MATH 431-2018 PROBLEM SET 4

DUE THURSDAY 18 OCTOBER 2018

(1) (An invariant of similarity)

(a) Let $z_0, z_1, z \in \mathbb{C}$ be three *distinct* complex numbers. They represent the vertices of a triangle

$$\Delta = \Delta(z, z_1, z_0) \subset \mathbb{C} \cong \mathbb{E}^2.$$

Define

$$\mathbb{A}(\Delta) = \mathbb{A}(z, z_1, z_0) := \frac{z - z_0}{z_1 - z_0}$$

Show that if f is an orientation-preserving similarity transformation, then

$$\mathbb{A}\big(f(\Delta)\big) = A(\Delta),$$

and if f is an orientation-reversing similarity transformation, then

$$\mathbb{A}\big(f(\Delta)\big) = \overline{A(\Delta)}.$$

(b) Show that $\mathbb{A}(z, 1, 0) = z$.

(c) Show that if Δ, Δ' are two triangles as above and

$$\mathbb{A}(\Delta) = \mathbb{A}(\Delta'),$$

then there is a unique orientation-preserving similarity transformation f such that $\Delta' = f(\Delta)$.

(d) (Effect of permutations) Call $\mathbb{A}(z, z_1, z_0) = \zeta$. Express $\mathbb{A}(z, z_0, z_1$ and $\mathbb{A}(z_1, z, z_0)$ as functions of ζ. Do the same for $\mathbb{A}(z_0, z_1, z), \mathbb{A}(z_0, z, z_1)$ and $\mathbb{A}(z_1, z_0, z)$.

(e) Characterize the *midpoint* in terms of the invariant $\mathbb{A}$.

(f) Let $A = \mathbb{A}(z, z_1, z_0)$. Express z as an affine combination of z_0 and z_1, that is, prove:

$$z \;=\; (1 - A)z_0 \;+\; A\, z_1$$

(2) (Stereographic projection)
   (a) Let $\mathbb{S}$ be the sphere centered at $(0,0,1)$ with radius 1 and let
   $$N = (0,0,2), S = (0,0,0) \in \mathbb{S}$$
   be the north and south pole respectively. Use the usual coordinates (x,y) on the xy-plane $z=0$ to identify it with $\mathbb{E}^2$. Show that given for any point $p \in \mathbb{S}\backslash\{N\}$, the (unique) line from N to p meets $\mathbb{E}^2$ in a unique point $\Sigma(p)$.
   (b) Show that Σ defines a homeomorphism
   $$\mathbb{S} \setminus \{N\} \xrightarrow{\Sigma} \mathbb{E}^2$$
   and compute its inverse.
   (c) A *great circle* on $\mathbb{S}$ is the intersection of $\mathbb{S}$ with an affine plane passing through its center. Show that if $C \subset \mathbb{S}$ is a great circle passing through N, then $\Sigma(C)$ is a Euclidean line passing through the origin $O \in \mathbb{E}^2$ (the point corresponding to the zero complex number $0 \in \mathbb{C}$).
   (d) The *equator* E on Ss is defined by $z = 1$. What is $\Sigma(E)$?
   (e) Show that other *not-so-great* circles on $\mathbb{S}$ map to lines $\mathbb{E}^2$ *not* passing through O.
   (f) Show that any circle on $\mathbb{S}$ not containing N maps to a circle in $\mathbb{E}^2$.
   (g) Let $\iota(\zeta) = |\zeta|^{-2}\zeta$ be inversion in the unit circle in $\mathbb{E}^2 \longleftrightarrow \mathbb{C}$. What is the transformation of $\mathbb{S}$ defined by $\Sigma^{-1} \circ \iota \circ \Sigma$?
   (h) What does N correspond to under Σ?

(3) On Thursday 27 September we proved that *inversion* ι in the unit circle U takes the circle $C(z,r)$ centered at z with radius r to the circle $C(z',r')$ whose center and radius are:

$$z' = \frac{z}{|z|^2 - r^2}, \qquad r' = \frac{r}{\big||z|^2 - r^2\big|} \tag{1}$$

This is, of course, assuming that $|z| \neq r$. Otherwise $C(z,r) \ni 0$ and since $\iota(0) = \infty$, the circle $C(z',r') = \iota C(z,r)$ is a line. Show that the point on the line $C(z',r') \setminus \{\infty\}$ closest to the origin equals $\iota(z)/2$. What happens when $C(z',r') \ni 0$?

Derivation of formula (1)

Inversion is defined by $\iota(\zeta) = 1/\overline{\zeta}$, and the metric circle by:

$$
\begin{aligned}
C(z,r) &:= \{\zeta \in \mathbb{C} \mid |\zeta - z| = r\} \\
&= \{\zeta \mid \zeta\overline{\zeta} =| \; |\zeta - z|^2 = r^2\}
\end{aligned}
$$

Changing variables with $\omega = \iota(\zeta)$ and using the fact that $\zeta = \iota(\omega)$:

$$
\begin{aligned}
\iota\big(C(z,r)\big) &:= \{\omega \in \mathbb{C} \mid \iota(\omega)\overline{\iota(\omega)} - r^2\} \\
&= \{\omega \mid (1/\overline{\omega} - z)(/\omega - z) - r^2 = 0\} \\
&= \{\omega \mid (1 - z\overline{\omega})(1 - z\omega) - \omega\overline{\omega}r^2 = 0\}
\end{aligned}
$$

Now expand and collect as a polynomial in ω and $\overline{\omega}$:

$$
(1 - z\overline{\omega})(1 - z\omega) - \omega\overline{\omega}r^2 = \Big(|z|^2 - r^2\Big)\omega\overline{\omega} - \overline{z}\omega - z\overline{\omega} + 1
$$

and divide by $|z|^2 - r^2$:

$$
\begin{aligned}
&\Big(\omega - \frac{z}{|z|^2 - r^2}\Big)\Big(\overline{\omega} - \frac{\overline{z}}{|z|^2 - r^2}\Big) \\
&\qquad - \frac{|z|^2}{\big(|z|^2 - r^2\big)^2} + \frac{1}{|z|^2 - r^2} \\
&= \left|\omega - \frac{z}{|z|^2 - r^2}\right| - \frac{r^2}{\big(|z|^2 - r^2\big)^2} \\
&= |\omega - z'|^2 - r'^2
\end{aligned}
$$

Thus $\iota\big(C(z,r)\big) = C(z', r')$ as claimed.