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On the Polynomial Cohomology of Affine Manifolds

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It is well known that the real cohomology of a compact Riemannian manifold M is isomorphic to the algebra of its harmonic forms. When M is a flat Riemannian manifold, i.e. a *Euclidean* manifold, a differential form is harmonic if and only if it is a parallel differential form. In the local Euclidean coordinate systems on M, such a differential form has constant coefficients. Thus, for a Euclidean manifold, there is canonical isomorphism between its real cohomology and its algebra of parallel differential forms.

One can also ask about differential forms on M which in local coordinates have *polynomial* coefficients. A parallel form is one which is expressed by polynomials of degree 0. Unlike parallel forms, however, a polynomial form is not in general closed. The inclusion of the complex of polynomial forms on M into the de Rham complex of M induces a map $i_*: H^*_{\text{poly}}(M) \to H^*(M)$ of the *polynomial cohomology* of M into the de Rham cohomology of M.

For a compact Euclidean manifold, a polynomial differential form on M must be parallel, so we get nothing new. However the notion of a polynomial form is invariant under more than just (isometric) Euclidean coordinate changes; rather the notion of polynomial form is invariant under *affine* coordinate changes. Hence if M is an affine manifold (see [FGH] for the precise definition), there is a well-defined complex $\mathscr{A}^*_{\text{poly}}(M)$ of polynomial forms, which is naturally a subcomplex of the C^∞ de Rham complex $\mathscr{A}^*(M)$. The inclusion $i: \mathscr{A}^*_{\text{poly}}(M) \subset \mathscr{A}^*(M)$ induces a map i^* on cohomology. Several properties of the map

$$i^*: H^*_{\text{poly}}(M) \to H^*(M)$$

are studied in [FGH] and [GH]. The purpose of this paper is to prove:

Theorem A. *Let M be a compact complete affine manifold such that $\pi_1(M)$ is virtually polycyclic. Then the natural map*

$$i^*: H^*_{\text{poly}}(M) \to H^*(M)$$

is an isomorphism.

In [FGH] this theorem is proved under the stronger assumption that $\pi_1(M)$ is nilpotent. In general it is not known whether the fundamental group of a complete affine manifold is virtually polycyclic; but all virtually polycyclic groups do occur. This is all discussed quite beautifully in [M]. We remark that in [FG] all three-dimensional compact complete affine manifolds are listed, and that all of these manifolds have polycyclic fundamental group.

The Bieberbach theorems give an extremely useful structure theorem for compact Euclidean M. For more general complete affine manifolds, such a structure theorem exists *provided that the fundamental group is virtually polycyclic.* The structure theorem (mainly due to Auslander) states that a complete affine manifold M with virtually polycyclic fundamental group may be represented as the quotient of an affine solvmanifold Γ/G by a finite group of affine transformations. A *(complete) affine solvmanifold* is a complete affine manifold of the form $\Gamma\backslash G$ where G is a Lie group with a complete left-invariant affine structure and $\Gamma \subset G$ is a discrete subgroup. Equivalently, an affine solvmanifold is a complete affine manifold E/Γ where Γ is a discrete subgroup of a simply transitive group of affine transformations. For more details the reader is referred to [FGH], [FG], §1, and [M].

In general it is difficult to determine the degrees of closed polynomial differential forms. However under the above assumptions on M, there will always exist a *parallel volume form* if M is orientable. In other words Lebesgue measure on E is - invariant and therefore defines a measure on M. This will be proved in [FG] and [GH2]. For a bound on the degrees of closed polynomial forms see [GH2] as well as [FGH].

We have greatly profitted from conversations with David Fried, Moe Hirsch, Calvin Moore, and Joe Wolf. We are especially grateful to Calvin Moore for supplying one of the key ideas in the proof.

Proof of Theorem A. Let M be a compact complete affine manifold with virtually polycyclic fundamental group. By passing to a covering we may assume that M is an affine solvmanifold $\Gamma\backslash G$. This means ([FGH]) that there exists a vector space E and a group G of affine transformations of E which acts simply transitively on E, such that the affine holonomy group Γ of M is a discrete cocompact subgroup of G. Let dev: $G \to E$ denote the evaluation map of G at the origin; this defines a developing map for a complete affine structure on G which is invariant under left-multiplications. In [FG], §1, it is shown that, given Γ (in other words, given M), the simply transitive "crystallographic hull" G may be chosen so that Γ and G have the same algebraic hull in $\mathrm{Aff}(E)$. It follows that the images of Γ and G under $\mathrm{Ad}: G \to \mathrm{Aut}(\mathfrak{g})$ have the same algebraic hull in $\mathrm{Aut}(\mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of G. By a theorem of Mostow [Mt] (see also [R], Corollary 7.29), the complex $\mathscr{A}^*_{\text{left-inv}}(G)$ consisting of left-invariant differential forms on G injects into the de Rham complex of $M = \Gamma\backslash G$, inducing an isomorphism of cohomology

$$H^*_{\text{left-inv}}(G) \to H^*(M).$$

In general what we have denoted $H^*_{\text{left-inv}}(G)$ is equal to the Lie algebra cohomology of $\mathfrak{g}$.

Thus we must identify the Lie algebra cohomology $H^*(\mathfrak{g})$ with the polynomial cohomology $H^*_{\text{poly}}(M)$ of M. Using the simply transitive action of G on E to pass back and forth between E and G, we reduce the proof to the following:
(*) every closed G-invariant differential form on E is cohomologous to a G-invariant polynomial form on E; furthermore for every exact form $d\eta$, where η is a G-invariant form on E, there exists a polynomial G-invariant form η' such that $d\eta = d\eta'$.

The proof of this will be based on the following lemmas. Part (ii) of the first lemma was supplied us by C. Moore.

Lemma B. *Let G be a connected solvable subgroup of a linear algebraic group and let $A(G)$ denote its algebraic hull. Then:*

(i) *$A(G)$ decomposes as a semidirect product $G \rtimes R$ where G is normal in $A(G)$ and R is a maximal reductive algebraic subgroup of $A(G)$;*

(ii) *Every closed left-invariant form on G is cohomologous to a left-invariant form on G which is also* $\mathrm{Ad}(R)$*-invariant; moreover if η is a left-invariant form, there exists a left-invariant and* $\mathrm{Ad}(T)$*-invariant form η' such that $d\eta = d\eta'$.*

Lemma C. *If $G \subset \mathrm{Aff}(E)$ acts simply transitively, then every tensor field on E invariant under the algebraic hull $A(G) \subset \mathrm{Aff}(E)$ is polynomial.*

Supposing these lemmas the proof of Theorem A for solvmanifolds concludes as follows. First we note the complex $\mathscr{A}^*_{\text{poly}}(M)$ is canonically isomorphic to the complex $\mathscr{A}^*_{\text{poly}}(E)^\Gamma$ of Γ-invariant polynomial forms on E ([GH], §1). Since a polynomial form on E is Γ-invariant if and only if it is invariant under $A(\Gamma) = A(G)$, we have $\mathscr{A}^*_{\text{poly}}(E)^\Gamma = \mathscr{A}^*_{\text{poly}}(E)^{A(G)}$. Furthermore by Lemma C, $\mathscr{A}^*_{\text{poly}}(E)^{A(G)} = \mathscr{A}^*(E)^{A(G)}$ the complex of all $A(G)$-invariant C^∞ differential forms on E.

We define an isomorphism $\mathscr{A}^*(E)^{A(G)} \to \mathscr{A}^*_{\text{left-inv}}(G)^{\mathrm{Ad}\,R}$ as follows. The developing map (i.e. evaluation at the origin) $\mathrm{dev}: G \to E$ is G-equivariant with respect to the action by left-multiplication on G and the affine action $G \subset \mathrm{Aff}(E)$ on E. It is also R-equivariant with respect to Ad on G and the action $R \subset A(G) \subset \mathrm{Aff}(E)$ on E. Since $A(G)$ is the semidirect product $G \rtimes R$, a differential form on E which is invariant under $A(G)$ pulls back (under dev) to a form on G which is invariant under both left-multiplications and $\mathrm{Ad}(R)$. It follows that dev^* is the desired isomorphism.

By Lemma B, the natural map $\mathscr{A}^*_{\text{left-inv}}(G)^{\mathrm{Ad}\,R} \to \mathscr{A}^*_{\text{left-inv}}(G)$ defines an isomorphism on cohomology. By Mostow's theorem, the map $\mathscr{A}^*_{\text{left-inv}}(G) \to \mathscr{A}^*(M)$ induces an isomorphism on cohomology. Composing all of these maps we obtain the inclusion $i: \mathscr{A}^*_{\text{poly}}(M) \subset \mathscr{A}^*(M)$ inducing an isomorphism on cohomology.

Proof of Lemma B. Let $g \in A(G)$. The property that $\mathrm{Ad}\,g$ stabilize $\mathfrak{g}$ is an algebraic condition on g. Since G obviously stabilizes $\mathfrak{g}$ and is Zariski-dense in $A(G)$ implies that every $g \in A(G)$ stabilizes $\mathfrak{g}$. Since G is connected, this means that G is normal in $A(G)$.

In general there is the decomposition of an algebraic group $A(G) = U \rtimes \hat{R}$; the unipotent radical U is the maximal normal connected unipotent subgroup and $\hat{R}$ is a maximal reductive subgroup (see e.g. [H]). Moreover when G is a connected solvable Lie subgroup of a linear algebraic group, the projection map $A(G)\to U$ with kernel $\hat{R}$ maps G onto U (this fact can be proved by a simple modification of the proof of Lemma 4.36 of [R], p. 72). It follows that $A(G)=G\cdot\hat{R}$. Taking R to be a connected algebraic subgroup of $\hat{R}$ such that $\hat{R}=R\times(\hat{R}\cap G)$ part (i) of Lemma B follows.

To prove (ii) we let $\mathscr{A}^p$ denote the space of all left-invariant exterior differential p-forms on G. Let $\mathscr{B}^p=d\mathscr{A}^{p-1}$ be the space of exact forms and $\mathscr{Z}^p=\operatorname{Ker} d\colon \mathscr{A}^p\to\mathscr{A}^{p+1}$ the space of closed forms. Evaluating a left-invariant form at the identity $e\in G$ and considering it as a tensor in $\Lambda^p\mathfrak{g}^*$ we readily see that the action of $\operatorname{Aut}(G)=\operatorname{Aut}(\mathfrak{g})$ on $\mathscr{A}^p$, $\mathscr{B}^p$, and $\mathscr{Z}^p$ is algebraic, i.e. $\mathscr{A}^p$, $\mathscr{B}^p$, and $\mathscr{Z}^p$ are rational $\operatorname{Aut}(G)$-modules.

In particular these spaces are modules over R. Consider the exact sequences:

(A) $0\to\mathscr{B}^p\to\mathscr{Z}^p\to H^p(\mathfrak{g})\to 0$

(B) $0\to\mathscr{Z}^p\to\mathscr{A}^p\to\mathscr{B}^{p+1}\to 0$

It is well known that G acts trivially (under Ad) on $H^p(\mathfrak{g})$; evidently its Zariski closure $A(G)$, and hence also $R\subset A(G)$, act trivially on $H^p(\mathfrak{g})$ as well. Since R acts reductively on all of these spaces there exists a splitting $s\colon H^p(\mathfrak{g})\to\mathscr{Z}^p$ to (A), as R-modules. If $\eta\in\mathscr{Z}^p$ has cohomology class $[\eta]\in H^p(\mathfrak{g})$, then $s([\eta])$ has the same cohomology class. Moreover $s([\eta])$ is R-invariant: for every $g\in R$, $g^*(s([\eta]))=s(g^*[\eta])=s([\eta])$. This proves the first assertion.

For the second assertion, suppose that $\eta\in\mathscr{A}^p$ has the property that $d\eta$ is R-invariant. Since R acts reductively on the spaces in (B) there exists an R-splitting $h\colon\mathscr{B}^{p+1}\to\mathscr{A}^p$. Then $h(d\eta)$ is an R-invariant form on G such that $d(h(d\eta))=\eta$, proving the second assertion. The proof of Lemma B is now complete.

Proof of Lemma C. Decompose $A(G)$ as $U\rtimes R$ where U is the unipotent radical of $A(G)$ and R is reductive. By Auslander ([A], §3), the unipotent radical U acts simply transitively on E. But in [FGH], §8, it is proved that any tensor field on E invariant under a simply transitive unipotent subgroup of $\operatorname{Aff}(E)$ must be polynomial. (In fact any tensor field invariant under a *transitive* unipotent affine action must be polynomial.) This completes the proof of Lemma C.

The proof of Theorem A is now complete when M is an affine solvmanifold $\Gamma\backslash G$ as above. To finish the proof of Theorem A, it suffices to prove that if $\hat{M}\to M$ is a finite regular covering and the theorem is true for $\hat{M}$, then it is also true for M. This is accomplished by the following standard lemma:

Lemma D. *Let $j\colon\mathscr{C}\to\mathscr{C}'$ be an inclusion of chain complexes (over the reals) which is equivariant with respect to a group Γ. Suppose that $\Gamma'\subset\Gamma$ is a normal subgroup of finite index such that $j_*\colon H(\mathscr{C}^{\Gamma'})\to H(\mathscr{C}'^{\Gamma'})$ is an isomorphism. Then $H(\mathscr{C}^\Gamma)\simeq(H(\mathscr{C}^{\Gamma'}))^{\Gamma/\Gamma'}$, $H(\mathscr{C}'^\Gamma)\simeq(H(\mathscr{C}'^{\Gamma'}))^{\Gamma/\Gamma'}$ and $j_*\colon H(\mathscr{C}^\Gamma)\to H(\mathscr{C}'^\Gamma)$ is an isomorphism.*

The proof of Lemma D uses a standard spectral sequence argument involving the comparison of two spectral sequences for $H(\mathscr{C}^\Gamma)$ and $H(\mathscr{C}'^\Gamma)$, as well as the fact that the cohomology of a finite group with coefficients in a real module vanishes. The details of the proof will not be given. To finish the proof of Theorem A, we let $\mathscr{C}$ be the complex of polynomial forms and $\mathscr{C}'$ the complex of C^∞ forms on E. The subgroup Γ' is chosen so that E/Γ' is a compact affine solvmanifold and such that Γ' is normal in Γ.

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