MATH 431-2018 PROBLEM SET 3

DUE THURSDAY 20 SEPTEMBER 2018

(1) (Midpoints)

(a) Let p, q be points in Euclidean space $\mathbb{E}^n$. The midpoint $\mathsf{mid}(p,q)$ of a line segment $\overline{pq}$ (in *Euclidean geometry*) is defined as the unique point $r \in \overleftrightarrow{pq}$ such that $d(p,r) = d(q,r)$. Define $\mathsf{mid}(p,q)$ $\overline{pq}$ in *affine geometry:* that is, just in terms of translations, parallelism, etc. but not involving distance.

(b) In terms of affine coordinates (where p, q are represented by vectors in $\mathbb{R}^n$), find a formula for $\mathsf{mid}(p,q)$.

(2) (Affine combinations)

Vectors in a vector space can be added. How can we do this in an affine space?

If $p_0, p_1, \dots, p_k \in \mathbb{A}^n$ are $k+1$ points in affine space, and $t_0, t_1, \dots t_k \in \mathbb{R}$ scalars such that

$$t_0 + t_1 + \dots t_k = 1, \tag{1}$$

we define an *affine combination* $\sum_{i=0}^n t_i p_i$ as follows.

Choose a point $O \in \mathbb{A}^n$ to be used as the *origin* and for each $j = 0, \dots, k$, let τ_j be the translation taking p_j to O. Then $\tau_j(p_i)$ is a vector in $\mathbb{R}^n$ (and when $i = j$, the *zero vector* $\mathbf{0}$). Thus it makes sense to form the linear combination (a vector)

$$\sum_{i=0}^{k} t_i \tau_j(p_i) \ \in \mathbb{R}^n$$

and then translate O by this vector (apply the translation $(\tau_j)^{-1}$) to obtain a point which we denote

$$^{(j)}\textstyle\sum_{i=0}^{k} t_i p_i \ \in \mathbb{A}^n.$$

(a) Show that $^{(j)}\sum_{i=0}^{k} t_i p_i$ is independent of j, so we denote this just by $\sum_{i=0}^{k} t_i p_i$.

(b) Show that if g is an affine transformation, then

$$g\Big(\sum_{i=0}^{k} t_i p_i\Big) \ = \ \sum_{i=0}^{k} t_i g(p_i).$$

(c) Does this characterize affine maps?

(d) An alternative approach is to use the *linearization* of affine spaces as follows. Represent $\mathbb{A}^n$ as the hyperplane $\mathbb{R}^n \times \{1\}$ in the Cartesian product $\mathbb{A}^n \times \mathbb{R}$. (More accurately, $\mathbb{A}^n$ identifies with $\mathbb{R}^n \oplus \{1\}$ in the *direct sum* $\mathbb{R}^n \oplus \mathbb{R} \cong \mathbb{R}^{n+1}$. Then an affine map $g = \big[A \mid \mathbf{b}\big]$ (that is, with linear part $A \in \mathsf{Mat}_n(\mathbb{R})$ and translational part $\mathbf{b} \in \mathbb{R}^n$) is represented by the $(n+1)$-square matrix

$$\begin{bmatrix} A & \mathbf{b} \\ 0 \dots 0 & 1 \end{bmatrix}.$$

which preserves the hyperplane $\mathbb{A}^n$ with last $(n+1$-th) coordinate equal to 1.

(e) If $p_0, \dots, p_k$ respectively correspond to vectors $\mathbf{p}_0, \dots, \mathbf{p}_k \in \mathbb{R}^n \times \{1\}$ in this hyperplane, that is:

$$\mathbf{p} = \begin{bmatrix} p \\ 1 \end{bmatrix},$$

then the usual linear combination $\sum_{i=0}^k t_i \mathbf{p}_i$ of vectors corresponds to the point $\sum_{i=0}^k t_i p_i$.

(f) Explain why condition (1) is necessary.

(3) Using the affine patch

$$\begin{aligned} \mathbb{A}^2 &\hookrightarrow \mathbb{P}^2 \\ (x, y) &\longmapsto [x : y : 1] \end{aligned}$$

which of the following sets of homogeneous coordinates represent the point $(0.2, -0.5) \in \mathbb{A}^2$?

(a) $[0.2 : -0.5 : 0]$
(b) $[2 : -5 : 1]$
(c) $[-4 : 10 : 2]$
(d) $[5 : 2 : 1]$
(e) $[-0.2 : 0.5 : -1]$

(4) Which of the following triples of homogeneous coordinates define a set of three collinear points in $\mathbb{P}^2$? For those ones, find the homogeneous coordinates for the line containing them.

(a) $[0.2 : -0.5 : 0]$, $[1 : 3 : 0]$, $[2 : 7 : 0]$
(b) $[0.2 : -0.5 : 0]$, $[1 : 3 : 0]$, $[2 : 7 : 1]$
(c) $[1 : 2 : -3]$, $[-1 : 1 : 0]$, $[0 : 4 : -4]$
(d) $[1 : 1 : 1]$, $[1 : 1 : -1]$, $[4 : 4 : 1]$
(e) $[1 : 1 : 1]$, $[1 : 1 : -1]$, $[1 : 4 : 4]$

(5) Here are four affine patches:

$$\mathbb{A}^2 \xrightarrow{\mathcal{A}_1} \mathbb{P}^2$$
$$(y, z) \longmapsto [1 : y : z]$$

$$\mathbb{A}^2 \xrightarrow{\mathcal{A}_2} \mathbb{P}^2$$
$$(x, z) \longmapsto [x : 1 : z]$$

$$\mathbb{A}^2 \xrightarrow{\mathcal{A}_3} \mathbb{P}^2$$
$$(x, y) \longmapsto [x : y : 1]$$

$$\mathbb{A}^2 \xrightarrow{\mathcal{A}_4} \mathbb{P}^2$$
$$(u, v) \longmapsto [u + 1 : u - v : u + v]$$

(a) Find an ideal point for each of these affine patches.
(b) Find the affine coordinates of the point $[1 : 2 : 3]$ in terms of these three affine patches. That is, compute $\mathcal{A}_i^{-1}\big([1 : 2 : 3]\big)$ for $i = 1, 2, 3, 4$.
(c) Let P be the parabola
$$\{(x, y) \in \mathbb{A}^2 \mid y = x^2\}$$
and consider the closure C of $\mathcal{A}_3(P) \subset \mathbb{P}^2$. Express C in homogeneous coordinates.
(d) Does P have an ideal point?
(e) Determine $(\mathcal{A}_i)^{-1}(C)$ for $i = 2, 3, 4$ and their ideal points (if any).