Complete affine manifolds: a survey

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An *affinely flat manifold* (or just *affine manifold*) is a manifold with a distinguished coordinate atlas with locally affine coordinate changes. Equivalently M is a manifold equipped with an affine connection with vanishing curvature and torsion. A *complete affine manifold* M is a quotient E/Γ where $\Gamma \subset \mathsf{Aff}(E)$ is a discrete group of affine transformations acting properly on E. This is equivalent to geodesic completness of the connection. In this case, the universal covering space of M is affinely diffeomorphic to E, and the group $\pi_1(M)$ of deck transformations identifies with the affine holonomy group Γ.

Flat Riemannian manifolds are special cases where Γ is a group of Euclidean isometries. The classical theorems of Bieberbach provide a very satisfactory picture of such structures: every compact flat Riemannian manifold is finitely covered by a flat torus E/Λ where $\Lambda \subset G$ is a lattice in the group G of translations of E. Furthermore every *complete* flat Riemannian manifold is a flat orthogonal vector bundle over its a *soul,* a totally geodesic flat Riemannian manifold. (See, for example, Wolf [29].)

An immediate consequence is $\chi(M) = 0$ if M is compact (or even if Γ is just nontrivial). This follows immediately from the intrinsic Gauß-Bonnet theorem of Chern [11], who conjectured that the Euler characteristic of a closed affine manifold vanishes. (Chern-Gauß-Bonnet applies only to *orthogonal connections* and not to linear connections.) In this generality, Chern's conjecture remains unsolved.

Affine manifolds are considerably more complicated than Riemannian manifolds, where metric completeness is equivalent to geodesic completeness. In particular, simple examples such as a Hopf manifold $\mathbb{R}^n \setminus \{0\}/\langle\gamma\rangle$, where γ is a linear expansion of $\mathbb{R}^n$ illustrate that closed affine manifolds need not be complete. For this reason we restrict only to *geodesically complete manifolds.*

Kostant and Sullivan [20] proved Chern's conjecture when M is complete. In other directions, Milnor [24] found flat oriented $\mathbb{R}^2$-bundles over surfaces with nonzero Euler class. Using Milnor's examples, Smillie [26] constructed flat affine connections on some manifolds of nonzero Euler characteristic. (Although the curvature vanishes, it seems hard to control the torsion.)

Auslander's flawed proof [4] of Kostant-Sullivan still contains interesting ideas. Auslander claimed that every closed complete affine manifold is finitely covered by a *complete affine solvmanifold* G/Γ, where $G \subset \mathsf{Aff}(E)$ is (necessarily solvable) closed subgroup of affine automorphisms of E. This generalizes Bieberbach's structure theorem for flat Riemannian manifolds. Whether every closed complete affine manifold has this form is a fundamental question in its own right, and this question is now known as the *"Auslander Conjecture."* ([16]). It has now been established in all dimensions $n < 7$ by Abels-Margulis-Soifer [2, 3].

Milnor's paper [25] clarified the situation. Influenced by Tits [28] he asked whether any discrete subgroup of $\mathsf{Aff}(E)$ which acts properly on E must be virtually polycyclic. If so then complete affine manifolds admit a simple structure, and can be classified by techniques similar to the Bieberbach theorems. Tits's theorem implies that either Γ is virtually polycyclic or it must contain a free subgroup of rank two. Thus Milnor's question is equivalent to whether $\mathbb{Z} \star \mathbb{Z}$ admits a proper affine action.

Margulis [21, 22] showed that indeed nonabelian free groups can act properly and affinely on affine spaces of all dimensions > 2. In dimension 3, Fried-Goldman [16] showed that if $\Gamma \subset \mathsf{Aff}(E)$ is discrete and acts properly, then either Γ is polycyclic *or* the linear holonomy homomorphism L maps Γ faithfully onto a discrete subgroup of a subgroup conjugate to the special orthogonal group $\mathsf{SO}(2,1) \subset \mathsf{GL}(3,\mathbb{R})$. In particular $\Sigma := \mathsf{H}^2/L(\Gamma)$ is a complete hyperbolic surface homotopy-equivalent to $M^3 = E/\Gamma$.

Already this implies Auslander's Conjecture in dimension 3: Since M^3 is closed, Γ has cohomological dimension 3, contradicting Γ being the fundamental group of a surface Σ. Much deeper is the fact that Σ cannot be closed (Mess [23]). Therefore Γ must itself be a free group.

Since Margulis's examples admit complete flat Lorentzian metrics, quotients E/Γ where Γ is free of rank > 2, have been called *Margulis spacetimes.*

Which groups admit proper affine actions in higher dimension remains an intriguing and mysterious question. The Bieberbach theorems imply that any discrete group of Euclidean isometries is finitely presented. The class of properly acting discrete affine groups contains $\mathbb{Z} \star \mathbb{Z}$, and is closed under Cartesian products and taking subgroups. Thus properly discontinuous affine groups needn't be finitely generated, and even finitely generated properly discontinuous affine groups needn't admit finite presentations (Stallings [27]). *The only hyperbolic groups known to admit proper affine actions are free.*

In his 1990 doctoral thesis [14], Drumm gave a geometric proof of Margulis's result and sharpened it. Using polyhedral hypersurfaces in $\mathbb{R}^3$ called *crooked planes,* he built fundamental polyhedra for proper afffine actions of discrete groups. Therefore his examples are homeomorphic to solid handlebodies. (This has been recently proved in general, for convex cocompact $L(\Gamma)$, by Choi-Goldman [12] and Danciger-Guéritaud-Kassel [13] independently.)

Using crooked planes, Drumm [15] showed that Mess's theorem is the *only* obstruction for the existence of a proper affine deformation: *Every* noncompact hyperbolic surface admits a proper affine deformation with a fundamental polyhedron bounded by crooked planes. Using much different dynamical methods, Goldman-Labourie-Margulis [18] identify the space of proper affine deformations of a convex cocompact Fuchsian group as an open convex cone in a vector space.

Our joint work [6, 7, 5, 8] with Charette and Drumm classifies Margulis spacetimes where $\Gamma \cong \mathbb{Z} \star \mathbb{Z}$ using crooked planes. Recently Danciger-Guéritaud-Kassel announced that *every* Margulis spacetime with convex cocompact $L(\Gamma)$ admits a crooked fundamental polyhedron.

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