MATH 431-2018 PROBLEM SET 5 - CORRECTED

DUE THURSDAY 15 NOVEMBER 2018

(1) (The circumcircle)

(a) (Bisectors) Let $z_0, z_1 \in \mathbb{C}$ be distinct complex numbers representing points in the Euclidean plane $\mathbb{E}^2$. Define the (metric or perpendicular) *bisector* $B(z_0, z_1)$ as the set of $z \in \mathbb{C}$ such that $d(z, z_0) = d(z, z_1)$.
Show that $B(z_0, z_1)$ is a line and derive a formula for it in terms of z_0 and z_1.

(b) If z_0, z_1, z_2 are not collinear, Show that $B(z_0, z_1)$ and $B(z_0, z_2)$ are *not* parallel. Show that their (unique) intersection point $B(z_0, z_1) \cap B(z_0, z_2)$ lies on $B(z_1, z_2)$.

(c) Show that a unique circle contains z_0, z_1, z_2.

(2) (Cross-ratio) If z_0, z_1, z, z_∞ are four distinct points in $\mathbb{P}^1$, find a unique projective transformation

$$\zeta \xmapsto{f} \frac{a\zeta + b}{c\zeta + d}$$

which maps

$$\begin{aligned} z_0 &\longmapsto 0, \\ z_1 &\longmapsto 1, \\ z_\infty &\longmapsto \infty \end{aligned}$$

Then $f(z)$ is called the *cross-ratio* $\mathcal{C}(z_0, z_1, z, z_\infty)$ of the quadruple (z_0, z_1, z, z_∞).

(a) Show that ϕ is a projective transformation of $\mathbb{P}^1$, then

$$\mathcal{C}(z_0, z_1, z, z_\infty) = \mathcal{C}\big(\phi(z_0), \phi(z_1), \phi(z), \phi(z_\infty)\big).$$

(b) If z_0, z, z_1, z_∞ lie on a circle, must $\mathcal{C}\big(\phi(z_0), \phi(z_1), \phi(z), \phi(z_\infty)\big)$ be real? Is the converse true?

(c) Remember the function

$$\mathbb{A}(z, z_1, z_0) := \frac{z - z_0}{z_1 - z_0}$$

from Problem Set 4? Show that

$$\mathcal{C}(z_0, z_1, z, \infty) = \mathbb{A}(z, z_1, z_0)$$

and, more generally,

$$\mathcal{C}(z_0, z_1, z, z_\infty) = \frac{\mathbb{A}(z_0, z_\infty, z)}{\mathbb{A}(z_0, z_\infty, z_1)}.$$

(d) Let $\{1,2,3,4\} \xrightarrow{\sigma} \{1,2,3,4\}$ be a permutation which interchanges two pairs in $\{1,2,3,4\}$ $\big($for example, $1 \xleftrightarrow{\sigma} 2$, $3 \xleftrightarrow{\sigma} 4$, which is commonly denoted $(12)(34)\big)$. Show that

$$\mathcal{C}(z_1, z_2, z_3, z_4) = \mathcal{C}(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)})$$

(e) Let $z := \mathbb{C}(z_1, z_2, z_3, z_4)$. Prove that:

$$\begin{aligned}
\mathcal{C}(z_1, z_2, z_3, z_4) &= z \\
\mathcal{C}(z_2, z_1, z_3, z_4) &= 1 - z \\
\mathcal{C}(z_4, z_2, z_3, z_1) &= 1/z \\
\mathcal{C}(z_2, z_4, z_3, z_1) &= (z-1)/z \\
\mathcal{C}(z_2, z_3, z_1, z_4) &= 1/(1-z) \\
\mathcal{C}(z_3, z_1, z_2, z_4) &= 1 - 1/z
\end{aligned}$$

(3) (Harmonic quadruples) A quadruple (z_1, z_2, z_3, z_4) is *harmonic* if $\mathcal{C}(z_1, z_2, z_3, z_4) = 1/2$, or equivalently $\mathcal{C}(z_1, z_3, z_4, z_2) = -1$.

(a) If $z_1, z_2, z_3 \in \mathbb{A}^1$ are affine points, then (z_1, z_2, z_3, ∞) is harmonic if and only if $z_3 = \mathsf{mid}(z_1, z_2)$.

(b) Suppose that $\ell \subset \mathbb{P}^2$ is a projective line containing four distinct points z_1, z_2, z_3, z_4 and $\ell_1, \ell_2, \ell_3, \ell_4$ are distinct lines (and distinct from ℓ) such that

$$\begin{array}{lll}
\ell, \ell_1, \ell_2 & \text{concur at} & z_1 \\
\ell, \ell_3, \ell_4 & \text{concur at} & z_3
\end{array}$$

and

$$\begin{aligned}
z_2 &= \overleftrightarrow{(\ell_1 \cap \ell_3)\ (\ell_1 \cap \ell_4)} \\
z_4 &= \overleftrightarrow{(\ell_1 \cap \ell_4)\ (\ell_1 \cap \ell_3)}
\end{aligned}$$

(See Figure 1.) Prove that $(z_1, z_2, z_3, z_4$ is a harmonic quadruple. (Hint: find a useful set of coordinates for these points and lines and apply projective transformations to reduce to an "obvious" (affine?) case.)

(c) Prove, conversely, if (z_1, z_2, z_3, z_4) is a harmonic quadruple on a line ℓ, then there exists a a configuration $\ell_1, \ell_2, \ell_3, \ell_4$ as above.

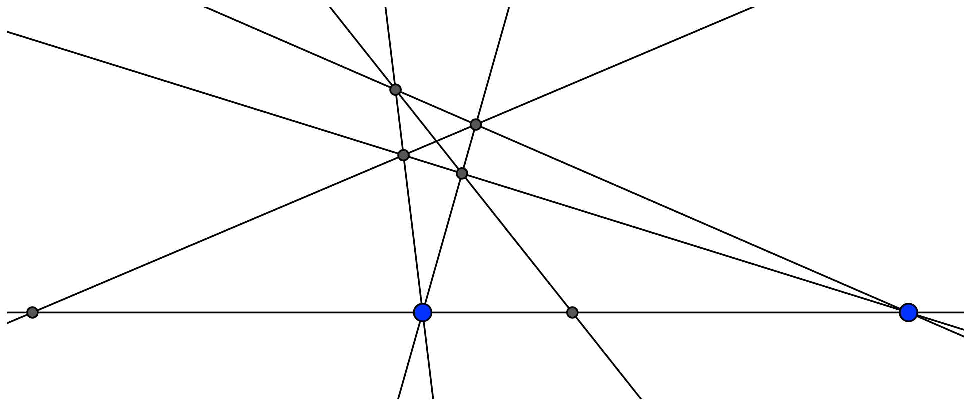

Figure 1. A Harmonic Quadruple

(4) Let $\mathbf{a}, \mathbf{b}$ be *linearly independent* vectors in $\mathbb{R}^3$ defining *non-collinear* points

$$a = [\mathbf{a}], b = [\mathbf{b}] \in \mathbb{P}^2$$

respectively. Show that the line $\overleftrightarrow{ab}$ containing a, b is defined in homogeneous coordinates $[X : Y : Z] \in \mathbb{P}^2$

$$(\mathbf{a} \times \mathbf{b})^\dagger \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0.$$

We generalize this to planes in $\mathbb{P}^3$ passing through three points $a, b, c \in \mathbb{P}^3$. Let V denote $\mathbb{R}^4$ and let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathsf{V}$ be nonzero vectors representing the homogeneous coordinates of the points a, b, c respectively. Define an alternating trilinear map

$$\mathsf{V} \times \mathsf{V} \times \mathsf{V} \xrightarrow{\mathsf{Orth}} \mathsf{V}^*$$

as follows. If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathsf{V}$, denote the determinant of the 4×4 matrix whose columns are $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ by $\mathsf{Det}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$.
For fixed vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathsf{V}$, the map

$$\mathbf{d} \longmapsto \mathsf{Det}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$$

is linear (hence a covector), denoted $\mathsf{Orth}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathsf{V}^*$.

(a) Let

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

be vectors in $\mathbb{R}^4$. Express $\mathsf{Orth}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ in terms of 3×3 determinants, that is,

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} := x_{13} \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{33} \end{vmatrix} - x_{23} \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix} + x_{33} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

$$= x_{11}x_{22}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{11}x_{23}x_{32} - x_{13}x_{22}x_{31} - x_{12}x_{21}x_{33}$$

(b) Prove or disprove: a, b, c are collinear if and only if

$$\mathsf{Orth}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0.$$

(c) Prove or disprove: Suppose $\mathbf{v} := \mathsf{Orth}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq 0$. Then $\mathbf{v}$ define homogeneous coordinates of the plane in $\mathbb{P}^3$ containing a, b, c.