Matrix Exponentials

If $X$ is a square matrix, then the infinite series

$$\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$$

always converges. The path

$$\mathbb{R} \xrightarrow{\Phi} \text{Mat}(n, \mathbb{R})$$

$$t \mapsto \exp(tX)$$

satisfies three basic properties:

$$\Phi(0) = 1$$
$$\Phi(-t) = \Phi(t)^{-1}$$
$$\Phi(s + t) = \Phi(s)\Phi(t)$$

and is called a one-parameter group. These generalize the basic law of exponents $e^{x+y} = e^x e^y$ and closely relates to the fact that $e^x$ solves the differential equation $f'(t) = f(t)$ with initial condition $f(0) = 1$.

We observed that if you translate by a vector $v$ and then translate by another vector $w$, then the composition is translation by the vector sum $v + w$. In a similar vein, the composition of two rotations (through the origin) of angles $\theta$ and $\phi$ is the rotation through the origin by angle $\theta + \phi$. Using complex numbers, this follows from the suggestive formula

$$e^{i\theta} = \cos(\theta)1 + \sin(\theta)i$$

representing rotation $\rho_\theta$ through angle $\theta$.

The proofs of these basic properties are outlined in the bonus problems at the end. (You can use some of those properties in the next problems. In general, compute a few powers $X^n$ and look for patterns.)
Prove or disprove:

(1) \[\exp(t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\]

(2) \[\exp(t \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}) = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\]

(3) \[\exp(t \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) = e^t \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\]

(4) \[\exp(t \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}) = e^t \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\]

(5) \[\exp(t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}\]

\[= \cos(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin(t) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\]

(6) \[\exp(t \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}) = \cos(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\]

(7) \[\exp(t \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}\]

(8) \[\exp(t \begin{bmatrix} \log(5) & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 7 \end{bmatrix}) = \begin{bmatrix} 5^t & 0 & 0 \\ 0 & e^{-6t} & 0 \\ 0 & 0 & e^{7t} \end{bmatrix}\]

(9) \[\exp(t \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}) = \begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}\]

(10) \[\exp(t \begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}) = \begin{bmatrix} \cos(3t) & -\sin(3t) & 0 \\ \sin(3t) & \cos(3t) & 0 \\ 0 & 0 & e^{7t} \end{bmatrix}\]

Hint: For (??), use the fact that

\[\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = U \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} U^{-1}\]

where

\[U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} .\]
Bonus Problems

Here are the steps for proving that if $X, Y$ are square matrices with $XY = YX$, then

$$ \exp(X) \exp(Y) = \exp(X + Y) $$

In particular,

$$ \Phi(t) := \exp(tX) $$

satisfies the three basic properties of one-parameter groups.

1. Define the binomial coefficient

$$ \binom{n}{p} := \begin{cases} \frac{n!}{p!(n-p)!} & \text{if } 0 \leq p \leq n \\ 0 & \text{otherwise} \end{cases} $$

and show that

$$ \binom{n+1}{p} := \binom{n}{p} + \binom{n}{p-1} $$

2. From this deduce Newton's Binomial Theorem:

$$ (X + Y)^n = \sum_{p=0}^{n} \binom{n}{p} X^p Y^{n-p} $$

assuming that $XY = YX$.

3. From this deduce that

$$ \exp(X) \exp(Y) = \exp(X + Y) $$

if $XY = YX$.

4. Find a counterexample to this when $XY \neq YX$.

5. Suppose that $\Phi(t)$ is a path of matrices satisfying $\phi(0)$ is the identity and

$$ \phi(s)\phi(t) = \phi(s + t) $$

for $s, t \in \mathbb{R}$. Let $X := \phi'(0)$ be the derivative of $\phi(t)$ at $t = 0$. Then

$$ \phi'(t) = X\phi(t) $$

6. From this deduce that $\phi(t) = \exp(tX)$. 