ρ_1, ρ_2 as above. We have

$$\rho_1 = b_3, \rho_2 = a_3, \rho_3 = 0, c_1 = 1, c_2 = 1 \tag{31}$$

by equation (18). From the condition that $C^2 = \mathrm{Id}$, and equation (18), we see that $\alpha_3 = \beta_3 = \gamma_3 = 1$. The equations (23) and (24) and equations (25), (26), (27), (28), (29), and (30) still apply here. Then with

$$(\lambda_1, \tau_1, \lambda_2, \tau_2) \in \mathcal{R}^2$$

fixed, we obtain values of α_1, β_2 by equations (25) and (27) and hence a unique solution for equations (23) and (24). Now, we plug the solution to equation (26) and equation (28) to obtain ρ_1 and ρ_2. Equation (30) is automatically satisfied. This determines the seven-tuple.–(**) □

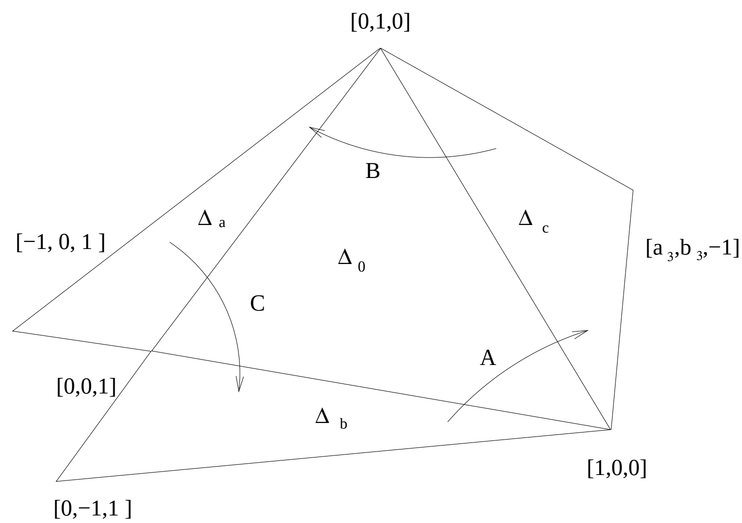

FIGURE 11. The four triangles needed to understand the deformation space of an annulus with one cone-point of order two.

6.7. **A disk with two cone-points (P3).** Now, we determine the topology of the deformation space of the elementary 2-orbifolds of type (P3). Let m and n be the order of the two cone-points. Assume $m \geq 3$ without loss of generality. By the same reasoning as (P2), the proof reduces again to determining the space $\mathcal{C}$ of equivalence classes under the diagonal group action of $\mathcal{C}'$ of 7-tuples $(\triangle_0, \triangle_a, \triangle_b, \triangle_c, A, B, C)$ where $\triangle_0$ is the standard triangle and $\triangle_a, \triangle_b, \triangle_c$ are adjacent triangles, A is a collineation sending $\triangle_b$ to $\triangle_c$ and B one sending $\triangle_a$ to $\triangle_c$ of order m and C one sending $\triangle_c$ to $\triangle_a$ of order n. That is, we show similarly to (P2) that $\mathcal{C}(P) = \mathcal{C}$ again by openness and closedness.

Assume for the moment $n \geq 3$ also. We can introduce unknowns $b_1, c_1, a_2, c_2, a_3, b_3$, ρ_1, ρ_2, ρ_3, σ_1, σ_2, $\alpha_i, \beta_i, \gamma_i$, $i = 1, 2, 3$, as above. The only change of the equations occur at the equations (27) and (28) which change to

$$\beta_2 = 1, \tag{32}$$

$$-\gamma_2 + \alpha_2(\rho_2 - 1) = 2\cos(2\pi/m). \tag{33}$$

The same procedure implies $\mathcal{F} : \mathcal{C}(P) \to \mathcal{R}_A$ is an principal $\mathbb{R}^2$-fibration where $\mathcal{R}_A$ is the space of invariants of A. Here $\mathcal{C}(P)$ has dimension four.

When $n = 2$, an argument similar to (**) $\mathcal{F}$ is a diffeomorphism $\mathcal{C}(P) \to \mathcal{R}_A$ so that $\mathcal{C}(P)$ is two-dimensional. (They confirm the conclusion of Proposition 6.3.)

6.8. **Spheres with three cone-points (P4).** For elementary 2-orbifolds of type (P4), first suppose that $r, m, n \geq 3$, then the same considerations as above apply if we change the equations (25) and (26) as well by

$$\alpha_1 = 1, \tag{34}$$

$$-\beta_1 + \gamma_1(\rho_1 - 1) = 2\cos(2\pi/r). \tag{35}$$

Thus $\mathcal{C}(P)$ is a 2-cell.

If $n = 2$, then both $r \geq 3$ and $m \geq 3$ since $\chi(\Sigma) < 0$. So the second part of (**) applies, and $\mathcal{C}(P)$ is a single point.

6.9. **Crowns with two prongs (A1).** From here on, our methods are based more on geometry than computations as in the previous cases.

Again, we reduce an element of $\mathcal{C}(P)$ to a point of a certain configuration space. For an elementary 2-orbifold P of type (A1), $\mathbf{dev}(\tilde{P})$ for the universal cover $\tilde{P}$ of P is a convex disk invariant under $h(\vartheta)$ for the deck transformation ϑ corresponding to its principal closed geodesic boundary component with the boundary orientation. A hyperbolic element $h(\vartheta)$ has three fixed points in $\mathbb{RP}^2$ and three lines through two of them, and four open triangles bounded by subsegments of them are $h(\vartheta)$-invariant. The developing image $\mathbf{dev}(\tilde{P})$ is convex, and lies inside the closure of one of these open triangles. Choose projective coordinates so that the open triangle is the standard coordinate triangle, that is, the repelling fixed point of $h(\vartheta)$ is $[0,0,1]$, the attracting fixed point $[0,1,0]$ and the saddle type fixed point $[1,0,0]$. (Figures are similar to elementary 2-orbifolds of type (P2) here.)

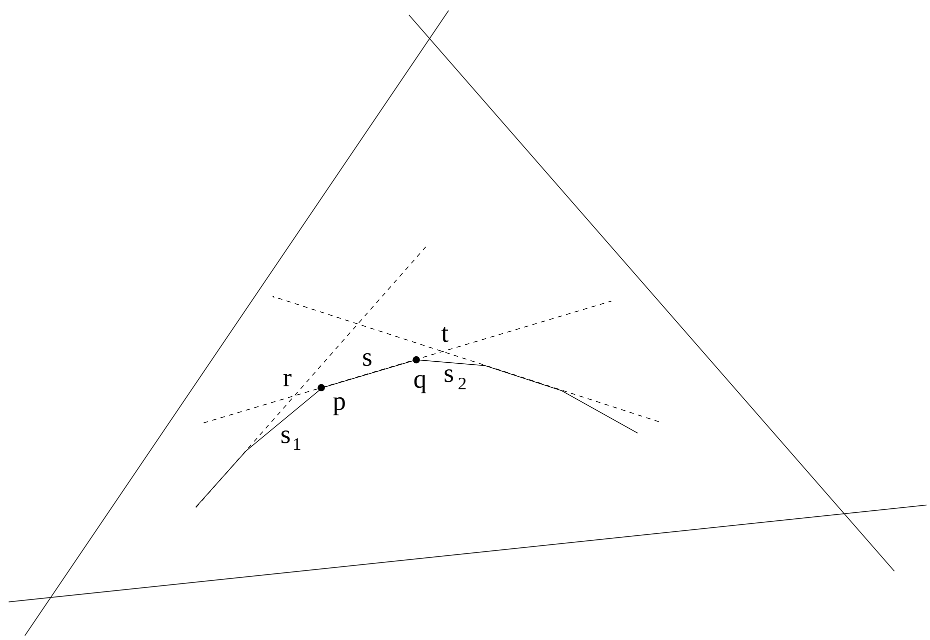

FIGURE 12. The annulus and reflections.

Given a convex $\mathbb{RP}^2$-structure on P, its universal covering space $\tilde{P}$ contains a disk K' with two boundary components l_1 and l_2 where l_1 covers a principal closed geodesic boundary of P under the universal covering map and l_2 covers the union of a segment in the singular locus and a boundary 1-orbifold. Let p' and q' be two endpoints of the boundary 1-orbifold in P, and let p and q be two endpoints of a segment in l_2 corresponding to p' and q' respectively; let s be the segment connecting p and q. Let s_1 and s_2 be the other two segments in l_2 starting from p and q respectively. Then $\vartheta(s_1) = s_2$.

The developing map **dev** embeds K' to a convex domain in the standard triangle so that l_1 maps to an open segment connecting $[0,0,1]$ to $[0,1,0]$, and l_2 to an arc consisting of segments bent in one direction connecting $[0,0,1]$ to $[0,1,0]$. Identify $\tilde{P}$ and associated objects with their developing images or holonomy images since **dev** is an embedding. There is a reflection r_1 whose fixed line contains s_1. The reflection in the line containing s_2 must equal

$$r_2 = \vartheta r_1 \vartheta^{-1}.$$

For each segment of the form $\vartheta^i(s_1)$, $i \in \mathbf{Z}$, has an associated reflection $\vartheta^i r_1 \vartheta^{-i}$. Since s corresponds to a 1-orbifold, r_1 and r_2 act on the line $l(s)$ containing s. Therefore, the isolated fixed points r and t of r_1 and r_2 respectively lie on $l(s)$. Since r_1 also acts on the line containing $\vartheta^{-1}(s)$, r must lie on $\vartheta^{-1}(l(s))$ the fixed line of $\vartheta^{-1}r_1\vartheta$. Thus, r is the unique intersection point of $l(s)$ and $\vartheta^{-1}(l(s))$. Similarly t is the unique fixed point of $l(s)$ and $\vartheta(l(s))$. By post-composing **dev** by a diagonal matrix, we may assume that r correspond to $[1,1,1]$ and t to $\vartheta([1,1,1])$. Thus p and q lie on $\overline{[1,1,1]\vartheta([1,1,1])}$. Recall that D is bounded by l_1 and l_2 the union of $\overline{\vartheta^i(p)\vartheta^i(q)}$ and $\overline{\vartheta^{i-1}(q)\vartheta^i(p)}$ for $i \in \mathbf{Z}$.

We showed that each element of $\mathcal{C}(P)$ corresponds to

$$(\vartheta, p, q), p, q \in \overline{[1,1,1]\vartheta([1,1,1])}$$

where ϑ is a positive projective transformation represented by a diagonal matrix so that $[0,0,1]$ correspond to the eigenvector with least eigenvalues, and $[1,0,0]$ to one with the largest eigenvalue.

We can identify the space of such configurations $\mathcal{C}$ with $\mathcal{C}(P)$: Any element of $\mathcal{C}(P)$ correspond to a unique such element by the above standardization process.

We can introduce the topology to the space $\mathcal{C}$ of above configurations as a subspace of $D \subset \mathrm{PGL}(3,\mathbb{R})$ and $A \times A$ where where D is the space of diagonal matrices with strictly decreasing positive eigenvalues and A is the interior of the standard triangle. Clearly, $\mathcal{C}$ is diffeomorphic to $\mathbb{R}^2 \times \mathbb{R}^2$. Then the map from $\mathcal{C}(P)$ to $\mathcal{C}$ defined by the above work is clearly continuous since ϑ, p, q is determined by the holonomy homomorphisms which depends continuously on the projective structures.

We now try to obtain the inverse ι of this map: From an element of $\mathcal{C}$, we form the domain K' in the standard triangle bounded by

$$\vartheta^n(\overline{[1,1,1]\vartheta([1,1,1])}), n \in \mathbf{Z}, \overline{[0,1,0][0,0,1]}.$$

More precisely, K' is defined to be the union of the interior and the segments

$$\vartheta^n(\overline{[1,1,1]\vartheta([1,1,1])}), n \in \mathbf{Z}$$

and the interior of the segment $\overline{[0,1,0][0,0,1]}$.

Let r be the reflection with isolated fixed point $[1,1,1]$ and the segment of fixed points $\overline{p\vartheta^{-1}(q)}$. K' and $r_1(K')$ meet in a side and adjacent sides extend each other as r preserves their ambient lines. Thus, $K' \cup r(K')$ is convex and so is $K' \cup r(K') \cup \vartheta r \vartheta^{-1}(K' \cup r(K'))$. Continuing in this manner, we see that a connected union of finitely many images of K' under $\langle \vartheta, r\rangle$-action is convex. Therefore, K^∞ equal to the union of all the images of of K' under $\langle \vartheta, r\rangle$-action is convex. Then

$$K^\infty / \langle \vartheta, r\rangle$$

is an elementary orbifold of type (A1), which is convex.

The map ι from $\mathcal{C}$ to $\mathcal{C}(P)$ defined in the above manner is also continuous: By definition of the topology, the composition

$$\mathcal{H} \circ \iota : \mathcal{C} \to U(P)_p/\mathrm{PGL}(3, \mathbb{R})$$

is continuous. Since $\mathcal{H}$ is a local homeomorphism, ι is continuous. (Here the proof is a little bit different from cases (P1),(P2),(P3), and (P4) since we can directly construct a convex projective structure from a point of the configuration space. However, it seems that there should be direct methods for (P1)-(P4) also.)

We obtained that $\mathcal{C}(P)$ is diffeomorphic to a 4-cell $\mathcal{C}$.

The cross-ratio $[[1,1,1], \vartheta([1,1,1]); p, q] \in (0,1)$ is the invariant of the boundary 1-orbifold. Fixing this cross-ratio, the space of choices of p and q on $\overline{[1,1,1]\vartheta([1,1,1])}$ is diffeomorphic to $\mathbb{R}$. This becomes a fiber of the fiber map

$$\mathcal{F} : \mathcal{C}(P) \to \mathcal{R} \times (0,1)$$

given by sending the triple (ϑ, p, q) to the invariants of ϑ and the cross ratio.

6.10. **One-pronged crowns (A2).** We now go to elementary 2-orbifolds of type (A2). Suppose that the order k of the corner-reflector is greater than or equal to 3. $\tilde{P}$ contains a disk K' with two boundary arcs l_1 and l_2, where l_1 covers the principal closed geodesic boundary, and l_2 covers the other boundary component. Identify K' with a convex domain in a standard triangle with ϑ acting on with properties as in (A1). Here ϑ is the holonomy of the deck transformation corresponding to the principal geodesic boundary component of P, and has a diagonal matrix with the attracting fixed point $[0,1,0]$ and the repelling fixed point $[0,0,1]$.

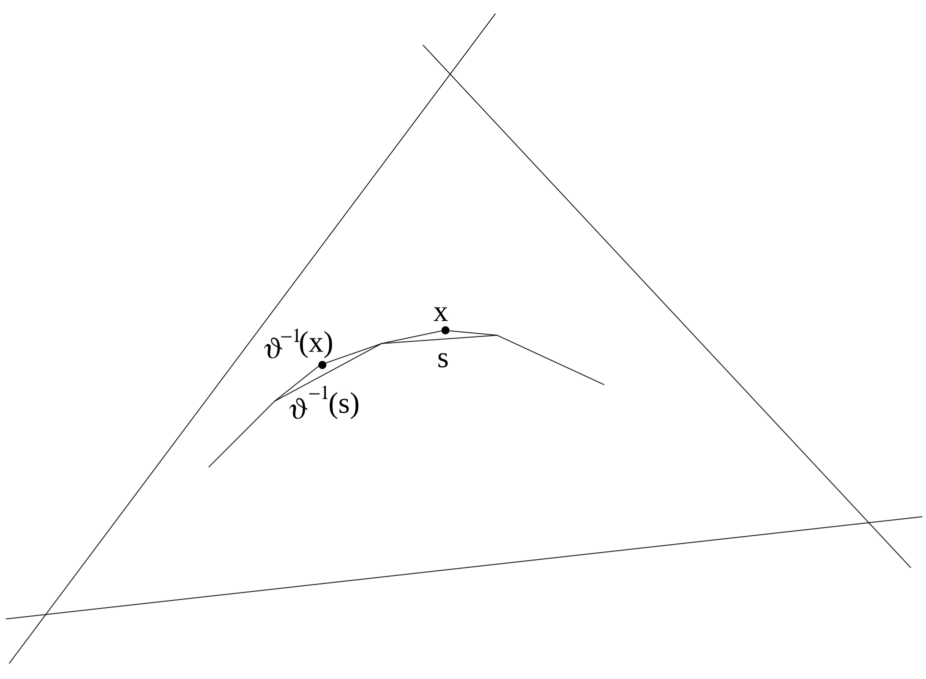

FIGURE 13. The annulus and reflections.

We also identify associated objects. l_2 is a union of segments of form $\vartheta^i(s)$, $i \in \mathbf{Z}$, for a segment s with endpoints $p, \vartheta(p)$. We post-compose **dev** with a diagonal matrix so that $p = [1,1,1]$ where ϑ does not change. The holonomy group contains a reflection r with a line of fixed points containing s. $\vartheta^n r \vartheta^{-n}$ is a reflection with a line of fixed points containing the segment $\vartheta^n(s)$. Let x be the isolated fixed point of r. Then $\vartheta^{-1}(x)$ is the isolated fixed point of $\vartheta^{-1} r \vartheta$. Let $l(x)$ and $l(\vartheta^{-1}(x))$ be the line from $[1,1,1]$ to points x and $\vartheta^{-1}(x)$ respectively. The corner-reflector has order $k \geq 3$, which is equivalent to the condition that the cross-ratio of lines satisfy

$$[l(x), l(\vartheta^{-1}(x)); s, \vartheta^{-1}(s)] = \frac{2}{\cos\frac{2\pi}{k} + 1} \tag{36}$$

as shown in [17]. Thus, x satisfies this condition.

We compute the solution space of (36): For each line l through $[1,1,1]$, we find a line $L(l)$ through $[1,1,1]$ so that

$$[l, L(l); s, \vartheta^{-1}(s)] = \frac{2}{\cos\frac{2\pi}{k} + 1}.$$

This is a projectivity L from the pencil $F([1,1,1])$ of lines through $[1,1,1]$ to the same pencil. Using the map $\hat{\vartheta}$ from $F([1,1,1])$ to the pencil $F(\vartheta([1,1,1])$ of lines through $\vartheta([1,1,1])$ induced by ϑ, we obtain a projectivity

$$\hat{\vartheta} \circ L : F([1,1,1]) \to F(\vartheta([1,1,1])).$$

By a classical result of Steiner (Chapters 6 and 7 of Coxeter [14]), the locus of $l \cap \hat{\vartheta} \circ L(l)$ for $l \in F([1,1,1])$ is a conic. Furthermore it passes through $[1,1,1]$ and $\vartheta([1,1,1])$.

For an element of $\mathcal{C}(P)$, we obtain a pair

$$(\vartheta, x)$$

where x lies on the conical arc through $[1,1,1]$ and $\vartheta([1,1,1])$ given by above equations.

As in case (A1), we can show that $\mathcal{C}(P)$ identifies with the space of such configurations: We follow (P2) in this case with a sketchy argument. We define a map

$$\iota : \mathcal{C} \to \mathbb{RP}^2(P)$$

first. We find a domain K' bounded by

$$\overline{[0,1,0][0,0,1]}, \vartheta^n(\overline{[1,1,1]\vartheta([1,1,1])}), n \in \mathbf{Z};$$

more precisely, K' is the union of the interior with the interior of $\overline{[0,1,0][0,0,1]}$ and the segments above. There is a reflection r with the isolated fixed point at x and with the segment of fixed points $\overline{[1,1,1]\vartheta([1,1,1])}$. $K'/\langle\vartheta\rangle$ is an annulus with principal boundary component a and a once-broken geodesic circle boundary component b. We can silver the broken geodesic using r and we obtain a point of $\mathbb{RP}^2(P)$.

Since $\mathcal{H} \circ \iota : \mathcal{C} \to U(P)_p/\mathrm{PGL}(3,\mathbb{R})$ is a function sending a configuration (ϑ, x) to (ϑ, r) where r is described above, it is continuous. Since $\mathcal{H}$ is a local homeomorphism ι is continuous. By the above description, the image of the composition is open. Thus, the image of ι in $\mathbb{RP}^2(P)$ is open. The set $\mathcal{C}(P)$ is a subset of the image of ι since a convex $\mathbb{RP}^2$-structure gives rise to a configuration by our construction. The closedness of $\mathcal{C}(P)$ in the image of ι follows since the image $\mathcal{C}(P)$ under $\mathcal{H}$ maps to a closed set in $U(P)_p/\mathrm{PGL}(3,\mathbb{R})$. We can show as in (P2), that $\mathcal{C}(P)$ is an open and closed subset of the image of ι and hence equal to the image.

Thus $\mathcal{C}(P)$ is diffeomorphic to a 3-cell $\mathcal{C}$. The mapping

$$\mathcal{F} : \mathcal{C}(P) \to \mathcal{R}_\vartheta$$

is a fibration with fibers the above arcs where $\mathcal{R}_\vartheta$ is the space of invariants of ϑ.

Suppose that the order of the corner-reflector is 2. Then using the above notation, we see that the point x lies on the line containing the segment $\vartheta^{-1}(s)$; the point $\vartheta^{-1}(x)$ on the line containing the segment s; and x is the unique intersection point of lines $l(\vartheta^{-1}(s))$ and $\vartheta(l(s))$. Thus, there is a unique choice of x given ϑ, and $\mathcal{C}(P)$ is a two-cell and

$$\mathcal{F} : \mathcal{C}(P) \to \mathcal{R}_\vartheta$$

is a homeomorphism.

6.11. **Disks with one boundary full 1-orbifolds (A3).** Again, we reduce an element of $\mathcal{C}(P)$ to a configuration: First in case (A3), let ϑ be the deck-transformation associated with the cone-point of order n, $n \geq 3$. The universal cover $\tilde{P}$ of P contains a disk where ϑ acts on so that its boundary map to the boundary of the underlying space X_P of P. Let s_1' be the boundary 1-orbifold and s_2' the segment in the singular locus of P. Thus the disk is a convex polygon with $2n$ sides

$$s_1, s_2, \vartheta^1(s_1), \vartheta^1(s_2), \ldots, \vartheta^{n-1}(s_1), \vartheta^{n-1}(s_2)$$

where s_1 map to s_1' and s_2 to s_2' in P. The holonomy group contains a reflection r with the line $l(s_2)$ of fixed points containing s_2. $\vartheta^i r \vartheta^{-1}$ has the line $\vartheta^i(l(s_2))$ of fixed points containing $\vartheta^i(s_2)$. Again, we find coordinates so that the isolated fixed point lie in $[0, 0, 1]$ and ϑ has a matrix form

$$\begin{bmatrix} \cos 2\pi/n & \sin 2\pi/n & 0 \\ -\sin 2\pi/n & \cos 2\pi/n & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

Since s_1 is a geodesic 1-orbifold, r and $\vartheta^{-1} r \vartheta$ act on the line $l(s_1)$ containing s_1. Let x be the isolated fixed point of r. Then $\vartheta^{-1}(x)$ is the isolated fixed point for $\vartheta^{-1} r \vartheta$. The fixed point x lies on $l(s_1)$, as r acts on $l(s_1)$, and on $l(\vartheta^{-1}(s_1))$, as r acts on $l(\vartheta^{-1}(s_1))$. Thus, x is the unique intersection point of $l(s_1)$ and $l(\vartheta^{-1}(s_1))$. The point $\vartheta(x)$ is the isolated fixed point of $\vartheta r \vartheta^{-1}$ with the line of fixed points $l(s_2)$. Since $\vartheta r \vartheta^{-1}$ acts on $l(s_1)$, $\vartheta(x)$ lies on $l(s_1)$. $\vartheta r \vartheta^{-1}$ acts on $l(\vartheta(s_1))$, $\vartheta(x)$ is the unique intersection point of $l(s_1)$ and $l(\vartheta(s_1))$. Post-compose **dev** with a matrix of form

$$\begin{bmatrix} k\cos\theta & k\sin\theta & 0 \\ -k\sin\theta & k\cos\theta & 0 \\ 0 & 0 & 1/k^2 \end{bmatrix}, k > 0, \tag{38}$$

to send x to $[1, 1, 1]$ and ϑ is unchanged. Let p and q be endpoints of s_1 with p separating x from q. The space of possible choices of p and q is diffeomorphic to $\mathbb{R}^2$. The cross-ratio $[[1, 1, 1], \vartheta([1, 1, 1]), p, q]$ is the invariant of the boundary 1-orbifold corresponding to s_1.

Our configuration space $\mathcal{C}$ is the set of points $p, q \in \overline{[1,1,1]\vartheta([1,1,1])}$. Given a point of $\mathcal{C}$, we can determine the reflection r, and as in case (A1), we show that a point of $\mathcal{C}$ gives rise to a convex orbifold of type (A3). We identify $\mathcal{C}$ with $\mathcal{C}(P)$ by a homeomorphism. Thus $\mathcal{C}(P)$ is diffeomorphic to $\mathbb{R}^2$ as ϑ is fixed now.

There is an $\mathbb{R}$-parameter family of choices of points to be labeled by p and q when the boundary invariant is fixed. This is a fiber of the fibration $\mathcal{F} : \mathcal{C}(P) \to (0,1)_{s_1'}$ where $(0,1)_{s_1'}$ is the space of invariants of the boundary 1-orbifold s_1'.

6.12. **Disks with one corner-reflectors (A4).** Let P be an orbifold of type (A4) with a corner-reflector of order n and a cone-point of order m where $1/n + 2/m < 1$. Then $m \geq 3$. The proof is completely analogous to the two preceding cases. If $n \geq 3$, then $\mathcal{C}(P)$ is diffeomorphic to $\mathbb{R}$. If $n = 2$, then $\mathcal{C}(P)$ is a single point.

6.13. **Pentagons (D2).** Let P be an orbifold of type (D2). Suppose that $n \geq 3$. Let $e_1', \dots, e_5'$ denote the edges and boundary orbifolds in the boundary of X_P ordered in an appropriate orientation. Let e_2', e_3', e_5' be the edges and e_1' and e_4' be boundary 1-orbifolds. Let e_2' and e_3' be the edges in the singular locus meeting in the corner-reflector of order n. $\tilde{P}$ contains a disk D bounded by five geodesics e_1, e_2, e_3, e_4, and e_5 so that e_i maps to e_i' for $i = 1, \dots, 5$. Let v_1, v_2, v_3, v_4, v_5 denote the vertices of D so that e_i has vertices v_i, v_{i+1} with cyclic indices. **dev** maps D into a convex pentagon in an affine patch of $\mathbb{RP}^2$. Post-compose **dev** with a collineation so that v_1, v_2, v_4, v_5 map to

$$[0,0,1], [1,0,1], [1,1,1], [0,1,1]$$

respectively. Then v_3 maps to a point $[s,t,1]$ where $s > 1$ and $0 < t < 1$ by convexity of the pentagon. Identify D and associated objects with their images in $\mathbb{RP}^2$. Let l_i denote the unique line containing e_i, $i = 1, \dots, 5$. For e_2, e_3, e_5 there are associated reflections r_2, r_3, and r_5 respectively. Their fixed lines are l_2, l_3, and l_5 respectively. Let x_2, x_3, and x_5 denote the respective isolated fixed points. e_1 and e_4 correspond to boundary 1-orbifolds. Since r_2 and r_5 act on l_1 and r_4 and r_5 on l_2, we have $x_2, x_5 \in l_1$ and $x_3, x_5 \in l_2$; that is, $x_5 = [1,0,0]$ and $x_2 = [t_2, 0, 1]$ and $x_3 = [t_3, 1, 1]$ with $t_2, t_3 > 1$. We compute the invariants of e_1 and e_4

$$\begin{aligned} \big[[t_2,0,1], \quad [1,0,0]; \quad [1,0,1], \quad [0,0,1]\big] &= \frac{t_2 - 1}{t_2} \\ \big[[t_3,1,1], \quad [1,0,0]; \quad [1,1,1], \quad [0,1,1]\big] &= \frac{t_3 - 1}{t_3}. \end{aligned} \tag{39}$$

Since the corner-reflector has order n,

$$\left[\overline{v_3[t_2,0,1]}, \overline{v_3[t_3,1,1]}; \overline{v_3[1,0,1]}, \overline{v_3[1,1,1]}\right] = \frac{2}{\cos\frac{2\pi}{n} + 1}. \tag{40}$$

We define our configuration space $\mathcal{C}$ to by v_2, v_4 satisfying equations (39) and v_3 satisfying equation (40). Therefore, we found a map from $\mathcal{C}(P)$ to the configuration space $\mathcal{C}$.

Let us explore the topology of the configuration space $\mathcal{C}$: Fix t_2 and t_3. Steiner's theorem (Chapters 6 and 7 of Coxeter [14]) implies the set of solutions v_3s is a conic through $[t_2,0,1], [t_3,1,1], [1,1,1]$, and $[1,0,1]$. The subarc of the conic in the triangle with vertices $[t_2, 0, 1], [t_3, 1, 1], [1, 0, 0]$ outside the quadrilateral is the solution space for v_3.

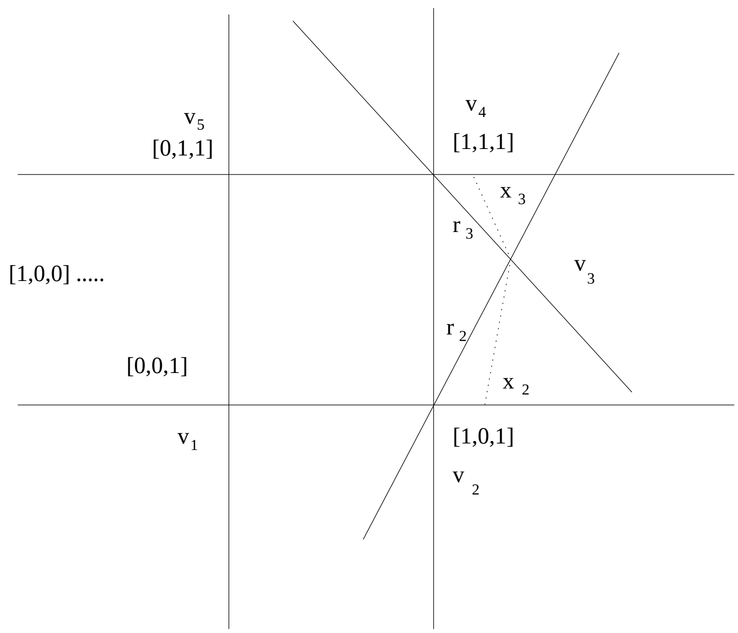

FIGURE 14. The deformation spaces of pentagons.

The above equation (40) admits a solution since every cross-ratio between 0 to 1 in the equation is realized by a point on the triangle. For each t_2 and t_3, there is an arc of solutions. Therefore, $\mathcal{C}$ is diffeomorphic to $\mathbb{R}^3$ and fibers over $(0, 1) \times (0, 1)$ with fibers equal to arcs given by the Steiner's theorem.

The map from $\mathcal{C}$ to $U(P)_p$ given by assigning to the configuration the reflections r_1, r_2, r_3 is obviously continuous. Also, given a point of $\mathcal{C}$, we can construct an $\mathbb{RP}^2$-structure on P. Therefore, as in case (P2), we can identify $\mathcal{C}(P)$ with $\mathcal{C}$, and the map $\mathcal{F} : \mathcal{C}(P) \to (0, 1)_{e_1} \times (0, 1)_{e_4}$ is a principal $\mathbb{R}$-fibration where $(0, 1)_{e_1}$ and $(0, 1)_{e_4}$ are the space of invariants of e_1 and e_4 respectively.

Next assume the cone-point order n equals 2. Then $x_5 = [0, 0, 1]$ and x_2 must be the intersection of the line through v_3 and $[1, 0, 1]$ with l_1. Furthermore x_3 is the intersection of the line through v_3 and $[1, 1, 1]$ with l_2. Let $x_2 = [t_2, 0, 1]$ and $x_3 = [t_3, 1, 1]$. Given t_2, t_3, let v_3 be the unique intersection point of the line through x_2 and $[1, 1, 1]$ with the line through x_3 and $[1, 0, 1]$. Thus $\mathcal{C}(P)$ is a 2-cell and

$$\mathcal{F} : \mathcal{C}(P) \to (0, 1)_{e_1} \times (0, 1)_{e_2}$$

is a diffeomorphism.

6.14. **Hexagons (D1).** Let P be a hexagon. The universal cover $\tilde{P}$ of P contains a disk with boundary edges $e_1, e_2, \ldots, e_6$ and vertices $v_1, v_2, \ldots, v_6$ so that e_1, e_3, and e_5 are in the interior of $\tilde{P}$ and correspond to singular edges of P and e_2, e_4, and e_6 are geodesic segments mapping to the boundary 1-orbifolds of P. Choose a developing map so that

$$\mathbf{dev}(v_1) = [0,0,1], \mathbf{dev}(v_2) = [1,0,1], \mathbf{dev}(v_3) = [1,1,1], \mathbf{dev}(v_6) = [0,1,1].$$

Then we have

$$\mathbf{dev}(v_4) = [s_4, t_4, 1], \mathbf{dev}(v_5) = [s_5, t_5, 1], 0 < s_4, s_5 < 1, 1 < t_4, t_5.$$

As above identify objects in $\tilde{P}$ with those in $\mathbb{RP}^2$ by $\mathbf{dev}$. There are reflections r_1, r_3, and r_5 whose lines of fixed points contain e_1, e_3, and e_5 respectively. Let x_1, x_3, x_5 denote the respective isolated fixed points. Let l_i denote the line containing e_i for $i = 1, \dots, 6$. Since l_2 and l_6 are r_1-invariant, x_1 is the intersection point of l_2 and l_6, that is, $x_1 = [0,1,0]$. Since l_2 and l_4 are r_2-invariant, x_3 is of form $[1, t_3, 1]$ for $t_3 > 1$, and similarly $x_5 = [0, t_5, 1]$ for $t_5 > 1$. The invariants for e_2 and e_6 are given by $(t_3 - 1)/t_3$ and $(t_5 - 1)/t_5$ respectively. Let l_{35} be the line through x_3 and x_5. Then l_{35} meets l_3 at v_4 and meets l_5 at v_5. The invariant for e_4 is given by the cross-ratio $[x_3, x_5; v_4, v_5]$.

The invariants for e_2 and e_6 determine t_3 and t_5. Given an invariant for e_4, choose v_4 and v_5 on the segment l_{35} connecting x_3 and x_5 giving us the correct invariant. The solutions are parametrized by $\mathbb{R}$.

Thus

$$\mathcal{F} : \mathcal{C}(P) \to \mathcal{C}(\partial P) = (0,1)_{e_2} \times (0,1)_{e_4} \times (0,1)_{e_6}$$

is a principal $\mathbb{R}$-fibration, and $\mathcal{C}(P)$ is a 4-cell.

6.15. **Quadrilaterals (D3).** Let P be a quadrilateral, and let e'_1 be the boundary 1-orbifold and e'_2, e'_3, and e'_4 the edges of P so that they are ordered with respect to an orientation. In the universal cover $\tilde{P}$, consider a disk D bounded by geodesic segments e_1, e_2, e_3, and e_4 mapping to e'_1, e'_2, e'_3, and e'_4 respectively. Let v_1, v_2, v_3, and v_4 denote the vertices so that v_i and v_{i+1} are the vertices of edges e_i for cyclic indices. Post-composing $\mathbf{dev}$ with a projective automorphism if necessary, v_1, v_2, v_3, and v_4 map to

$$[0,0,1], [1,0,1], [1,1,1], [0,1,1]$$

respectively. Identify D and associated objects by $\mathbf{dev}$ as before. Let l_i denote the lines containing e_i, $i = 1, \dots, 4$. Let r_2, r_3, and r_4 denote the reflections in the holonomy group whose lines of fixed points are l_2, l_3 and l_4 respectively. Let x_2, x_3, and x_4 denote the respective isolated fixed points. Since e_1 correspond to a 1-orbifold, r_2 and r_4 act on l_1. Thus, $x_2, x_4 \in l_1$.

First, suppose that $n, m \geq 3$. Then the cross-ratios are as follows:

$$[\overline{[0,1,1]x_3}, \overline{[0,1,1]x_4}; \overline{[0,1,1][1,1,1]}, \overline{[0,1,1][0,0,1]}] = \frac{2}{\cos 2\pi/m + 1}, \tag{41}$$

$$[\overline{[1,1,1]x_2}, \overline{[1,1,1]x_3}; \overline{[1,1,1][1,0,1]}, \overline{[1,1,1][0,1,1]}] = \frac{2}{\cos 2\pi/n + 1}, \tag{42}$$

and

$$[x_2, x_4; [1,0,1], [0,0,1]]$$

parametrizes e_1. In this case, the pentagon with vertices $x_2, [1,0,1], x_3, [1,1,1], x_4$ is convex. For any choice of $x_2, x_4 \in l_1$, there is a line through $[1,0,1]$ containing x_3

satisfying (41). Similarly there is a line through $[1,1,1]$ containing x_3 satisfying (42). Thus, any x_2, x_4 determines a unique x_3. The space of the choices of x_2, x_4 keeping the boundary invariant constant is diffeomorphic to $\mathbb{R}$. Thus, $\mathcal{C}(P)$ is a 2-cell and the map

$$\mathcal{F} : \mathcal{C}(P) \to (0,1)_{e_1}$$

is a principal $\mathbb{R}$-fibration for the space $(0,1)_{e_1}$ of invariants of e_1.

Suppose that $n = 2$ and $m \geq 3$. While $x_3 \in l_2$, x_2 lies on the line containing $[0,1,1]$ and $[1,1,1]$. Since x_2 lies on l_1, it follows that $x_2 = [0,1,0]$. Given any x_4, there exists $x_3 \in l_2$ satisfying

$$[\overline{[0,1,1]x_3}, \overline{[0,1,1]x_4}; \overline{[0,1,1][1,1,1]}, \overline{[0,1,1][0,0,1]}] = \frac{2}{\cos 2\pi/m+1}.$$

Thus, $\mathcal{C}(P)$ is diffeomorphic to $\mathbb{R}$ and $\mathcal{F} : \mathcal{C}(P) \to (0,1)$ is a homeomorphism.

6.16. **Triangles (D4).** In the last case (D4), [17] implies that $\mathcal{C}(P)$ is diffeomorphic to $\mathbb{R}^+$, and earlier by Kac-Vinberg [24]. This completes the proof of Proposition 6.3. □

6.17. **Proof of Theorem A or 6.1.** We assume the following inductive assumption: Let Σ be the 2-orbifold whose components have negative Euler characteristic obtained from sewing a 2-orbifold Σ' whose components have negative Euler characteristic. We suppose that Σ' satisfies the fibration property. If we show that Σ has the fibration property, the proof of Theorem 6.1 follows from Proposition 6.3.

First, we suppose that Σ is obtained from Σ' by pasting along simple closed curves. Let Σ' have two boundary principal closed geodesics l_1 and l_2 corresponding to l in Σ. Take a convex $\mathbb{RP}^2$-structure on Σ' with l_1 and l_2 with matching invariants and find an isomorphism between neighborhoods of l_1 and l_2 in an appropriate ambient 2-orbifold. Identifying and truncating these neighborhoods produces a convex $\mathbb{RP}^2$-structure on Σ. There is a principal fibration

$$\mathcal{F}' : \mathcal{C}(\Sigma') \to \mathcal{C}(\partial\Sigma')$$

with the action of a cell of dimension equal to

$$\dim \mathcal{C}(\Sigma') - \dim \mathcal{C}(\partial\Sigma').$$

Take the diagonal subset $\triangle$ of $\mathcal{C}(\partial\Sigma')$ consisting of elements of where invariants of l_1 agree with that of l_2. Since $\mathcal{F}'$ is a fibration, $\mathcal{F}'^{-1}(\triangle)$ is a cell of dimension two less than that of $\mathcal{C}(\Sigma')$.

Proposition 3.11 states that there exists an $\mathbb{R}^2$-action Φ on $\mathcal{C}(\Sigma)$ and a Φ-invariant fibration

$$\mathcal{SP} : \mathcal{C}(\Sigma) \to \mathcal{F}'^{,-1}(\triangle)$$

where Φ acts transitively, freely, and properly on the fibers. We have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{C}(\Sigma) & \xrightarrow{\mathcal{F}} & \mathcal{C}(\partial\Sigma) & = & \mathcal{C}(\partial\Sigma) \\ \downarrow \mathcal{SP} & & & & \downarrow g \\ \mathcal{F}'^{-1}(\triangle) \subset \mathcal{C}(\Sigma') & \xrightarrow{\mathcal{F}'} & \triangle \subset \mathcal{C}(\partial\Sigma') & \xrightarrow{f} & \mathcal{C}(\partial\Sigma' - l_1 - l_2) \end{array} \tag{43}$$

where f is a function defined on $\triangle$ by forgetting about the values on l_1 and l_2, and g a function sending invariants of $\partial\Sigma$ to corresponding invariants of $\partial\Sigma' - l_1 - l_2$. Since f forgets the value of the invariants at l_1 and l_2, f is a principal fibration with the $\mathbb{R}^2$-action. g is a homeomorphism since there is a one-to-one correspondence between the boundary components.

From the diagram, $\mathcal{F}$ can be identified with $f \circ \mathcal{F}' \circ \mathcal{SP}$. Therefore, $\mathcal{F}$ is a fibration with the action of a cell of dimension

$$2 + \dim \mathcal{C}(\Sigma') - \dim \mathcal{C}(\partial\Sigma') + 2.$$

Since $\mathcal{F}'^{-1}(\triangle)$ is of codimension 2 in the ambient space, $\mathcal{C}(\Sigma)$ is a cell of dimension $\dim \mathcal{C}(\Sigma')$. We verify the dimension formula: The Euler characteristic of the underlying space of Σ' equals $\chi(\Sigma)$. Furthermore Σ' and Σ have equal numbers of cone-points, corner-reflectors (of equal orders), and boundary full 1-orbifolds respectively. Thus, the fibration property (i) holds for Σ'.

Since we lose two boundary components l_1 and l_2, we obtain $\dim \mathcal{C}(\partial\Sigma) = \dim \mathcal{C}(\partial\Sigma') - 4$. Since the above cell is of dimension $\dim \mathcal{C}(\Sigma) - \dim \mathcal{C}(\partial\Sigma)$, the property (ii) holds for Σ.

Suppose that Σ is obtained from Σ' by cross-capping along a simple closed curve l'. By Proposition 3.11, $\mathcal{SP}$ maps $\mathcal{C}(\Sigma)$ diffeomorphically to $\mathcal{C}(\Sigma')$. The commutative diagram

$$\begin{array}{ccccc} \mathcal{C}(\Sigma) & \xrightarrow{\mathcal{F}} & \mathcal{C}(\partial\Sigma) & = & \mathcal{C}(\partial\Sigma) \\ \downarrow \mathcal{SP} & & & & \downarrow g \\ \mathcal{C}(\Sigma') & \xrightarrow{\mathcal{F}'} & \mathcal{C}(\partial\Sigma') & \xrightarrow{f} & \mathcal{C}(\partial\Sigma' - l'), \end{array} \tag{44}$$

where f is the forgetful principal $\mathbb{R}^2$-fibration and g a homeomorphism, holds. Therefore (i) and (ii) easily follow by verifying the dimension formula.

When Σ is obtained from Σ' be silvering a simple closed curve l', the proof is exactly same as the above if we change the meanings of the symbols correspondingly to the silvering case.

Suppose that Σ is obtained from Σ' by folding along a simple closed curve l'. Then l' is purely hyperbolic. Since the subspace of $\mathcal{R}$ consisting of purely hyperbolic elements is diffeomorphic to an arc, the subspace $\triangle$ of foldable convex $\mathbb{RP}^2$-structures is a cell of dimension $\dim \mathcal{C}(\Sigma') - 1$ by the fibration property. The commutative diagram above (43) holds where f is a principal $\mathbb{R}$-fibration and g is a homeomorphism again. Since $\mathcal{SP}$ is a principal $\mathbb{R}$-fibration, (i) and (ii) easily follow.

Suppose that Σ is obtained from Σ' by pasting along two full 1-orbifolds l_1 and l_2. Again (i) and (ii) follow by the commutative diagram (43) for this case.

Suppose that Σ is obtained from Σ' by folding or silvering a 1-orbifold l'. There is a unique way to do this. Thus $\mathcal{C}(\Sigma)$ is diffeomorphic to $\mathcal{C}(\Sigma')$, and (i) and (ii) follows by diagram 44 for this case. □

6.18. **A theorem of Thurston.** Since the holonomy of hyperbolic $\mathbb{RP}^2$-surfaces are hyperbolic, the space of invariants of a boundary closed curve is a one-dimensional subspace of $\mathcal{R}$ diffeomorphic to $\mathbb{R}$: The lengths of the closed curves provide satisfactory invariants. Invariants of boundary full 1-orbifolds are again simply lengths. Therefore, we define $\mathcal{T}(\partial\Sigma)$ to be the product of these lines of invariants.

For the sake of completeness we include the projective proof of the following theorem:

Theorem 6.9 (Thurston)**.** *Let Σ be a 2-orbifold of negative Euler characteristic and empty boundary. Then the deformation space $\mathcal{T}(\Sigma)$ of hyperbolic $\mathbb{RP}^2$-structures is diffeomorphic to a cell of dimension $-3\chi(X_\Sigma) + 2k + l$ where X_Σ is the underlying space, and k is the number of cone-points and l is the number of corner-reflectors.*

Proof. Since we repeat the proof of Theorem 6.1, we give a sketchy argument.

Let a 2-orbifold Σ, each component of which has negative Euler characteristic, be in a class $\mathcal{P}$ if the following hold:

(i) The deformation space of hyperbolic $\mathbb{RP}^2$-structures $\mathcal{T}(\Sigma)$ is diffeomorphic to a cell of dimension

$$-3\chi(X_\Sigma) + 2k + l + 2n$$

where k is the number of cone-points, l the number of corner-reflectors, n is the number of boundary full 1-orbifolds.

(ii) There exists a principal fibration

$$\mathcal{F} : \mathcal{T}(\Sigma) \to \mathcal{T}(\partial\Sigma)$$

with the action by a cell of dimension $\dim \mathcal{T}(\Sigma) - \dim \mathcal{T}(\partial\Sigma)$.

Let Σ be a 2-orbifold whose components are orbifolds of negative Euler characteristic, and it splits into an orbifold Σ' in $\mathcal{P}$. We suppose that (i) and (ii) hold for Σ', and show that (i) and (ii) hold for Σ. Since Σ eventually decomposes into a union of elementary 2-orbifolds where (i) and (ii) hold, we would have completed the proof by Proposition 5.1.

Since we need to match lengths and find gluing maps preserving lengths, the arguments are similar to the rest of the proof of Theorem 6.1.

We need the result of Proposition 3.11 for hyperbolic cases:

(A)(I)(1): Let the 2-orbifold Σ'' be obtained from pasting along two closed curves b, b' in a 2-orbifold Σ'. The map resulting from splitting

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \Delta \subset \mathcal{T}(\Sigma')$$

is a principal $\mathbb{R}$-fibration, where Δ is the subset of $\mathcal{C}(\Sigma')$ where b and b' have equal invariants.

(A)(I)(2): Let Σ'' be obtained from Σ' by cross-capping. The resulting map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a diffeomorphism.

(A)(II)(1): Let Σ'' be obtained from Σ' by silvering. The clarifying map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a diffeomorphism.

(A)(II)(2): Let Σ'' be obtained from Σ' by folding a boundary closed curve l'. The unfolding map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \Delta \subset \mathcal{T}(\Sigma')$$

is a principal $\mathbb{R}$-fibration.

(B)(I): Let Σ'' be obtained by pasting along two full 1-orbifolds b and b' in Σ'. The splitting map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \Delta \subset \mathcal{T}(\Sigma')$$

is a diffeomorphism where Δ is a subset of $\mathcal{T}(\Sigma')$ where the invariants of b and b' are equal.

(B)(II): Let Σ'' be obtained by silvering or folding a full 1-orbifold. The clarifying or unfolding map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a diffeomorphism.

Again, using diagrams (43) and (44) adopted to each cases, we prove the theorem. □

Appendix A. Developing maps of elementary orbifolds

This appendix presents various developing maps of elementary 2-orbifolds which are generated by Maple. We will divide the orbifolds into various one or two polygonal domains and develop them. In the following, the *depth* means the maximum word length of the deck transformations written using the standard generators. (s, t) indicates certain invariants analogous to the invariants for a pair-of-pants as given in [19] (see §6). The pictures for (P1) are not given, and examples for (D4) are given in [19]. The maple files are available from mathx.kaist.ac.kr\~schoi. One can change many parameters in the maple files to get more examples.

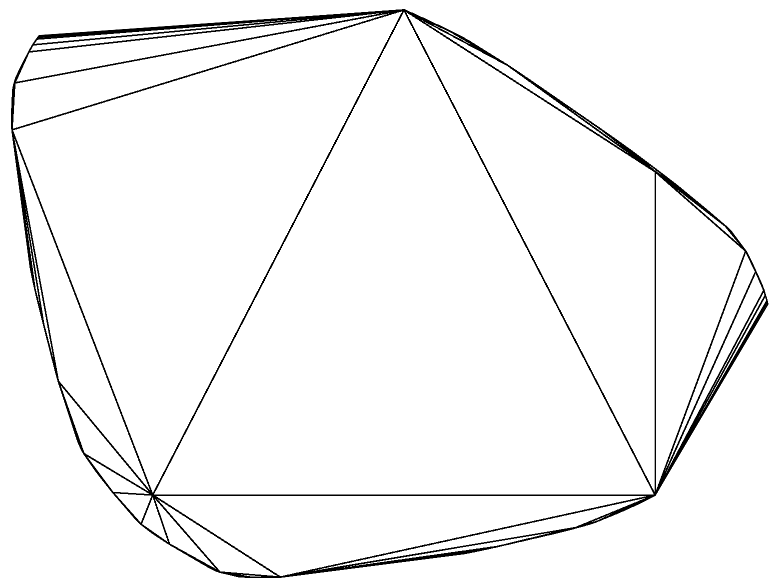

FIGURE 15. An annulus with a cone-point of order 5, boundary invariants $(1/3.1, 4.1)$ and $(1/4.1, 5.1)$, $(s, t) = (2, 1)$, depth 4, type (P2), and symbol $A(;5)$.

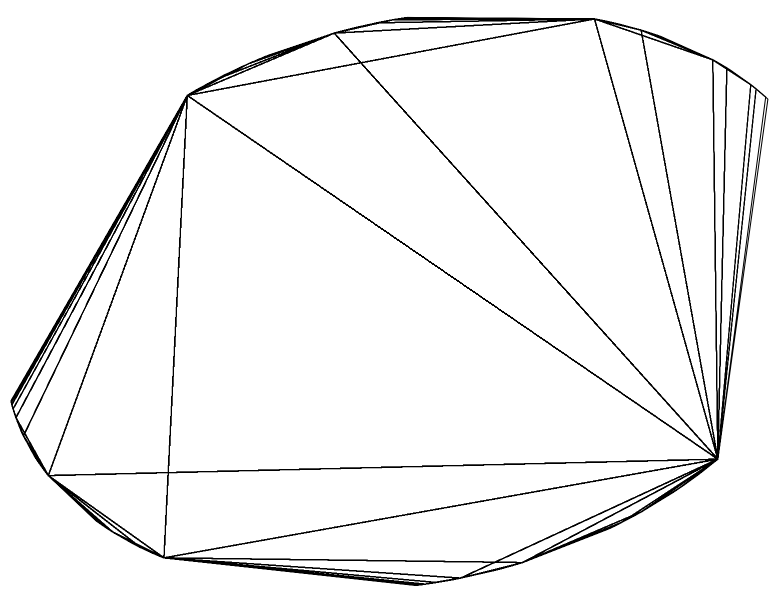

FIGURE 16. An annulus with a cone-point of order 2, boundary invariants $(1/2, 3)$, $(1/4, 5)$, $(s, t) = (2, 1)$, depth 4, type (P2), and symbol $A(;2)$.

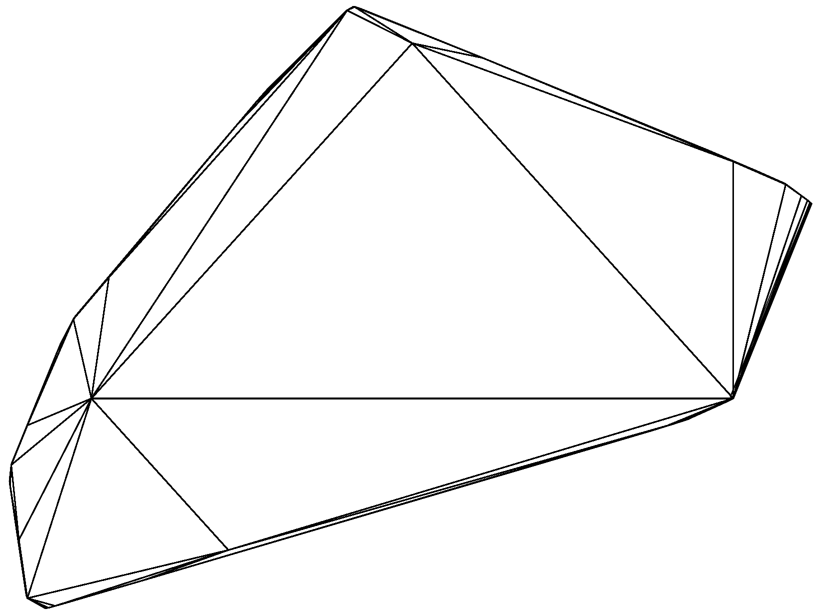

FIGURE 17. A disk with two cone-points of orders 3 and 5, the boundary invariant $(1/5, 6)$, $(s, t) = (1, 1)$, depth 4, type (P3), and symbol $D(;3, 5)$.

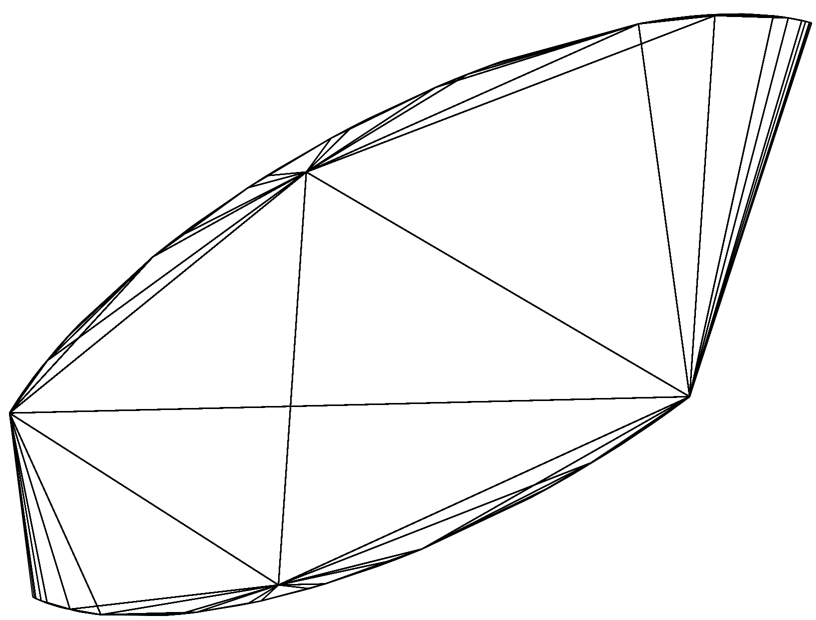

FIGURE 18. A disk with cone-points of orders 2 and 7, the boundary invariant $(1/3, 4)$, depth 4, type (P3), and symbol $D(;2, 7)$.

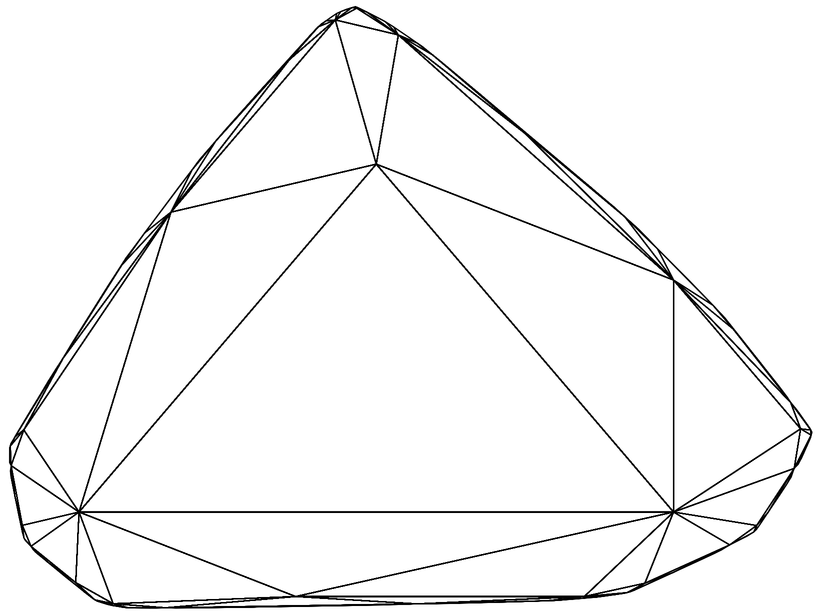

FIGURE 19. A sphere with three cone-points of orders $3, 5, 5$. $(s, t) = (2, 2)$. depth 4, type (P4), and symbol $\mathbf{S}^2(;3, 5, 5)$.

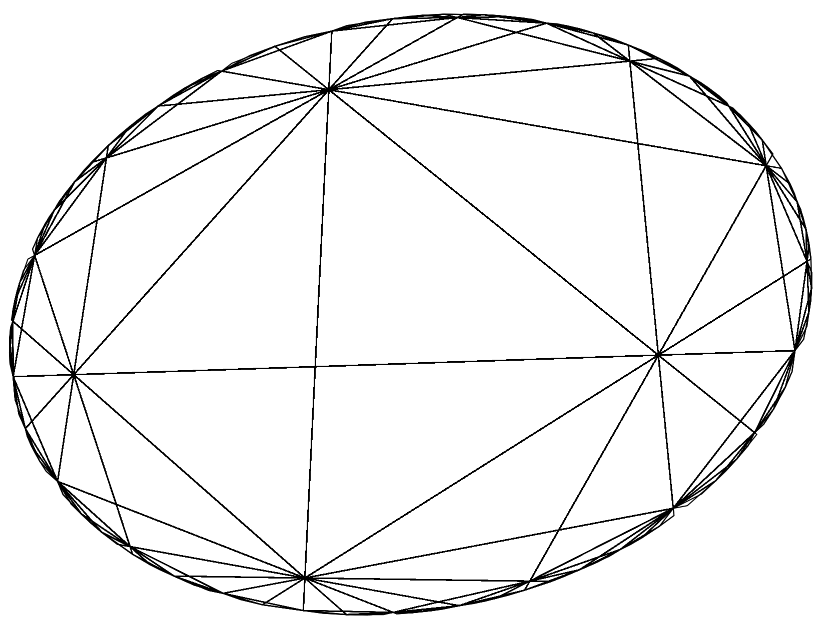

FIGURE 20. A sphere with order two cone-points orders $2, 5, 7$ depth 4, type (P4), and symbol $\mathbf{S}^2(;2, 5, 7)$.

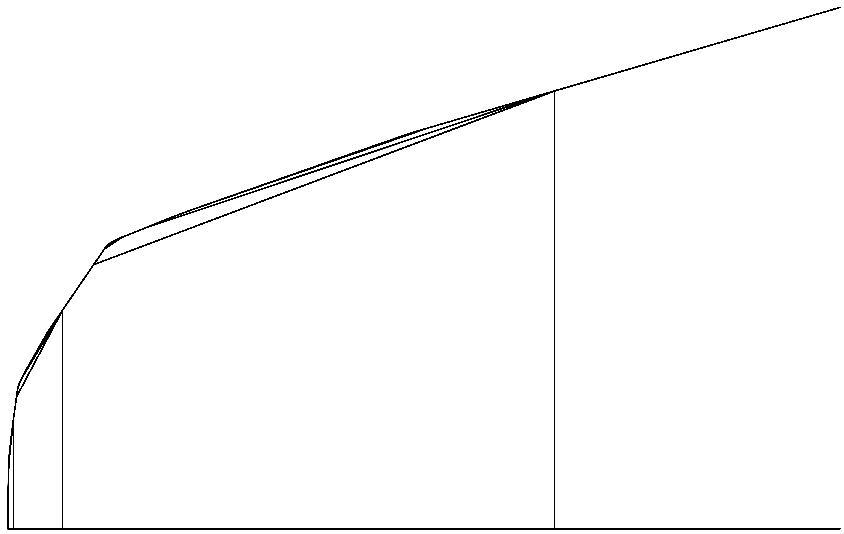

Figure 21. An annulus with a boundary 1-orbifold, boundary invariants 0.4 and $(0.5, 5)$, depth 5, type (A1), and symbol $A(2, 2;)$.

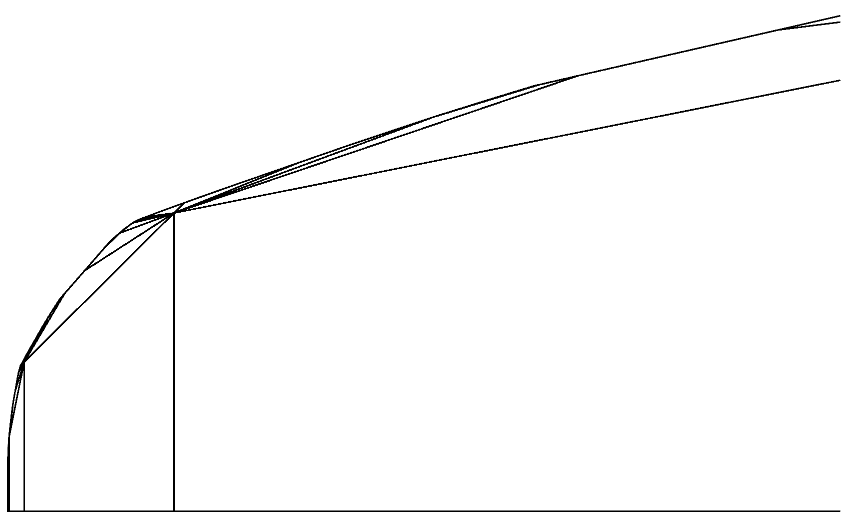

Figure 22. An annulus with a corner-reflector of order 3 , the boundary invariant $(0.5, 5)$, depth 5, type (A2), and symbol $A(3;)$.

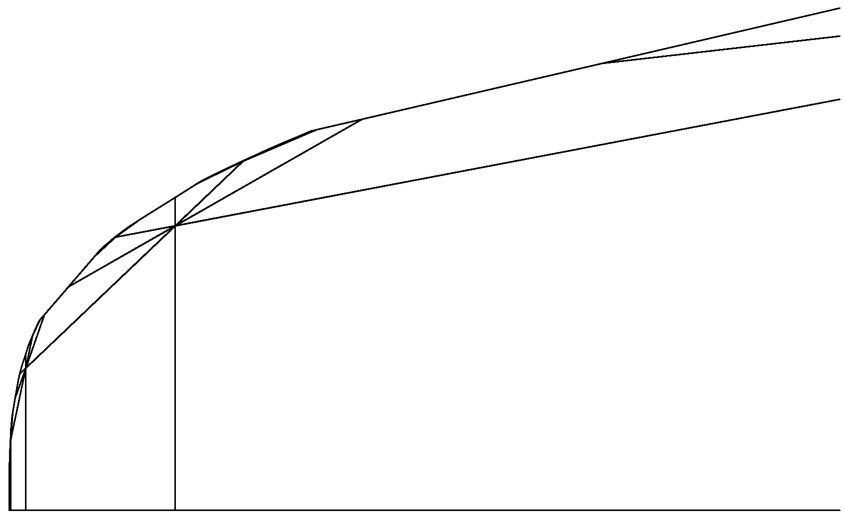

FIGURE 23. An annulus with a corner-reflector of order 2, the boundary invariant $(0.5, 5)$, depth 5, type (A2), and symbol $A(2;)$

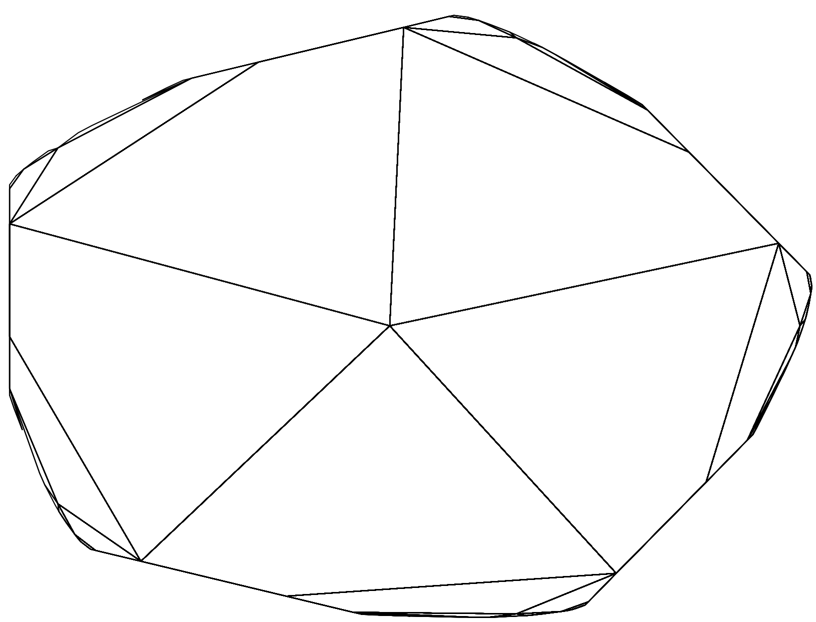

FIGURE 24. A disk with one segment of mirror points, and boundary 1-orbifold and a cone-point of order 5, boundary invariant 0.3, depth 4, type (A3), and symbol $D^2(2, 2; 5)$.

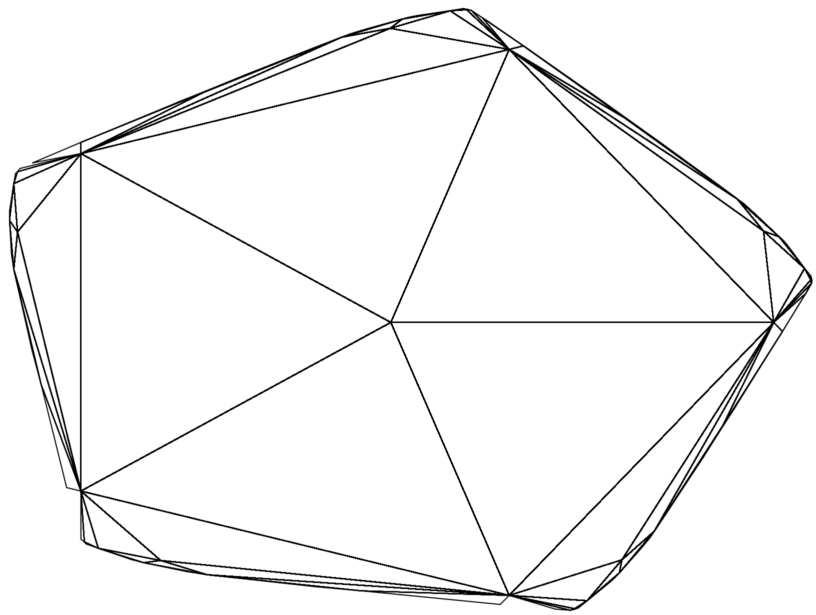

FIGURE 25. A disk with one corner-reflector of order 3 and one cone-point of order 5, depth 6, type (A4), and symbol $D^2(3;5)$.

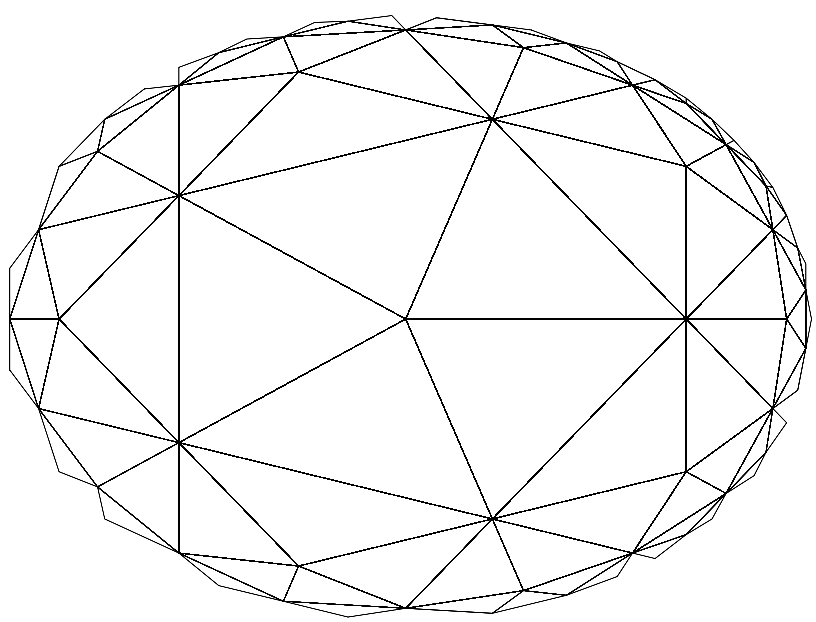

FIGURE 26. A disk with one corner-reflector of order 2 and one cone-point of order 5, depth 6, type (A4), and symbol $D^2(2;5)$.

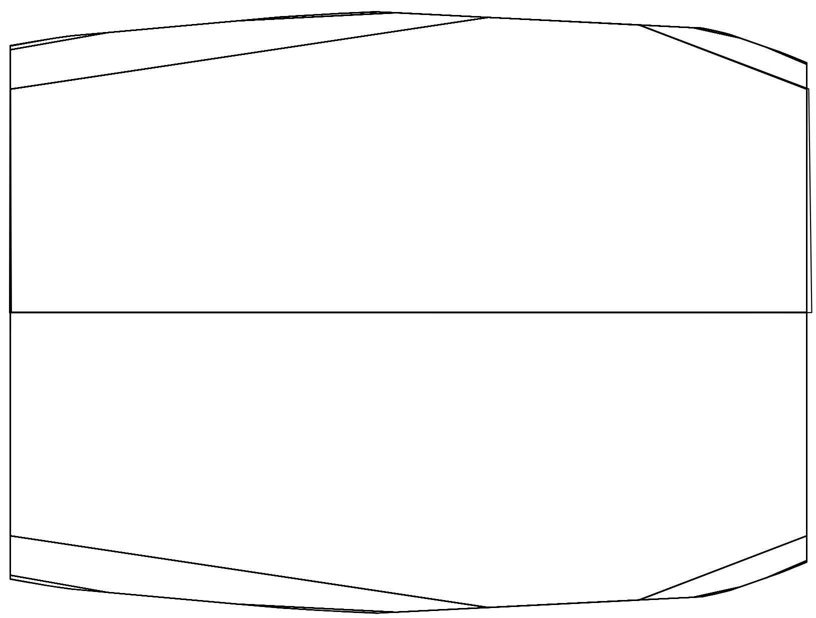

Figure 27. A hexagon, boundary invariants $0.2, 0.4, 0.3$, depth 5, type (D1), and symbol $D^2(2,2,2,2,2,2;)$.

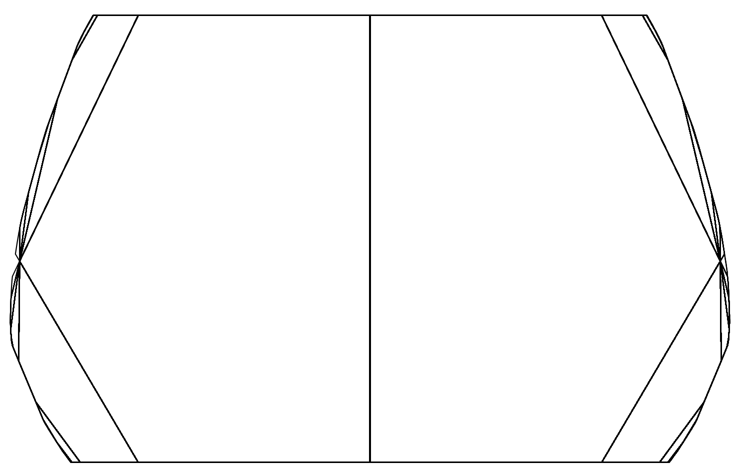

Figure 28. A pentagon with a corner-reflector of order 5, boundary invariants $0.4, 0.3$, depth 5, type (D2), and symbol $D^2(2,2,2,2,5;)$.

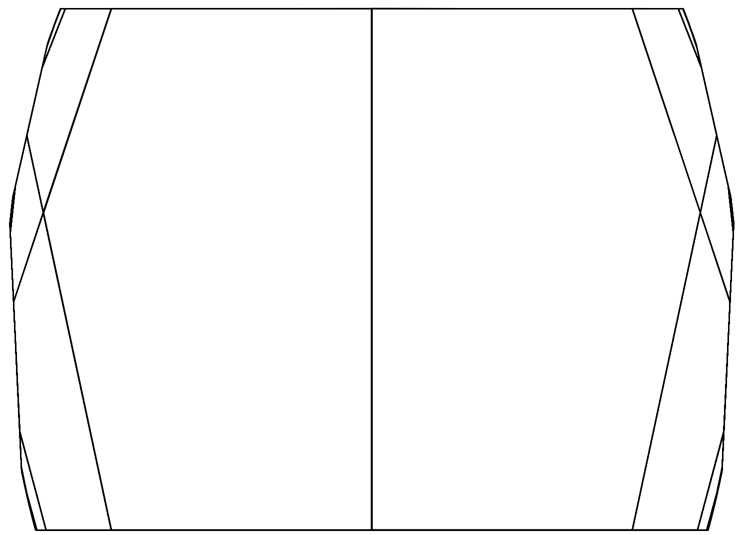

FIGURE 29. A pentagon with a corner-reflector order 2, boundary invariants $0.4, 0.3$, depth 5, type (D2), and symbol $D^2(2,2,2,2,2;)$.

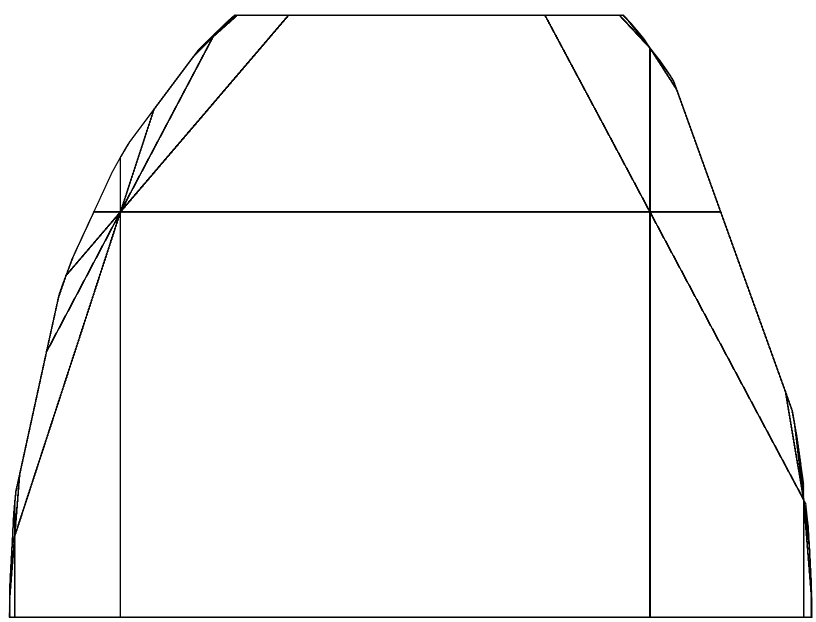

FIGURE 30. A square with corner reflectors of order 3 and 5, the boundary invariant 0.15, depth 5, type (D3), and symbol $D^2(2,2,3,5;)$.

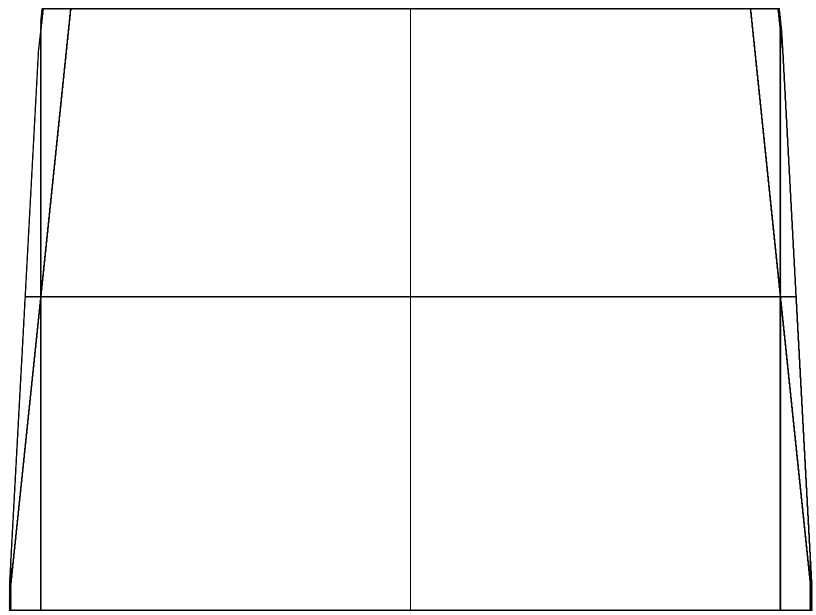

FIGURE 31. A square with corner reflectors of order 2 and 3, the boundary invariant 0.15, depth 4, type (D3), and symbol $D^2(2,2,2,3;)$.

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Department of Mathematics, KAIST, Taejeon 305-701, South Korea
E-mail address: schoi@math.kaist.ac.kr

Department of Mathematics, University of Maryland, College Park, MD, 20742 USA
E-mail address: wmg@math.umd.edu