Due Tuesday, 27 September

(1) Let \( \ell \subset \mathbb{E}^2 \) be a line. Show that there is a unique point on \( \ell \) which is closest to the origin \( O \), as follows.
   (a) Suppose that \( p_0 \in \mathbb{E}^2 \) is a point on \( \ell \) and \( v \in \mathbb{R}^2 \) is a vector parallel to \( \ell \). Then \( v \) specifies the direction of \( \ell \) and \( v \neq 0 \).
      Find an expression for the points \( p(t) \) on \( \ell \) in parametric form, where \( t \in \mathbb{R} \) is the parameter, and \( v \) and \( p_0 \) are given.
   (b) Compute the distance \( f(t) = d(p(t), O) \) in terms of \( t, p_0, v \).
   (c) Show that \( f(t) \) has a unique minimum at \( t = t_0 \) and compute \( t_0 \) in terms of \( p_0, v \).
   (d) Prove that the vector from \( O \) to the closest point \( p(t_0) \) is perpendicular to the vector \( v \) specifying the direction of \( \ell \).
   (e) Describe, in terms of complex numbers, all lines for which \( p(t_0) = O \).
   (f) Suppose that \( p(t_0) \neq O \). Show that \( \ell \) is completely determined by \( p(t_0) \).
   (g) Using complex numbers, find an expression for \( \ell \) in terms of the complex number corresponding to \( p(t_0) \).

(2) Write \( p(t_0) = P(\ell) \). Use the above expression for \( \ell \) in terms of \( P(\ell) \) to see how this parametrization transforms under:
   (a) A scaling \( z \mapsto \lambda z \), where \( \lambda > 0 \);
   (b) A rotation \( z \mapsto e^{i\theta} z \);
   (c) A reflection \( z \mapsto \overline{z} \);
   (d) A translation \( z \mapsto z + \zeta \), where \( \zeta \in \mathbb{C} \).

(3) Using the affine patch
\[
\mathbb{A}^2 \hookrightarrow \mathbb{P}^2
\]
\[
(x, y) \mapsto [x : y : 1]
\]
which of the following sets of homogeneous coordinates represent the point \( (0.2, -0.5) \in \mathbb{A}^2 \)?
   (a) \([0.2 : -0.5 : 0]\)
   (b) \([2 : -5 : 1]\)
(c) \([-4 : 10 : 2]\)
(d) \([5 : 2 : 1]\)
(e) \([-0.2 : 0.5 : -1]\)

(4) Which of the following triples of homogeneous coordinates define a set of three collinear points in $\mathbb{P}^2$? For those ones, find the homogeneous coordinates for the line containing them.
(a) \([0.2 : -0.5 : 0], [1 : 3 : 0], [2 : 7 : 0]\)
(b) \([0.2 : -0.5 : 0], [1 : 3 : 0], [2 : 7 : 1]\)
(c) \([1 : 2 : -3], [-1 : 1 : 0], [0 : 4 : -4]\)
(d) \([1 : 1 : 1], [1 : 1 : -1], [4 : 4 : 1]\)
(e) \([1 : 1 : 1], [1 : 1 : -1], [1 : 4 : 4]\)

(5) Here are four affine patches:

\[ \mathbb{A}^2 \xrightarrow{A_1} \mathbb{P}^2 \]
\[ (y, z) \longmapsto [1 : y : z] \]

\[ \mathbb{A}^2 \xrightarrow{A_2} \mathbb{P}^2 \]
\[ (x, z) \longmapsto [x : 1 : z] \]

\[ \mathbb{A}^2 \xrightarrow{A_3} \mathbb{P}^2 \]
\[ (x, y) \longmapsto [x : y : 1] \]

\[ \mathbb{A}^2 \xrightarrow{A_4} \mathbb{P}^2 \]
\[ (u, v) \longmapsto [u + 1 : u - v : u + v] \]

(a) Find an ideal point for each of these affine patches.
(b) Find the affine coordinates of the point \([1 : 2 : 3]\) in terms of these three affine patches. That is, compute $A^{-1}_i([1 : 2 : 3])$ for $i = 1, 2, 3, 4$.
(c) Let $P$ be the parabola
\[ \{(x, y) \in \mathbb{A}^2 \mid y = x^2\} \]
and consider the closure $C$ of $\mathbb{A}_3(P) \subset \mathbb{P}^2$. Express $C$ in homogeneous coordinates.
(d) Does $P$ have an ideal point?
(e) Determine $(A_i)^{-1}(C)$ for $i = 2, 3, 4$ and their ideal points (if any).