More matrix exponentials

(1) For matrices $E$ below, solve the matrix equation $e^L = E$. That is, find all matrices $L$ such that $e^L = E$. (Hint: there may be no solutions, unique solutions, or infinitely many solutions. All the matrices are required to be real.)

(a) $E = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, $E = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$, $E = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}$

(b) $E = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $E = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$

Quaternions

(2) Find all quaternion solutions $x \in \mathbb{H}$ of $x^2 = 2$.

(3) Prove or disprove: If $x \in \mathbb{H}$ is a quaternion, then $\exp(tx)$ is real for all $t \in \mathbb{R}$, then $x$ is real.

(4) Prove or disprove: If $x \in \mathbb{H}$ is a quaternion, then $\exp(tx)$ is real for some nonzero $t \in \mathbb{R}$, then $x$ is real.

(5) Prove or disprove: Let $\ell$ be the line defined by $x + z = 1$ and $y = 0$. We rotate points $(0, y, 0)$ on the $y$-axis around $\ell$, using quaternions. Identify $\mathbb{R}^3$ with the vector space $\mathbb{H}_0$ of purely imaginary quaternions, so that the $y$-axis consists of all scalar multiples $y \hat{j}$ of $j$. Let $\theta \in \mathbb{R}$ define an angle and $\rho_{\ell}^\theta$ denote rotation through angle $\theta$ in $\ell$. Then rotating the $y$-axis around $\ell$ gives the line comprising (for $y \in \mathbb{R}$):

$$\rho_{\ell}^\theta(y\hat{j}) = \hat{i} + e^{\theta(\hat{i} - \hat{k})/(2\sqrt{2})} (y\hat{j} - \hat{i}) e^{-\theta(\hat{i} - \hat{k})/(2\sqrt{2})}$$
Generalizing cross-products to $\mathbb{R}^4$

If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are nonzero vectors representing points 

$$a := [a], \ b := [b] \in \mathbb{P}^2,$$

then the covector $(\mathbf{a} \times \mathbf{b})^\dagger$ is nonzero if and only if $a \neq b$. In that case it represents the homogeneous coordinates of the line 

$$\overleftrightarrow{ab} \subset \mathbb{P}^2$$

containing $a$ and $b$. (Here $A^\dagger$ denotes the transpose of the matrix $A$.)

We generalize this to planes in $\mathbb{P}^3$ passing through three points $a, b, c \in \mathbb{P}^3$. Let $V$ denote $\mathbb{R}^4$ and let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ be nonzero vectors representing the homogeneous coordinates of the points $a, b, c$ respectively. Define an alternating trilinear map 

$$V \times V \times V \xrightarrow{\text{Orth}} V^*$$

as follows. If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in V$, denote the determinant of the $4 \times 4$ matrix whose columns are $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ by $\text{Det}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$.

For fixed vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$, the map

$$\mathbf{d} \mapsto \text{Det}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$$

is linear, and hence defines a covector, denoted $\text{Orth}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in V^*$.

(6) Let

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

be vectors in $\mathbb{R}^4$. Express $\text{Orth}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ in terms of $3 \times 3$ determinants, that is,

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} := x_{13} \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} - x_{23} \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix} + x_{33} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

$$= x_{11}x_{22}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{11}x_{23}x_{32} - x_{13}x_{22}x_{31} - x_{12}x_{21}x_{33}$$

(7) Prove or disprove: $a, b, c$ are collinear if and only if

$$\text{Orth}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0.$$

(8) Prove or disprove: If $\text{Orth}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq 0$, then it represents the homogeneous coordinates of the plane in $\mathbb{P}^3$ containing $a, b, c$. 
Rotations and $SO(n)$

The special orthogonal group, denoted $SO(n)$ consists of all orthogonal $n \times n$ matrices of determinant 1. Equivalently, $SO(n)$ consists of orientation-preserving linear isometries of $\mathbb{R}^n$ (Euclidean $n$-space).

Every element of $SO(2)$ is a rotation about the origin:

$$\exp(\theta J) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$  

Similarly, every element of $SO(3)$ is rotation about a line (its axis) $A \subset \mathbb{R}^3$. In terms of the orthogonal direct-sum decomposition $\mathbb{R}^3 = A^\perp \oplus A$ this rotation is just the direct sum of $\exp(\theta J)$ on $A^\perp$ (with respect to an orthonormal basis) and the identity 1 on $A$. However, higher dimensions are more complicated:

(8) The matrix

$$M := \begin{bmatrix} 4/5 & 3/5 & 0 & 0 \\ -3/5 & 4/5 & 0 & 0 \\ 0 & 0 & \cos(2) & -\sin(2) \\ 0 & 0 & \sin(2) & \cos(2) \end{bmatrix}$$

is orthogonal and lies in $SO(4)$

(9) $M$ does not fix any point in $\mathbb{P}^3$. (Hint: projective fixed points correspond to eigenvectors.)

(10) Find a matrix $L$ such that $e^L = M$.

(11) Find two projective lines in $\mathbb{P}^3$ which are invariant under this projective transformation. Do those lines intersect?

Homogeneous Coordinates for Lines in $\mathbb{P}^3$

Points in $\mathbb{P}^3$ correspond to (projective equivalence classes) of nonzero vectors in $\mathbb{R}^4$. That is, the point in $\mathbb{P}^3$ with homogeneous coordinates $[X : Y : Z : W]$ is the line $[v]$ spanned by the nonzero vector

$$v := \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} \in \mathbb{R}^4.$$  

Similarly, planes in $\mathbb{P}^3$ correspond to (projective equivalence classes) of covectors

$$\phi := [a \ b \ c \ d] \in (\mathbb{R}^4)^*.$$
where $[\phi] = [a : b : c : d]$ is the hyperplane defined in homogeneous coordinates by $\phi(v) = 0$, that is,

$$(aX + bY + cZ + dW = 0).$$

That is, the point $[X : Y : Z : W]$ lies on the plane $[a : b : c : d]$ if and only if $\phi(v)$ is satisfied.

Thus lines and planes in $\mathbb{P}^3$ are defined in homogeneous coordinates by vectors in the vector space $V := \mathbb{R}^4$ and covectors in its dual vector space $V^* = (\mathbb{R}^4)^*$. Moreover, the orthogonal complement $v^\perp$ of the line $\mathbb{R}v \in \mathbb{R}^4$ is the hyperplane in $\mathbb{R}^4$ defined by the covector $v^\dagger$, which is the transpose of $v$.

How can you describe lines in $\mathbb{P}^3$ in a similar way by homogeneous coordinates?

**Exterior Outer Products**

Recall that $\mathfrak{so}(n)$ denotes the set of $n \times n$ skew-symmetric matrices, that is $X \in \text{Mat}_n$ such that $X + X^\dagger = 0$. The exterior outer product is the alternating bilinear map:

$$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathfrak{so}(n)$$

$$(v, w) \longmapsto v \wedge w := wv^\dagger - v^\dagger w.$$ 

The following facts are easy to verify:

- $(u \wedge v) : w \longmapsto (v \cdot w)u - (u \cdot w)v$
- If $n = 3$, then $(u \wedge v)(w) = (u \times v) \times w$.
- $w$ and $v$ are linearly dependent if and only if $w \wedge v = 0$.
- If $w$ and $v$ are linearly independent, then the projective equivalence class $[w \wedge v] \in \mathbb{P}(\mathfrak{so}(n))$ depends only the plane $\mathbb{R}\langle w, v \rangle$ spanned by $w, v$.
- The orthogonal complement of the plane $\mathbb{R}\langle w, v \rangle \subset V$ lies in the kernel $\text{Ker}(w \wedge v)$:

$$\mathbb{R}\langle w, v \rangle^\perp \subset \text{Ker}(w \wedge v).$$

Therefore every 2-dimensional linear subspace $L$ (plane through the origin) of $\mathbb{R}^n$ determines an element of the projective space $\mathbb{P}(\mathfrak{so}(n))$. The corresponding homogeneous coordinates are the Plücker coordinates of the plane, or the corresponding projective line $\mathbb{P}(L) \subset \mathbb{P}(\mathbb{R}^n)$.
Plücker coordinates in $\mathbb{P}^3$

Let $V = \mathbb{R}^4$ and $\Lambda = \mathfrak{so}(4)$ the 6-dimensional vector space of $4 \times 4$ skew-symmetric matrices. Then lines in $\mathbb{P}^3 = \mathbb{P}(V)$ correspond to 2-dimensional linear subspaces of $V$, which in turn correspond to projective equivalence classes of certain nonzero elements $v \wedge w \in \Lambda$. Which elements of $\Lambda$ correspond to lines in $\mathbb{P}^3$?

Since $\dim(V) = 4$, the plane $\mathbb{R} \langle v, w \rangle \neq V$ so there exists a nonzero vector $n$ normal to this plane. By the above, $n$ lies in the kernel of the skew-symmetric matrix $v \wedge w$. Thus lines in $\mathbb{P}^3$ determine nonzero singular matrices in $\Lambda$.

By the spectral theorem for real skew-symmetric matrices, the eigenvalues are purely imaginary and occur in complex conjugate pairs. For example, when $n = 3$, every element in $\mathfrak{so}(3)$ must have a zero eigenvalue (if zero occurs with higher multiplicity the matrix itself must be zero). This implies that every element of $\text{SO}(3)$ is a rotation, for example.

When $n = 4$, then if 0 is not an eigenvalue, then the set of eigenvalues must be of the form

$$\{r_1i, -r_1i, r_2i, -r_2i\},$$

where $r_1, r_2 \in \mathbb{R}$ are nonzero. In particular such a matrix has determinant $r_1^2r_2^2 > 0$. Since $\text{Det}(v \wedge w) = 0$, but $v \wedge w \neq 0$, the multiplicity of 0 as an eigenvalue is exactly two, so the matrix $v \wedge w$ has rank 2.

When $n$ is even, skew-symmetric matrices in $\mathfrak{so}(n)$ have the following curious property. In general the determinant of an $n \times n$ is a degree $n$ polynomial in its entries. When $n$ is even, there is a degree $n/2$ polynomial $\mathcal{P}$ on $\mathfrak{so}(n)$ (called the Pfaffian) such that if $M \in \mathfrak{so}(n)$, then

$$\text{Det}(M) = \mathcal{P}(M)^2$$

for $M \in \mathfrak{so}(n)$. That is, in even dimensions, the determinant of a skew-symmetric matrix is a perfect square. For example, when $n = 2$, the general skew-symmetric matrix is

$$M = \begin{bmatrix} 0 & -y \\ y & 0 \end{bmatrix}$$

which has determinant $y^2$. Thus $\mathcal{P}(M) = y$, for example.

When $n = 4$, the Pfaffian is a quadratic polynomial. The general element of $\mathfrak{so}(4)$ is:

$$M := \begin{bmatrix} 0 & m_{12} & m_{13} & m_{14} \\ -m_{12} & 0 & m_{23} & m_{24} \\ -m_{13} & -m_{23} & 0 & m_{34} \\ -m_{14} & -m_{24} & -m_{34} & 0 \end{bmatrix}.$$
which has determinant
\[
\text{Det}(M) = \left( m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34} \right)^2
\]
so the Pfaffian is (up to a choice of $-1$):
\[
\mathcal{P}(M) = m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34}.
\]
The vector space $\Lambda$ has dimension 6, with coordinates
\[
m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}.
\]
Thus projective equivalence classes of nonzero $4 \times 4$ skew-symmetric matrices is the projective space
\[
\mathbb{P}(\Lambda) \cong \mathbb{P}^5
\]
with homogeneous coordinates
\[
[m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34}].
\]
The nonzero singular matrices (namely, those of rank two), are those for which $\mathcal{P}(M) = 0$, which is just the homogeneous quadratic polynomial condition:
\[
m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34} = 0
\]
This defines a quadric hypersurface $\mathcal{Q}$ in $\mathbb{P}^5$. Since it is defined by one equation in a 5-dimensional space, this quadric has dimension 4.

Intuitively, we would expect that the space of lines in $\mathbb{P}^3$ has dimension 4. A generic line $\ell \subset \mathbb{P}^3$ is not ideal and does not pass through the origin. In that case there is a point
\[
p(\ell) \in \mathbb{R}^3 \setminus \{0\}
\]
closest to the origin $0 \in \mathbb{R}^3$. These points form a 3-dimensional space $\mathbb{R}^3 \setminus \{0\}$.

Any point $p \in \mathbb{R}^3 \setminus \{0\}$ is the closest point $p(\ell)$ for some $\ell$. Namely, look at the plane $W(p)$ containing $p$ and normal to the vector from 0 to $p$. Any line $\ell$ on $W(p)$ passing through $p$ satisfies $p(\ell) = p$. The set of all lines $\ell$ with $p(\ell) = p$ forms a $\mathbb{P}^1$, which is one-dimensional. Thus lines in $\mathbb{P}^3$ are parametrized by a $4 = 3 + 1$-dimensional space.

This space is the quadric $\mathcal{Q}$ defined above.

Just as quadric surfaces in $\mathbb{P}^3$ can be parametrized as tori $S^1 \times S^1$, the 4-dimensional quadric hypersurface in $\mathbb{P}^5$ can be parametrized by $S^2 \times S^2$. Namely make the elementary linear substitution
\[
X := (m_{14} + m_{23})/2, \quad A := (m_{14} - m_{23})/2,
\]
\[
Y := (m_{13} - m_{24})/2, \quad B := (m_{13} + m_{24})/2,
\]
\[
Z := (m_{12} + m_{34})/2, \quad C := (m_{12} - m_{34})/2.
\]
so that
\[
P(M) = m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34} \\
= X^2 - A^2 + Y^2 - B^2 + Z^2 - C^2.
\]

Thus \( Q \) is the quadric in \( \mathbb{P}^5 \) consisting of points with homogeneous coordinates \([X : Y : Z : A : B : C]\) satisfying
\[
\]

Since the coordinates are real at least one is nonzero, this common sum-of-squares is positive. By rescaling we may suppose that that \( X^2 + Y^2 + Z^2 = 1 \) and \( A^2 + B^2 + C^2 = 1 \). Each of these equations describes a unit sphere in a 3-dimensional Euclidean space. Furthermore the coordinates \((A, B, C)\) and \((X, Y, Z)\) are independent of one another (we are looking at a \textit{direct-sum decomposition} of \( \mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3 \)), so that the quadric \( Q \) looks like \( S^2 \times S^2 \).

**Orthogonal Complement and Involution**

Since \( P \) is a homogeneous quadratic function on the vector space \( \Lambda \), it arises from a symmetric bilinear form \( P \) on \( \Lambda \) by the usual correspondences:

\[
P(X) = P(X, X),
\]
\[
P(X, Y) := \frac{1}{2}(P(X + Y) - P(X) - P(Y))
\]

Explicitly,
\[
P(M, N) = \frac{1}{2}(m_{14}n_{23} + m_{23}n_{14} - m_{13}n_{24} - m_{24}n_{13} + m_{12}n_{34} + m_{34}n_{12}).
\]

The usual inner product (dot product) on \( \mathfrak{so}(4) \) is given by
\[
M \cdot N = -\frac{1}{2} \text{tr}(MN)
\]
\[
= m_{12}n_{12} + m_{13}n_{13} + m_{14}n_{14} + m_{23}n_{23} + m_{24}n_{24} + m_{34}n_{34}
\]

Since the bilinear forms \( P \) and the above dot product define linear isomorphisms \( \Lambda \overset{\cong}{\rightarrow} \mathcal{V}^* \), they are related by a linear isomorphism \( \Lambda \overset{\cong}{\rightarrow} \Lambda \) defined by:

\[
\begin{pmatrix}
0 & m_{34} & -m_{24} & m_{23} \\
-m_{34} & 0 & m_{14} & -m_{13} \\
m_{24} & -m_{14} & 0 & m_{12} \\
-m_{23} & m_{13} & -m_{12} & 0
\end{pmatrix}
\]
that is,
\[
\mathcal{I}(m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) := (m_{34}, -m_{24}, m_{23}, m_{14}, -m_{13}, m_{12})
\]

Clearly \(\mathcal{I} \circ \mathcal{I} = \text{I}\); such a transformation is called an \textit{involution}.

Geometrically, if \(M \in \mathcal{Q}\) corresponds to a 2-dimensional linear subspace \(L \subset \mathcal{V}\), then \(\mathcal{I}(M)\) corresponds to its orthogonal complement \(L^\perp \subset \mathcal{V}\).

If \(p \in \mathbb{P}^3\) is a point corresponding to a 1-dimensional linear subspace \(L \subset \mathcal{V}\), then its dual plane \(p^* \subset \mathbb{P}^3\) corresponds to the orthogonal complement \(L^\perp\). (The homogeneous coordinates of \(p^*\) form the \textit{transpose} of the vector formed by the homogeneous coordinates of \(p\).) Then \(\mathcal{I}\) maps lines through \(p\) to the lines contained in the plane \(p^*\).

Here is a basic example. Take \(p\) to be the origin \((0,0,0)\) in the standard affine patch; then \(p^*\) is the ideal plane. The line through 0 in the direction \((a,b,c)\) has Plücker coordinates

\[
M := \begin{bmatrix}
0 & 0 & 0 & a \\
0 & 0 & 0 & b \\
0 & 0 & 0 & c \\
-a & -b & -c & 0
\end{bmatrix}.
\]

Its dual is the ideal line, which in the ideal plane \(\mathbb{P}^2_{\infty}\) has homogeneous coordinates \([a : b : c]\) (that is, the line defined in homogeneous coordinates \(aX + bY + cZ = 0\). In \(\mathbb{P}^3\) this line has Plücker coordinates:

\[
\mathcal{I}(M) := \begin{bmatrix}
0 & c & -b & 0 \\
-c & 0 & a & 0 \\
b & -a & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

**Relation to Orth**

We can relate this to the alternating trilinear function \textbf{Orth} (the four-dimensional cross product). It can be defined in terms of the involution \(\mathcal{I}\) and exterior outer product \(\wedge\):

\[
\textbf{Orth}(u, v, w) = \mathcal{I}((w \wedge u)(v))
\]

Three points \([u], [v], [w] \in \mathbb{P}^3\) (where \(u, v, w \in \mathcal{V}\) are nonzero vectors are collinear if and only if \(\text{Orth}(u, v, w) = 0\). Otherwise they span a plane in \(\mathbb{P}^3\) represented by \(\text{[Orth}(u, v, w)]\).