Let $A^1$ be the affine line (that is, the vector space $\mathbb{R}^1$ without the special significance of 0). Recall that an affine transformation is a transformation

$$A^1 \xrightarrow{T} A^1$$

$$x \mapsto Px + Q$$

where $P \neq 0$ and $P, Q$ are scalars. If $a, b, c \in A^1$ and $a \neq b$, then define

$$F(a, b, c) := \frac{c - a}{b - a}$$

(1) Show that $F$ is invariant under affine transformations: that is, if $T, a, b, c$ are as above, then

$$F(T(a), T(b), T(c)) = F(a, b, c).$$

(2) Suppose that $r = F(a, b, c)$. Find an affine transformation $T$ (that is, coefficients $P, Q$ as above), such that

$$T(a) := 0$$
$$T(b) := 1$$
$$T(c) := r$$

(3) **Prove or disprove:** If $(a, b, c)$ and $(a', b', c')$ are two ordered triples in $A^1$ with $a \neq b$ and $a' \neq b'$, then there is a unique affine transformation $T$ such that

$$T(a) := a'$$
$$T(b) := b'$$
$$T(c) := c'.$$
(4) Prove or disprove: $c$ is the midpoint of $\overline{ab}$ if and only if $F(a, b, c) = \frac{1}{2}$.

(5) All this works if the scalars are complex numbers. In that case the field of complex numbers $\mathbb{C}$ corresponds to the affine plane $\mathbb{A}^2$ where the complex number $z = x + iy$ identifies with the point $p(z) := (x, y) \in \mathbb{A}^2$, where $x, y \in \mathbb{R}$.

Let $u, v, z \in \mathbb{C}$ with $u \neq v$. Then $F(u, v, z) \in \mathbb{C}$.

Prove or disprove: The points $p(u), p(v), p(z) \in \mathbb{A}^2$ are collinear if and only if the complex number $F(u, v, w) \in \mathbb{R}$. 