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A Generalization of Bieberbach's Theorem

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Introduction

In 1912 Bieberbach proved that every compact flat Riemannian manifold M is finitely covered by a flat torus. More precisely, M has the form $(\Gamma\backslash G)/H$ where G is a group of translations of Euclidean space, $\Gamma\subset G$ is a discrete subgroup, and H is a finite group of isometries of the space of right cosets $\Gamma\backslash G$. For a proof see e.g. Wolf [18].

The condition that M has a flat Riemannian metric can be separated into two conditions. First, M has an affine structure – a distinguished covering by coordinate charts, whose coordinate changes are affine. Second, M has a Riemannian metric whose coefficients in the affine charts are constants.

In this paper we relax the second condition. A *polynomial Riemannian metric* on the affine manifold M is a Riemannian metric whose local expression in affine coordinates has the form $\sum g_{ij}(x)\,dx^i\,dx^j$ where the g_{ij} are polynomial functions on Euclidean space E. By letting the g_{ij} be rational functions on E, we arrive at the more general notion of a *rational Riemannian metric*. (It is not assumed that these expressions define Riemannian metrics on all of E.)

The object of this paper is to determine which affine manifolds have polynomial Riemannian metrics and to give examples of affine manifold having rational Riemannian metrics.

Theorem 1. *Let M be a compact affine manifold. Then M admits a polynomial Riemannian metric if and only if M is finitely covered by a complete affine nilmanifold.*

An affine manifold is *complete* provided its universal covering is E with its canonical affine structure. An *affine nilmanifold* is an affine manifold of the form $\Gamma\backslash G$ where G is a simply-connected nilpotent Lie group with a left-invariant affine structure and $\Gamma\subset G$ is a discrete subgroup.

It follows from Theorem 1 that $M=(\Gamma\backslash G)/H$ where H is a finite group acting freely on the complete nilmanifold $\Gamma\backslash G$ by affine automorphisms and Γ is a discrete uniform subgroup of G. The proof shows that the polynomial

Riemannian metric on M comes from a left-invariant metric on G. It is therefore locally homogeneous. (For rational Riemannian metrics on a compact affine manifold we only conjecture such local homogeneity.)

Here is a simple example; it can easily be generalized to higher dimensions. A simply transitive affine action of the vector group $G \simeq \mathbf{R}^2$ on the vector space $E = \mathbf{R}^2$ is defined by

$$f\colon G \to \mathrm{Aff}(E),$$

$$f(s,t)\colon (x,y) \mapsto \left(x+s,\, y+sx+\left(\frac{s^2}{2}+t\right)\right).$$

This action is generated by the flows of the commuting vector fields $\partial/\partial y$ and $\partial/\partial x + x\,\partial/\partial y$. The resulting left-invariant complete affine structure on G defines a complete affine structure on the torus $\Gamma\backslash G$ (where Γ is any lattice) which is not isomorphic to any flat Riemannian affine structure (Kuiper [12], see also Kobayashi and Nomizu [10], p. 211). For different choices of Γ one finds different affine structures on the torus. The space of 1-forms on E that are invariant under G has as a basis dx and $dy - x\,dx$. The symmetric 2-form

$$(dx)^2 + (dy - x\,dx)^2 = (x^2+1)\,dx^2 - 2x\,dx\,dy + dy^2$$

is therefore a polynomial Riemannian metric on E invariant under the action f of G. The metric evidently induces a polynomial Riemannian metric on the affine two-torus $\Gamma\backslash G$.

Suppose that one generator of Γ is a translation $(x, y) \mapsto (x, y+2)$. Then the glide-reflection $(x, y) \mapsto (-x, y+1)$ induces an affine isometry of this torus, of period two. The orbit space is a Klein bottle with a complete affine structure admitting a polynomial Riemannian metric.

Proofs of Theorems

We briefly review the basic concepts, referring the reader to Fried, Goldman, and Hirsch [5] for details.

Let $E = \mathbf{R}^n$ be a fixed real vector space. The group of affine automorphisms of E will be denoted $\mathrm{Aff}(E)$; the group of linear automorphisms will be denoted $\mathrm{GL}(E)$. Let $L\colon \mathrm{Aff}(E) \to \mathrm{GL}(E)$ denote the natural homomorphism.

M will always denote an affine manifold of dimension $n \geqq 1$, connected and without boundary. Denote the universal covering of M by $\tilde{M}$. There is an affine immersion $D\colon \tilde{M} \to E$ and a homomorphism $h\colon \pi \to \mathrm{Aff}(E)$ for which D is equivariant, where $\pi = \pi_1(M)$ is the group of deck transformations of $\tilde{M}$. We call D the *developing map* and h the *affine holonomy homomorphism* of the affine manifold M. Together they determine the affine structure of M; conversely, the affine structure determines D up to composition with some $p \in \mathrm{Aff}(E)$, and it determines h up to conjugation by p. When D is bijective M is called a *complete* affine manifold.

The subgroup $\Gamma = h(\pi) \subset \mathrm{Aff}(E)$ is called the *affine holonomy group* of M, and the subgroup $L(\Gamma) \subset \mathrm{GL}(E)$ is the *linear holonomy group*. It is not hard to show that an affine manifold M has a polynomial Riemannian metric g_M if and only if there is a polynomial 2-form g on E invariant under Γ such that $g_{\tilde{M}} = D^* g$ is a Riemannian metric on $\tilde{M}$. This latter condition means that g is positive definite on the developing image $D(\tilde{M})$.

Let G be a simply-connected Lie group. An affine structure on G is *left-invariant* if every left-multiplication is an affine map with respect to affine coordinates on G. When a left-invariant affine structure is complete, there is a simply transitive affine action of G on E. The developing map can be taken to be the map $p \mapsto p(0)$, the evaluation of this action at the origin. Conversely, every simply transitive affine action of G determines a left-invariant affine structure on G. It is known (see Milnor [13]) that if G admits such a structure, then G is solvable. The converse is conjectured, but is not known even for nilpotent Lie groups G.

We now state the main step in the proof of Theorem 1.

Proposition 2. *Let M be an affine manifold admitting a complete polynomial Riemannian metric. Then:*

(a) *the affine structure on M is complete*

(b) *the affine holonomy group has a finitely generated nilpotent subgroup of finite index.*

Proof of Part (a). It suffices to prove that $\tilde{M}$ is complete. Therefore we assume that M is simply connected and that there is an affine immersion $D: M \to E$. We use D to define local affine coordinates on M. Let g denote the polynomial 2-form on E such that the polynomial Riemannian metric on M is $g_M = D^* g$.

The condition that M be complete as an affine manifold is well-known [3] to be equivalent to the condition of *geodesic completeness*: every affine map from an interval into M extends to an affine map of all of $\mathbf{R}$. It is easy to see that this condition holds provided that every affine map $s: [0, b) \to M$ extends to a continuous map on $[0, b]$. Since the polynomial Riemannian metric on M is complete, it suffices to prove that the path s has finite g_M-length, or equivalently, that the path $D \circ s$ has finite length with respect to g. Now $D \circ s$ is the restriction of an affine map $s_0: \mathbf{R} \to E$. Then the g-length of the part of s_0 corresponding to s is $\int_0^b g(s_0'(t), s_0'(t))^{1/2}\, dt$, which is finite. This completes the proof.

In the proof above, we used the polynomial nature of the metric only to obtain a continuous 2-form on E. Using the well-known fact that a local isometry of a complete Riemannian manifold is a covering, the same proof gives the following result:

Proposition 3. *Suppose the affine manifold M has a complete Riemannian metric whose lift to $\tilde{M}$ is $D^* g$ for some continuous 2-form g on E. Then the affine structure is complete. If g is defined only on some open subset Ω of E, then the developing map D is a covering of $\tilde{M}$ onto a connected component of Ω.*

Proof of Part (b) *of Proposition 2.* From part (a) we know that M is complete, so we take $\tilde{M} = E$. Therefore the polynomial Riemannian metric g_M is covered by a polynomial Riemannian metric g on E. The affine holonomy group Γ is a group of isometries of this metric; since it also acts by deck transformations on E, it must be a discrete subgroup of $\mathrm{Aff}(E)$.

We consider $\mathrm{Aff}(E)$ as embedded in $\mathrm{GL}(E \times R)$ in the usual way: an affine map $x \mapsto Ax+b$ becomes the linear map $(x,t) \mapsto (Ax+tb,t)$. Thus $\mathrm{Aff}(E)$ is a subgroup of $\mathrm{GL}(E \times R)$, namely the subgroup which preserves projection onto $\mathbf{R}$. In particular, it is *algebraic*.

Thus we may take the *algebraic hull* $A(\Gamma)$ (i.e. Zariski closure of Γ) as a subgroup of $\mathrm{Aff}(E)$.

Evidently $A(\Gamma)$ also preserves the polynomial Riemannian metric g; the condition that an affine transformation preserve a polynomial tensor field is a polynomial condition on the affine transformation. Thus $A(\Gamma)$ is a Lie group of isometries of the metric g. It follows that *all the isotropy subgroups of $A(\Gamma)$ are compact.*

Let $K \subset A(\Gamma)$ be a maximal isotropy subgroup. Being compact, K is reductive. Since every reductive group of affine transformations has a fixed point (see Milnor [13]), it follows that K is a maximal reductive subgroup.

Let $U \subset A(\Gamma)$ be the unipotent radical. Then by a result of C. Chevalley $A(\Gamma)$ is the semidirect product of U and any maximal reductive subgroup (Hochschild [8], theorem 14.2). It follows that $A(\Gamma)$ is the semidirect product $K \ltimes U$. Observe that U is connected (the exponential map is surjective) and Γ is a discrete subgroup of $A(\Gamma)$. Therefore part (b) of Proposition 2 follows from:

Proposition 4. *Let G be a Lie group which is the semidirect product of a compact subgroup K and a connected nilpotent normal subgroup U. If $\Gamma \subset G$ is a discrete subgroup, then Γ has a finitely generated nilpotent subgroup of finite index.*

Proof. Let p: $G \to K$ be the canonical projection with kernel U. Consider any neighborhood W of the identity $e \in K$ and define $\Gamma_0 \subset \Gamma$ to be the subgroup generated by $\Gamma \cap p^{-1} W$.

We assert that Γ_0 has finite index in Γ. We may assume that $p(\Gamma)$ is dense in K; otherwise replace K by the closure of $p(\Gamma)$. Since K is compact, it is covered by finitely many left-translates of W by elements of K. Since $p(\Gamma)$ is dense, there exist $y_1, y_2, \ldots, y_r$ in Γ such that $K = \bigcup_{i=1}^{r} p(y_i) W$. Equivalently $G = \bigcup_{i=1}^{r} y_i p^{-1} W$. It follows that the cosets $y_i \Gamma_0$ cover Γ, proving that Γ_0 has finite index in Γ.

By a theorem of Zassenhaus (see Raghunathan [14], proposition 8.11), there is a neighborhood W of the identity $p(e)$ in K with the following properties:

(1) W is closed under the commutator operation $[x,y] = x^{-1} y^{-1} xy$;

(2) for every sequence $x_1, x_2, \ldots$ in $p^{-1} W$, the sequence x_1, $[x_1, x_2]$, $[[x_1, x_2], x_3]$, $[[[x_1, x_2], x_3], x_4], \ldots$ converges to the identity e in G;

(3) the convergence is uniform for all such sequences $x_1, x_2, \ldots$

Fix such neighborhood W; define Γ_0 as above.

Let $F \subset \Gamma_0$ be a finite set. Since Γ is discrete it follows that for some integer n depending on F, we have $[[\ldots[x_1, x_2], \ldots], x_n] = 1$ for all $x_1, x_2, \ldots, x_n \in F$.

This implies that *the subgroup generated by F is n-step nilpotent.* To see this, we prove by induction on n that a group H is n-step nilpotent provided it has a finite generating set F such that every n-fold commutator as above of elements of F is the identity. Since every $(n-1)$-fold commutator centralizes the generating set it must be in the center Z of H. The induction hypothesis now implies that H/Z is $(n-1)$-step nilpotent, whence H is n-step nilpotent.

We have proved that every finitely generated subgroup of Γ_0 is nilpotent (and thus solvable). Another theorem of Zassenhaus (see Raghunathan [14], corollary 8.4) says that in such a case Γ_0 itself must be solvable. Thus Γ_0 is a discrete solvable subgroup of a Lie group having finitely many components. It follows from a result of Auslander and Baumslag (see Milnor [13]) that Γ_0 is polycyclic, and therefore finitely generated. Therefore Γ_0 is nilpotent. This completes the proof of Proposition 4, and hence the proof of Proposition 2.

Proof of Theorem 1. Suppose that M is a compact affine manifold with a polynomial Riemannian metric g_M. By compactness, g_M is complete. By Proposition 2, M is a complete affine manifold which admits a finite covering M' with nilpotent affine holonomy group. In Fried, Goldman, and Hirch [5] it is proved that such affine manifolds M' are complete affine nilmanifolds $\Gamma\backslash G$. Let H be the group of deck transformations of $M' \to M$; then H acts affinely and $M = (\Gamma\backslash G)/H$ as claimed.

Conversely, suppose that M is a compact affine manifold finitely covered by a complete affine nilmanifold $M' = \Gamma'\backslash G$. There is a subgroup Γ_0 of Γ' which is normal in $\pi_1(M)$ and which has finite index. Then $\Gamma_0\backslash G$ is a complete affine manifold M_0 which is a regular finite regular covering space of M. Let H denote the group of deck transformations of $M_0 \to M$. Then H is an affine transformation group on H_0.

Now G has a left-invariant Riemannian metric. In [5], section 8, it is proved that every left-invariant tensor field on G is polynomial in affine charts. Therefore G has a left-invariant polynomial Riemannian metric. It follows that $\Gamma_0\backslash G$ has such a metric also. Averaging this metric by H gives a polynomial Riemannian metric on M. The proof of Theorem 1 is complete.

In the foregoing proof of the existence of polynomial metrics, compactness of M was used only to ensure completeness of the affine structure. If we simply postulate this completeness instead and observe that left-invariant metrics are homogeneous, we obtain:

Theorem 4. *Let M be a complete affine manifold which is finitely covered by an affine nilmanifold. Then M has a polynomial Riemannian metric which is complete and locally homogeneous.*

Remarks

(1) Bieberbach's theorem can be derived from Theorem 1 as follows. The algebraic hull $A(\Gamma)$ is a group of isometries of the Euclidean metric on $\mathbf{R}^n$; the unipotent radical must then be the simply transitive group of translations. The finite covering constructed is necessarily a flat torus and Bieberbach's theorem follows easily.

(2) The class of affine manifolds with polynomial Riemannian metrics can be characterized in another way in terms of the eigenvalues of the holonomy. In [5] it is proved that the class of compact affine manifolds whose holonomy is unipotent coincides with the class of compact complete affine nilmanifolds. It follows that an affine manifold has a covering in this class if and only if the eigenvalues of every element of the linear holonomy $L(\Gamma)$ are roots of unity. D. Fried [4] has proved the following stronger statement: a compact affine manifold is finitely covered by a complete affine nilmanifold if and only if the eigenvalues of every element of $L(\Gamma)$ have norm 1. Combining Fried's result with Theorem 1 we conclude: *a compact affine manifold has a polynomial Riemannian metric if and only if all of the eigenvalues of elements of $L(\Gamma)$ lie on the unit circle in $\mathbb{C}$.*

Rational Riemannian Metrics

Now suppose that M is a compact affine manifold with a rational Riemannian metric g_M which is affine coordinates has the form

$$q(x)^{-1} \sum p_{ij}(x)\, dx_i\, dx_j$$

where $q(x)$ and all the $p_{ij}(x)$ are polynomials. Then the proof of Theorem 1 goes through, completely as before, *provided we assume that $q(x)$ has no zeroes on E.* Thus a compact affine manifold admitting such a metric is finitely covered by a complete affine nilmanifold, and on the corresponding nilpotent Lie group the metric lifts to a left-invariant metric. It follows from §8 of [5] that such a metric must be polynomial.

The condition that g_M comes from a 2-form defined on all of E is satisfied whenever the algebraic hull $A(\Gamma)$ acts transitively on E. In [6] we prove this is the case if M is a compact affine manifold with a parallel volume form (i.e. the standard Euclidean n-form $dx_1 \dots dx_n$ in affine coordinates). Then, assuming the existence of a rational Riemannian metric, M must be complete; in fact M must be covered by a compact complete affine nilmanifold and the metric must be of the type described.

The obstruction to the existence of a parallel volume is precisely the element of $H^1(M; \mathbf{R})$ given by the logarithm of the absolute value of the determinant of the linear holonomy homomorphism. Thus if the first Betti number of M is zero, then *every* affine structure must admit parallel volume, every rational Riemannian metric must be polynomial. This also shows that there are manifolds M such that every affine structure admits a polynomial Riemannian metric. Take any closed manifold with $b_1(M)=0$ and such that $\pi_1(M)$ has a nilpotent subgroup of finite index (for example a suitable flat Riemannian manifold). By [5] it follows that every affine manifold homeomorphic to M must be finitely covered by a complete affine nilmanifold; the polynomial Riemannian metrics are then just the left-invariant Riemannian metrics on the corresponding Lie group invariant under the finite group of deck transformations.

When $q(x)$ is allowed to vanish in E, then Proposition 3 implies that the developing map is a covering onto a connected component of $E-q^{-1}(0)$. Nonetheless a rich variety of examples may occur:

Hopf Manifolds

Let $\gamma \in \mathrm{GL}(\mathbf{R}^n)$ be a *linear expansion*, i.e. a linear map all of whose eigenvalues have norm greater than 1. Then the cyclic group $\Gamma=\{\gamma^n : n \in Z\}$ acts properly discontinuously and freely on $\mathbf{R}^n-\{0\}$ with quotient an affine manifold M diffeomorphic to $S^{n-1}\times S^1$.

Suppose first that γ is a homothety, i.e. scalar multiplication by some $\lambda>1$. Then $d(x_i \circ \gamma)=\lambda\, dx_i$ and γ takes spheres of radius r to spheres of radius λr. It follows that $(x_1^2+\ldots+x_n^2)^{-1}\ ((dx_1)^2+\ldots+(dx_n)^2)$ defines a Γ-invariant Riemannian metric on $\mathbf{R}^n-\{0\}$ which is rational. Thus the quotient M admits a rational Riemannian metric.

In general when γ is an arbitrary expansion, the Hopf manifold M may or may not admit a rational Riemannian metric. The following theorem shows that the set of expansions γ for which the corresponding Hopf manifold M has a rational Riemannian metric is dense, and its complement is dense.

Theorem 5. *Let $\gamma \in \mathrm{GL}(\mathbf{R}^n)$ be an expansion and let $M=(\mathbf{R}^n-\{0\})/\Gamma$ be the corresponding Hopf manifold. Then M admits a rational Riemannian metric if and only if*

(1) *γ is diagonalizable over $\mathbb{C}$*

and

(2) *The collection of eigenvalues $\{\lambda_1,\ldots,\lambda_k\}\subset\mathbb{C}$ satisfies: $\dfrac{\log|\lambda_i|}{\log|\lambda_j|}$ is rational for all i and j.*

(It is interesting to note that conditions (1) and (2) are precisely the condition for a cyclic group Γ that $A(\Gamma)/\Gamma$ be compact.)[1]

Proof. Suppose that γ satisfies hypotheses (1) and (2); that is, for some basis of $\mathbf{R}^n=\mathbf{R}^{2k-n}\times\mathbb{C}^{n-k}$ the linear map γ is represented by a diagonal matrix

$$\begin{bmatrix} \lambda^{(a_1)} & \cdot & \cdot & & \cdot \\ \cdot & \cdots\ \cdot & \cdot & & \cdot \\ \cdot & \lambda^{(a_{2k-n})} & \cdot & & \cdot \\ \cdot & \cdot & e^{(i\zeta_1)}\,\lambda^{(a_{2k-n+1})} & & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & & e^{(i\zeta_{n-k})}\,\lambda^{(a_k)} \end{bmatrix}$$

where the a_j are positive integers, $\lambda>0$; $\zeta_j \in \mathbb{R}$.

Define a "radius function" $r: \mathbf{R}^{2k-n}\times\mathbb{C}^{n-k}\to\mathbf{R}$ by the following recipe:

$$\begin{aligned} &r(x_1,\ldots,x_{2k-n},z_{2k-n+1},\ldots,z_k)\\ &\quad=\{(x_1)^{2b_1}+\ldots+(x_{2k-n})^{2b_{2k-n}}+|z_{2k-n+1}|^{2b_{2k-n+1}}+\ldots+|z_k|^{2b_k}\}^{\frac{1}{2}} \end{aligned}$$

[1] Note added in proof: If M is a compact affine manifold a necessary condition (although not sufficient) for M to admit a rational Riemannian metric is that $A(\Gamma)/\Gamma$ be compact.

where the b_j are positive integers chosen so that $a_1 b_1 = \ldots = a_k b_k = m$ is some (probably large) positive integer. Then γ takes "spheres of radius r" to "spheres of radius $\lambda^{2m} r$," i.e. r satisfies

$$r \circ \gamma = \lambda^{2m} r.$$

Then the expression

$$\begin{aligned}(x_1^{2b_1} + \ldots + |z_k|^{2b_k})^{-1} (2b_1(2b_1-1)\, x_1^{2b_1-2} (dx_1)^2 + \ldots \\ + 2b_k(2b_k-1)\, |z_k|^{2b_k-2}\, d\bar{z}_k\, dz_k)\end{aligned}$$

defines a Γ-invariant rational Riemannian metric on $\mathbf{R}^n - \{0\}$ and hence a rational Riemannian metric on M.

Conversely suppose that M is a Hopf manifold with holonomy Γ and let g_M be a rational Riemannian metric on M. Let g be the induced metric on $\mathbf{R}^n - \{0\}$. Now a linear map γ is diagonalizable over $\mathbb{C}$ (i.e. is semisimple) if and only if the smallest algebraic subgroup $A(\Gamma)$ of $\mathrm{GL}(\mathbf{R}^n)$ containing Γ has no nontrivial unipotent elements (see e.g. [9]). Hence if a generator γ of Γ is *not* semisimple, then $A(\Gamma)$ contains a nontrivial unipotent subgroup U. Since $A(\Gamma)$ acts isometrically on $(\mathbf{R}^n - \{0\}, g)$, for every $p \neq 0$ the isotropy group $A(\Gamma)_p$ is compact. Now U (being unipotent) has a fixed point $p \neq 0$. Thus $A(\Gamma)_p \cap U$ is a nontrivial compact subgroup of a unipotent group. This contradiction proves (1).

Thus γ may be represented over $\mathbb{C}$ by a diagonal matrix with real eigenvalues $\lambda_1, \ldots, \lambda_{2k-n}$ and complex eigenvalues $\lambda_{2k-n+1}, \bar{\lambda}_{2k-n+1}, \ldots, \lambda_k, \bar{\lambda}_k$. Now the compact group $G = 0(1)^{2k-n} \times U(1)^{n-k}$ centralizes Γ and so defines a compact group action on M. By averaging over G we can then replace the original rational Riemannian metric g_M by a rational Riemannian metric h_M which is invariant under G. Let h be the induced metric on $\mathbf{R}^n - \{0\}$.

In particular h defines a rational Riemannian metric on the Hopf manifold M' whose holonomy group Γ' is generated by the diagonal matrix with entries $|\lambda_1|, \ldots, |\lambda_{2k-n}|, |\lambda_{2k-n+1}|, |\lambda_{2k-n+1}|, \ldots, |\lambda_k|, |\lambda_k|$.

Let $1 \leqq i < j \leqq n$. Then the $x_i x_j$-plane E_{ij} is invariant under Γ' and the quotient $(E_{ij} - \{0\})/\Gamma$ is a Hopf two-torus with holonomy generated by

$$\gamma_{ij} = \begin{bmatrix} |\lambda_i| & 0 \\ 0 & |\lambda_j| \end{bmatrix}.$$

Now if $\dfrac{\log|\lambda_i|}{\log|\lambda_j|}$ is irrational, the group generated by γ_{ij} is Zariski dense in the group D of diagonal matrices in $\mathrm{GL}(\mathbf{R}^2)$. It is not hard to prove that every D-invariant rational Riemannian metric on $\mathbf{R}^2$ must be of the form

$$\frac{a}{x^2} dx^2 + \frac{b}{xy} dx\, dy + \frac{c\, dy^2}{y^2} \quad \text{with} \quad b^2 < 4ac.$$

But such a metric is not defined everywhere on $\mathbf{R}^2 - \{0\}$, a contradiction. This concludes the proof of Theorem 5.

Homogeneous Spaces

If G is a Lie group with a left-invariant affine structure and Γ is a discrete subgroup then the space of right cosets $\Gamma \backslash G$ inherits an affine structure from G. The right-invariant vector fields on G form a Lie algebra of affine vector fields on G (since they are infinitesimal left-multiplications). Since the coefficients of the right-invariant vector fields are polynomials (of degree 1) it follows by duality that the coefficients of the right-invariant 1-forms are rational functions. By choosing a basis of right-invariant 1-forms and summing their squares (this is equivalent to decreeing the dual basis of vector fields to be orthonormal), we obtain a *right-invariant* rational Riemannian metric on G.

However this rational Riemannian metric on G does not in general define a metric on $\Gamma \backslash G$, since it is not necessarily left-invariant. To obtain an affine manifold with a rational Riemannian metric in this way, we need both the affine structure and the metric to be invariant under the *same* group of multiplications. Hence if G carries a *bi-invariant affine structure* and a discrete subgroup Γ, then the coset space $\Gamma \backslash G$ (as well as G/Γ) is an affine manifold with a rational Riemannian metric.

To construct bi-invariant affine structures, take any finite-dimensional real associative algebra A. Let A^* denote the new associative algebra whose underlying vector space is $A+\mathbf{R}$ where the multiplication is given by

$$(x,s)\cdot(y,t)=(xy+tx+sy,\, st).$$

(In other words, we have "adjoined" the two-sided identity $(0,1)$. Note that A may or may not have an identity to start with.) Let E denote the affine hyperplane of all $(x,1)$ in A^*; clearly E is closed under multiplication. The set of invertible elements of E form an open subset G, which has an affine structure induced by the inclusion of G in E. Since both left multiplication and right multiplication define affine transformations of E, the affine structure on G is bi-invariant. (Compare Milnor [13].)

The bracket in the Lie algebra of G is given by the commutator $[X,Y] = XY-YX$ in A; here we use the natural identification of A with the tangent space to G at the identity. Conversely the original product in A arises from covariant differentiation of left-invariant vector fields. It can be proved that every bi-invariant affine structure arises in this way (see Vey [15], or Goldman [7]).

The above construction yields many rational Riemannian metrics on certain solvmanifolds with affine structures; we also obtain affine structures and rational Riemannian metrics on compact quotients $\Gamma \backslash \mathrm{GL}(n;\mathscr{k})$ where $\mathscr{k}=\mathbf{R}$, $\mathbb{C}$, or the quaternions.

There are also left-invariant affine structures on Lie groups G which are not right-invariant, but nonetheless admit left-invariant rational Riemannian metrics. In [5] the following example is given. G is the subgroup of $\mathrm{GL}(3;\mathbf{R})$ consisting of matrices

$$\begin{bmatrix} e^{2s} & 0 & u \\ 0 & 1 & v \\ 0 & 0 & e^{s} \end{bmatrix} \qquad (s,u,v\in\mathbf{R}).$$

Evidently G acts simply transitively on the half-space $H=\{(x,y,z): z>0\}$. Then G inherits an affine structure from H via the evaluation map f_p at any point p of H; we pick $p=(0,0,1)$. This affine structure is invariant under left-multiplications. The rational 1-forms $z^{-1}dz$, $z^{-2}(dx-z^{-1}x\,dz)$, $dy-yz^{-1}dz$ in H correspond under f_p to a basis of the left-invariant 1-forms on G; thus the expression

$$(z^{-1}dz)^2+(z^{-2}dx-z^{-3}x\,dz)^2+(dy-yz^{-1}dz)^2$$

corresponds to a left-invariant rational Riemannian metric on G. For every discrete uniform subgroup Γ of G, we obtain a compact affine solvmanifold $\Gamma\backslash G$ with a rational Riemannian metric.

Cones

(D) Another class of examples arise from homogeneous sharp convex cones (see Vinberg [17]). Let E be a real vector space and $\Omega\subset E$ a convex cone containing no complete line. Let G be a subgroup of $\mathrm{GL}(E)$ which preserves Ω; then there is a G-invariant Riemannian metric on Ω canonically associated to Ω (for another exposition of this fact see Vey [16], §1). If G acts transitively on Ω, this Riemannian metric is a rational Riemannian metric (Koszul [11]). For each discrete subgroup Γ the quotient $M=\Gamma\backslash\Omega$ is an affine manifold with a natural rational Riemannian metric.

The well-known affine structures on the product of a closed surface Σ of genus greater than one and S^1 (see [5] or [13]) are of this form. Here $\Omega=\{(x,y,z): x^2+y^2-z^2<0\}$, $G=SO(2,1)$ and the rational Riemannian metric is

$$\begin{aligned}(x^2+y^2-z^2)^{-2}\{&(-3x^2+y^2-z^2)\,dx\,dx+4xy\,dx\,dy+(x^2-3y^2-z^2)\,dy\,dy\\ &-4xz\,dx\,dz-4yz\,dy\,dz+(x^2+y^2-5z^2)\,dz\,dz\}.\end{aligned}$$

From its invariance under G, this metric restricts to one of constant negative curvature on each hyperboloid $x^2+y^2-z^2=\text{constant}$, which is embedded as a fiber Σ; indeed this metric is just the product metric on $\Sigma\times S^1$.

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