MATH 431-2018 PROBLEM SET 6

DECEMBER 2, 2018
DUE TUESDAY 11 DECEMBER 2018

1. Rotations and quaternions

Consider the line ℓ through $p_0 := (1,0,0)$ and parallel to the vector

$$\mathbf{v} := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

that is, defined implicitly by $x - 1 = y = z$. How do you find the rotation $\mathsf{Rot}_\ell(\theta)$ about ℓ through angle θ?

(1) Let $\ell_O := \mathbb{R}\mathbf{v}$ be the line $x = y = z$ parallel to ℓ and passing through the point O (the "origin") corresponding to the zero vector $\mathbf{0} \in \mathbb{R}^3$. Given the rotation $\mathsf{Rot}_{\mathbf{v}}(\theta)$ about ℓ_O through angle θ, compute $\mathsf{Rot}_\ell(\theta)$.

(2) $\mathsf{Rot}_{\mathbf{v}}(\theta)$ can be computed in several ways. The *quaternionic formula*

$$w \xmapsto{\mathsf{Rot}_v(\theta)} \exp\left(\frac{\theta}{2}\, v\right) w\, \exp\left(-\frac{\theta}{2}\, v\right),$$

where $v \in \mathbb{H}_0 \cong \mathbb{R}^3$ is the purely imaginary quaternion corresponding to the vector $\mathbf{v}$, and $w \in \mathbb{H}_0$ corresponds to an arbitrary vector in $\mathbb{R}^3$. Use this formula to compute $\mathsf{Rot}_\ell(\theta)$.

(3) Another approach involves finding a rotation ρ which takes the unit vector $\frac{1}{\sqrt{3}}\mathbf{v}$ to a fixed unit vector, say $\mathbf{i}$, and then conjugating $\mathsf{Rot}_{\mathbf{i}}(\theta)$ by ρ. Find a rotation which takes $\frac{1}{\sqrt{3}}\mathbf{v}$ to $\mathbf{i}$.

Here are some more problems about quaternions:

(4) Find all quaternion solutions $x \in \mathbb{H}$ of $x^2 = 2$.

(5) Find all quaternion solutions $x \in \mathbb{H}$ of $x^2 = -2$.

(6) Prove or disprove: If $x \in \mathbb{H}$ is a quaternion, then $\exp(tx)$ is real for all $t \in \mathbb{R}$, then x is real.

(7) Prove or disprove: If $x \in \mathbb{H}$ is a quaternion, then $\exp(tx)$ is real for some *nonzero* $t \in \mathbb{R}$, then x is real.

2. Quadrics

2.1. **Three Types of Unruled Quadrics.** Define the *ellipsoid, elliptic paraboloid, and two-sheeted hyperboloid:*

$$E_{a,b,c} := \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \right\}$$
$$P_{a,b} := \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = z \right\}$$
$$H_{a,b,c} := \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = -1 \right\}$$

where $a, b, c \neq 0$ (they are usually taken to be positive). Visualize these surfaces.

In projective space $\mathbb{P}^3$ define the *quadric*

$$Q := \left\{ [X : Y : Z : W] \in \mathbb{P}^3 \,\middle|\, X^2 + Y^2 + Z^2 = W^2 \right\}.$$

(1) In the usual affine patch $(x,y,z) \longmapsto [x : y : z : 1]$, find the ideal points of $E_{a,b,c}, P_{a,b}, H_{a,b,c}$.
(2) Find three affine patches $\mathcal{A}$ into $\mathbb{P}^3$ such that $E_{a,b,c}$, $P_{a,b}$ and $H_{a,b,c}$ are each $\mathcal{A}^{-1}(Q)$. (Hint: use the formulas
$a^2 - b^2 = (a-b)(a+b), \quad 4ab = (a+b)^2 - (a-b)^2$.)
(3) (Bonus problem) Prove or disprove: The affine patch $E_{a,b,c} \longrightarrow Q$ is a homeomorphism. (Recall that a *homeomorphism* is a continuous bijection whose inverse is continuous. In other words, it is a mapping which *preserves the topology,* the way the points are "organized" into a space. It can stretch, squeeze and otherwise distort the geometry, but it can't tear, collapse or break the space. Being continuous means preserving the underlying "topological fabric.")
(4) (Bonus problem) Find a homeomorphism of $E_{a,b,c} \longrightarrow S^2$ where S^2 is the 2-dimensional sphere (the unit sphere in $\mathbb{R}^3$.

2.2. **Surfaces of revolution and cylindrical coordinates.** A *surface of revolution* is a surface obtained by rotating a plane curve about a straight line in that plane. A simple example is revolving a line about a parallel line to obtain a *cylinder.* To fix notation, let $(x_0, y_0) \in \mathbb{R}^2$ be a point in the xy-plane. To obtain a cylinder, rotate the line $\ell_{(1,0)}$ about the z-axis $\ell_{(0,0)}$: rotation through angle angle θ takes $\ell_{1,0}$ to the line $\ell_{(\cos(\theta),\sin(\theta))}$ and the cylinder equals $\left\{x^2 + y^2 = 1\right\}$.

(5) Express the ellipsoid $E_{1,1,c}$ and the paraboloid $P_{1,1}$ as surfaces of revolution.
(6) How is the cone $\left\{x^2 + y^2 = z^2\right\}$ a surface of revolution?

(7) Define *cylindrical coordinates* (r, θ, z) on $\mathbb{E}^3$ by

$$r := \sqrt{x^2 + y^2}$$
$$\theta := \tan^{-1}(y/x)$$
$$z := z$$

and

$$x := r\cos(\theta)$$
$$y := r\sin(\theta)$$
$$z := z.$$

Express the above surfaces in cylindrical coordinates.

(8) (Bonus problem) Prove or disprove: Every surface defined implicitly by an equation $f(r, z) = 0$ is a surface of revolution about the z-axis.

2.3. **Ruled Quadrics.** Sometimes quadrics contain straight lines. Then the quadric is said to be *ruled.* In that case the quadric corresponds to the surface in projective space:

$$Q' := \{[X : Y : Z : W] \in \mathbb{P}^3 \mid X^2 + Y^2 = Z^2 + W^2\}$$

A simple example is the *hyperbolic paraboloid* or *saddle:*

$$S' := \{(x, y, z) \mid xy = z\}$$

The intersection of S' with the xy-plane $z = 0$ is defined by $xy = z = 0$, which decomposes as the union of two lines: the y-axis $x = z = 0$ and the x-axis $y = z = 0$.

(9) For any point $p_0 = (x_0, y_0, x_0y_0)$, find two lines through the point and lying on S. (Hint compute the tangent plane to S at p_0. In the preceding example, what is the relation between S, the origin $(0, 0, 0)$ and the xy-plane?)

(10) Find an affine patch $\mathcal{A}$ such that $S = \mathcal{A}^{-1}(Q')$.

(11) Find a set of affine coordinates (u, v, w) so that S is given by the equation

$$w = u^2 - v^2.$$

Another example is the *one-sheeted hyperboloid:*

$$H' := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$$

which is a surface of revolution in several different ways. ls In the usual affine patch

$$\mathcal{A}(x, y, z) = [x : y : z : 1]$$

(with viewing hyperplane $W = 1$), $H' = \mathcal{A}^{-1}(Q')$. Notice that it is invariant under the one-parameter group of rotations about the z-axis:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \exp \begin{bmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

H' is obtained by revolving the hyperbola

$$y^2 - z^2 - 1 = x = 0$$

around the z-axis.

(12) Write H' in cylindrical coordinates.
(13) Find the ideal points of Q' in the affine patch $\mathcal{A}$.
(14) For each $\theta \in \mathbb{R}$ representing an angle (that is, only defined modulo 2π),

$$\ell_\theta^\pm := \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} + \mathbb{R} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \pm 1 \\ 1 \end{bmatrix}$$

determines two lines (from different choices $\pm$) which lie on H'.
(15) H' is also the surface of revolution obtained by revolving any one of these lines about the z-axis.

These families of lines are called *rulings* and this quadric is ruled in two different ways.

Similarly, this quadric is a surface of revolution in two different ways. Try making a model of H' out of string and two flat circular (or elliptical) rings.

2.4. **Topology of a ruled quadric.** The projective surface Q' is actually a *torus,* a surface homeomorphic to a a bagel, doughnut, or inner tube. This can be seen as follows. Write the equation defining Q' in the form

$$X^2 + Y^2 = Z^2 + W^2$$

and note that this quantity is positive. (Being a sum of squares it is always nonnegative, and if it is zero, then $X^2 + Y^2 = Z^2 + W^2 = 0$, which implies $X = Y = Z = W = 0$, a contradiction.) By scaling the homogeneous coordinates, we can assume that $X^2 + Y^2 = 1$ and $Z^2 + W^2 = 1$, and equation defines a pair of circles (one in the X, Y-plane and the other in the Z, W-plane).

Here is an explicit formula. Let $\theta, \phi \in \mathbb{R}$ represent angles (so they are only defined modulo 2π). Write

$$X_\theta := \cos(\theta)X - \sin(\theta)Y \qquad Y_\theta := \sin(\theta)X + \cos(\theta)Y$$
$$Z_\phi := \cos(\phi)Z - \sin(\phi)W \qquad W_\phi := \sin(\phi)Z + \cos(\phi)W$$

and note that

$$X_\theta^2 + Y_\theta^2 = X^2 + Y^2 \qquad Z_\phi^2 + W_\phi^2 = Z^2 + W^2$$

This will enable us to understand the topology.

Let $T \subset \mathbb{R}^4$ denote the subset defined by

$$X^2 + Y^2 = Z^2 + W^2 = 1.$$

Prove or disprove the following statements.

(16) (Bonus problem) The map

$$S^1 \times S^1 \longrightarrow T \subset \mathbb{R}^4$$
$$(\theta, \phi) \longmapsto \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ \cos(\phi) \\ \sin(\phi) \end{bmatrix}$$

is a homeomorphism (a topological equivalence).

(17) (Bonus problem) The map

$$S^1 \times S^1 \longrightarrow Q' \subset \mathbb{P}^3$$
$$(\theta, \phi) \longmapsto \big[\cos(\theta) : \sin(\theta) : \cos(\phi) : \sin(\phi)\big]$$

is a homeomorphism (Hint: Look at what happens to (π, π).)

(18) (Bonus problem) Q' is homeomorphic to $S^1 \times S^1$.

3. Lines in projective space

3.1. **Lines and planes in projective space.**

3.1.1. *Four-dimensional cross-products.* If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are nonzero vectors representing points

$$a := [\mathbf{a}],\ b := [\mathbf{b}] \in \mathbb{P}^2,$$

then the covector $(\mathbf{a} \times \mathbf{b})^\dagger$ is nonzero if and only if $a \neq b$. In that case it represents the homogeneous coordinates of the line

$$\overleftrightarrow{ab} \subset \mathbb{P}^2$$

containing a and b. (Here $A^\dagger$ denotes the *transpose* of the matrix A.)

In Problem Set 5, Exercise 4, we extended this, using the Orth trilinear form, to points in 3-space. Namely, if $a, b, c \in \mathbb{P}^3$ are points represented by nonzero vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^4$, the the homogeneous coordinates of the plane spanned by a, b, c is represented by the covector $\mathsf{Orth}(\mathbf{a}, \mathbf{b}, \mathbf{c})$. The degenerate case $\mathsf{Orth}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$ occurs if and only if a, b, c are collinear.

3.2. **Rotations and the orthogonal group.** The *special orthogonal group,* denoted $\mathsf{SO}(n)$ consists of all orthogonal $n \times n$ matrices of determinant 1. Equivalently, $\mathsf{SO}(n)$ consists of orientation-preserving linear isometries of $\mathbb{R}^n$ (Euclidean n-space). Every element of $\mathsf{SO}(2)$ is a rotation about the origin:

$$\exp(\theta \mathbf{J}) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Similarly, every element of $\mathsf{SO}(3)$ is rotation about a line (its *axis*) $A \subset \mathbb{R}^3$. In terms of the orthogonal direct-sum decomposition

$$\mathbb{R}^3 = A^\perp \oplus A$$

this rotation is just the direct sum of $\exp(\theta \mathbf{J})$ on $A^\perp$ (with respect to an orthonormal basis) and the identity $\mathbb{1}$ on A. However, higher dimensions are more complicated:

(8) The matrix

$$M := \begin{bmatrix} 4/5 & 3/5 & 0 & 0 \\ -3/5 & 4/5 & 0 & 0 \\ 0 & 0 & \cos(2) & -\sin(2) \\ 0 & 0 & \sin(2) & \cos(2) \end{bmatrix}$$

is orthogonal and lies in $\mathsf{SO}(4)$

(9) M does not fix any point in $\mathbb{P}^3$. (Hint: projective fixed points correspond to eigenvectors.)

(10) Find a matrix L such that $e^L = M$.

(11) Find two projective lines in $\mathbb{P}^3$ which are invariant under this projective transformation. Do those lines intersect?

3.3. **Homogeneous Coordinates for Lines.** Points in $\mathbb{P}^3$ correspond to (projective equivalence classes) of nonzero vectors in $\mathbb{R}^4$. That is, the point in $\mathbb{P}^3$ with homogeneous coordinates $[X : Y : Z : W]$ is the line $[\mathbf{v}]$ spanned by the nonzero vector

$$\mathbf{v} := \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} \in \mathbb{R}^4.$$

Similarly, planes in $\mathbb{P}^3$ correspond to (projective equivalence classes) of covectors

$$\phi := \begin{bmatrix} a & b & c & d \end{bmatrix} \in (\mathbb{R}^4)^*,$$

where $[\phi] = [\![a : b : c : d]\!]$ is the hyperplane defined in homogeneous coordinates by $\phi(\mathbf{v}) = 0$, that is,

$$aX + bY + cZ + dW = 0. \qquad (\star)$$

That is, the point $[X : Y : Z : W]$ lies on the plane $[\![a : b : c : d]\!]$ if and only if $(\star)$ is satisfied.

Thus points and planes in $\mathbb{P}^3$ are defined in homogeneous coordinates by vectors in the vector space $\mathsf{V} := \mathbb{R}^4$ and covectors in its dual vector space $\mathsf{V}^* = (\mathbb{R}^4)^*$. Moreover, the orthogonal complement $\mathbf{v}^\perp$ of the line $\mathbb{R}\mathbf{v} \in \mathbb{R}^4$ is the hyperplane in $\mathbb{R}^4$ defined by the covector $\mathbf{v}^\dagger$, which is the *transpose* of $\mathbf{v}$.

How can you describe *lines* in $\mathbb{P}^3$ in a similar way by homogeneous coordinates?

Exterior Outer Products

Recall that $\mathfrak{o}(n)$ denotes the set of $n \times n$ *skew-symmetric* matrices, that is $X \in \mathsf{Mat}_n$ such that $X + X^\dagger = 0$. The *exterior outer product* is the alternating bilinear map:

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathfrak{o}(n) \\ (\mathbf{v}, \mathbf{w}) &\longmapsto \mathbf{v} \wedge \mathbf{w} := \mathbf{w}\mathbf{v}^\dagger - \mathbf{v}^\dagger \mathbf{w}. \end{aligned}$$

The following facts are easy to verify:

- $(\mathbf{u} \wedge \mathbf{v}) : \mathbf{w} \longmapsto (\mathbf{v} \cdot \mathbf{w})\mathbf{u} - (\mathbf{v} \cdot \mathbf{u})\mathbf{w}$
- If $n = 3$, then $(\mathbf{u} \wedge \mathbf{v})(\mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.
- $\mathbf{w}$ and $\mathbf{v}$ are linearly dependent if and only if $\mathbf{w} \wedge \mathbf{v} = 0$.
- If $\mathbf{w}$ and $\mathbf{v}$ are linearly independent, then the projective equivalence class $[\mathbf{w} \wedge \mathbf{v}] \in \mathsf{P}\big(\mathfrak{o}(n)\big)$ depends only the plane $\mathbb{R}\langle \mathbf{w}, \mathbf{v} \rangle$ spanned by $\mathbf{w}, \mathbf{v}$.
- The orthogonal complement of the plane $\mathbb{R}\langle \mathbf{w}, \mathbf{v} \rangle \subset \mathsf{V}$ lies in the kernel $\mathsf{Ker}(\mathbf{w} \wedge \mathbf{v})$:

  $$\mathbb{R}\langle \mathbf{w}, \mathbf{v} \rangle^\perp \subset \mathsf{Ker}(\mathbf{w} \wedge \mathbf{v}).$$

Therefore every 2-dimensional linear subspace L (plane through the origin) of $\mathbb{R}^n$ determines an element of the projective space $\mathsf{P}\big(\mathfrak{o}(n)\big)$. The corresponding homogeneous coordinates are the *Plücker coordinates* of the plane, or the corresponding projective line $\mathsf{P}(L) \subset \mathsf{P}(\mathbb{R}^n)$.

Plücker coordinates in $\mathbb{P}^3$

Let $\mathsf{V} = \mathbb{R}^4$ and $\Lambda = \mathfrak{o}(4)$ the 6-dimensional vector space of 4×4 skew-symmetric matrices. Then lines in $\mathbb{P}^3 = \mathsf{P}(\mathsf{V})$ correspond to 2-dimensional linear subspaces of V, which in turn correspond to projective equivalence classes of certain nonzero elements $\mathbf{v} \wedge \mathbf{w} \in \Lambda$. Which elements of Λ correspond to lines in $\mathbb{P}^3$?

Since $\dim(\mathsf{V}) = 4$, the plane $\mathbb{R}\langle \mathbf{v}, \mathbf{w} \rangle \neq \mathsf{V}$ so there exists a nonzero vector $\mathbf{n}$ *normal* to this plane. By the above, $\mathbf{n}$ lies in the kernel of the skew-symmetric matrix $\mathbf{v} \wedge \mathbf{w}$. Thus lines in $\mathbb{P}^3$ determine nonzero *singular* matrices in Λ.

By the spectral theorem for real skew-symmetric matrices, the eigenvalues are purely imaginary and occur in complex conjugate pairs. For example, when $n = 3$, every element in $\mathfrak{o}(3)$ must have a zero eigenvalue (if zero occurs with higher multiplicity the matrix itself must be zero). This implies that every element of $\mathsf{SO}(3)$ is a rotation, for example.

When $n = 4$, then if 0 is not an eigenvalue, then the set of eigenvalues must be of the form

$$\{r_1 i, -r_1 i, r_2 i, -r_2 i\},$$

where $r_1, r_2 \in \mathbb{R}$ are nonzero. In particular such a matrix has determinant $r_1^2 r_2^2 > 0$. Since $\mathsf{Det}(\mathbf{v} \wedge \mathbf{w}) = 0$, but $\mathbf{v} \wedge \mathbf{w} \neq 0$, the multiplicity of 0 as an eigenvalue is exactly *two,* so the matrix $\mathbf{v} \wedge \mathbf{w}$ has *rank* 2.

When n is even, skew-symmetric matrices in $\mathfrak{o}(n)$ have the following curious property. In general the determinant of an $n \times n$ is a degree n polynomial in its entries. When n is even, there is a degree $n/2$ polynomial $\mathcal{P}$ on $\mathfrak{o}(n)$ (called the *Pfaffian*) such that if $M \in \mathfrak{o}(n)$, then

$$\mathsf{Det}(M) = \mathcal{P}(M)^2$$

for $M \in \mathfrak{o}(n)$. That is, in even dimensions, the determinant of a skew-symmetric matrix is a *perfect square.* For example, when $n = 2$, the general skew-symmetric matrix is

$$M = \begin{bmatrix} 0 & -y \\ y & 0 \end{bmatrix}$$

which has determinant y^2. Thus $\mathcal{P}(M) = y$, for example.

When $n = 4$, the Pfaffian is a quadratic polynomial. The general element of $\mathfrak{o}(4)$ is:

$$M := \begin{bmatrix} 0 & m_{12} & m_{13} & m_{14} \\ -m_{12} & 0 & m_{23} & m_{24} \\ -m_{13} & -m_{23} & 0 & m_{34} \\ -m_{14} & -m_{24} & -m_{34} & 0. \end{bmatrix}$$

which has determinant

$$\mathsf{Det}(M) = \big(m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34}\big)^2$$

so the Pfaffian is (up to a choice of -1):

$$\mathcal{P}(M) = m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34}.$$

The vector space Λ has dimension 6, with coordinates

$$m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}.$$

Thus projective equivalence classes of nonzero 4×4 skew-symmetric matrices is the projective space

$$\mathsf{P}(\Lambda) \cong \mathbb{P}^5$$

with homogeneous coordinates

$$[m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34}].$$

The nonzero singular matrices (namely, those of rank two), are those for which $\mathcal{P}(M) = 0$, which is just the homogeneous quadratic polynomial condition:

$$m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34} = 0$$

This defines a *quadric hypersurface* $\mathcal{Q}$ in $\mathbb{P}^5$. Since it is defined by one equation in a 5-dimensional space, this quadric has dimension 4.

Intuitively, we would expect that the space of lines in $\mathbb{P}^3$ has dimension 4. A generic line $\ell \subset \mathbb{P}^3$ is not ideal and does not pass through the origin. In that case there is a point

$$p(\ell) \in \mathbb{R}^3 \setminus \{0\}$$

closest to the origin $0 \in \mathbb{R}^3$. These points form a 3-dimensional space $\mathbb{R}^3 \setminus \{0\}$.

Any point $p \in \mathbb{R}^3 \setminus \{0\}$ is the closest point $p(\ell)$ for some ℓ. Namely, look at the plane $W(p)$ containing p and normal to the vector from 0 to p. Any line ℓ on $W(p)$ passing through p satisfies $p(\ell) = p$. The set of all lines ℓ with $p(\ell) = p$ forms a $\mathbb{P}^1$, which is one-dimensional. Thus lines in $\mathbb{P}^3$ are parametrized by a $4 = 3 + 1$-dimensional space.

This space is the quadric $\mathcal{Q}$ defined above.

Just as quadric surfaces in $\mathbb{P}^3$ can be parametrized as tori $S^1 \times S^1$, the 4-dimensional quadric hypersurface in $\mathbb{P}^5$ can be parametrized by $S^2 \times S^2$. Namely make the elementary linear substitution

$$\begin{aligned} X &:= (m_{14} + m_{23})/2, \quad & A &:= (m_{14} - m_{23})/2, \\ Y &:= (m_{13} - m_{24})/2, \quad & B &:= (m_{13} + m_{24})/2, \\ Z &:= (m_{12} + m_{34})/2, \quad & C &:= (m_{12} - m_{34})/2. \end{aligned}$$

so that

$$\begin{aligned} \mathcal{P}(M) &= m_{14}m_{23} - m_{13}m_{24} + m_{12}m_{34} \\ &= X^2 - A^2 + Y^2 - B^2 + Z^2 - C^2. \end{aligned}$$

Thus $\mathcal{Q}$ is the quadric in $\mathbb{P}^5$ consisting of points with homogeneous coordinates $[X : Y : Z : A : B : C]$ satisfying

$$X^2 + Y^2 + Z^2 = A^2 + B^2 + C^2.$$

Since the coordinates are real at least one is nonzero, this common sum-of-squares is positive. By rescaling we may suppose that that $X^2 + Y^2 + Z^2 = 1$ and $A^2 + B^2 + C^2 = 1$. Each of these equations describes a unit sphere in a 3-dimensional Euclidean space. Furthermore the coordinates (A, B, C) and (X, Y, Z) are independent of one another (we are looking at a *direct-sum decomposition* of $\mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3$), so that the quadric $\mathcal{Q}$ looks like $S^2 \times S^2$.

3.3.1. *Orthogonal Complement and Involution.* Since $\mathcal{P}$ is a homogeneous quadratic function on the vector space Λ, it arises from a symmetric bilinear form $\mathcal{P}$ on Λ by the usual correspondences:

$$\begin{aligned} \mathcal{P}(X) &= \mathcal{P}(X, X), \\ \mathcal{P}(X, Y) &:= \frac{1}{2}\Big(\mathcal{P}(X + Y) - \mathcal{P}(X) - \mathcal{P}(Y)\Big) \end{aligned}$$

Explicitly,

$$\mathcal{P}(M, N) = \frac{1}{2}(m_{14}n_{23} + m_{23}n_{14} - m_{13}n_{24} - m_{24}n_{13} + m_{12}n_{34} + m_{34}n_{12}).$$

The usual inner product (dot product) on $\mathfrak{o}(4)$ is given by

$$\begin{aligned} M \cdot N &= -\frac{1}{2}\mathsf{tr}(MN) \\ &= m_{12}n_{12} + m_{13}n_{13} + m_{14}n_{14} + m_{23}n_{23} + m_{24}n_{24} + m_{34}n_{34} \end{aligned}$$

Since the bilinear forms $\mathcal{P}$ and the above dot product define linear isomorphisms $\Lambda \xrightarrow{\cong} \Lambda^*$, they are related by a linear isomorphism $\Lambda \xrightarrow{\mathcal{I}} \Lambda$ defined by:

$$M \longmapsto \begin{bmatrix} 0 & m_{34} & -m_{24} & m_{23} \\ -m_{34} & 0 & m_{14} & -m_{13} \\ m_{24} & -m_{14} & 0 & m_{12} \\ -m_{23} & m_{13} & -m_{12} & 0. \end{bmatrix},$$

that is,

$$\mathcal{I}(m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) := (m_{34}, -m_{24}, m_{23}, m_{14}, -m_{13}, m_{12})$$

Clearly $\mathcal{I} \circ \mathcal{I} = I$; such a transformation is called an *involution.*

Geometrically, if $M \in \mathcal{Q}$ corresponds to a 2-dimensional linear subspace $L \subset \mathsf{V}$, then $\mathcal{I}(M)$ corresponds to its orthogonal complement $L^\perp \subset \mathsf{V}$.

If $p \in \mathbb{P}^3$ is a point corresponding to a 1-dimensional linear subspace $L \subset \mathsf{V}$, then its dual plane $p^* \subset \mathbb{P}^3$ corresponds to the orthogonal complement $L^\perp$. (The homogeneous coordinates of p^* form the *transpose* of the vector formed by the homogeneous coordinates of p.) Then $\mathcal{I}$ maps lines through p to the lines contained in the plane p^*.

Here is a basic example. Take p to be the origin $(0, 0, 0)$ in the standard affine patch; then p^* is the ideal plane. The line through 0 in the direction (a, b, c) has Plücker coordinates

$$M := \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \\ -a & -b & -c & 0 \end{bmatrix}.$$

Its dual is the ideal line, which in the ideal plane $\mathbb{P}^2_\infty$ has homogeneous coordinates $[\![a : b : c]\!]$ (that is, the line defined in homogeneous coordinates $aX + bY + cZ = 0$. In $\mathbb{P}^3$ this line has Plücker coordinates:

$$\mathcal{I}(M) := \begin{bmatrix} 0 & c & -b & 0 \\ -c & 0 & a & 0 \\ b & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3.3.2. *Relation to* Orth. We can relate this to the alternating trilinear function Orth (the four-dimensional cross product). It can be defined in terms of the involution $\mathcal{I}$ and exterior outer product $\wedge$:

$$\mathsf{Orth}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathcal{I}\big((\mathbf{w} \wedge \mathbf{u})(\mathbf{v})\big)$$

Three points $[\mathbf{u}], [\mathbf{v}], [\mathbf{w}] \in \mathbb{P}^3$ (where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathsf{V}$ are nonzero vectors are collinear if and only if $\mathsf{Orth}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0$. Otherwise they span a plane in $\mathbb{P}^3$ represented by $[\mathsf{Orth}(\mathbf{u}, \mathbf{v}, \mathbf{w})^\dagger]$.

4. Appendix: The algebra of Quaternions

4.1. **Generalizing complex numbers.** Let $\mathbb{H}$ denote a four-dimensional vector space with basis denoted $\mathbf{1}, \hat{\imath}, \hat{\jmath}, \hat{k}$. Let $\mathbb{H}_0$ be the 3-dimensional vector space based on $\hat{\imath}, \hat{\jmath}, \hat{k}$, regarded as vectors in $\mathbb{R}^3$. The *bilinear* map

$$\begin{aligned} \mathbb{H} \times \mathbb{H} &\longrightarrow \mathbb{H} \\ (\mathbf{q}_1, \mathbf{q}_2) &\longmapsto \mathbf{q}_1\mathbf{q}_2 := -(\mathbf{q}_1 \cdot \mathbf{q}_2)\mathbf{1} + (\mathbf{q}_1 \times \mathbf{q}_2) \end{aligned}$$

is called *quaternion multiplcation. Quaternion conjugation* is the *linear* map:

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathbb{H} \\ \mathbf{q} := r\mathbf{1} + \mathbf{q}_0 &\longmapsto \bar{\mathbf{q}} := r\mathbf{1} - \mathbf{q}_0 \end{aligned}$$

where $r \in \mathbb{R}$ is the *real part* $\mathsf{Re}(\mathbf{q})$ and $\mathbf{q}_0 \in \mathbb{H}_0$ is the *imaginary part* $\mathsf{Im}(\mathbf{q})$. Then

$$\mathbf{q}\bar{\mathbf{q}} = \|\mathbf{q}\|^2 = r^2 + \|\mathbf{q}_0\|^2 \geq 0$$

and equals zero if and only if $\mathbf{q} = 0$. Thus if $\mathbf{q} \in \mathbb{H}$ is nonzero, then it is multiplicatively invertible, with its inverse defined by:

$$\mathbf{q}^{-1} := \|qq\|^{-2}\bar{\mathbf{q}}$$

just like for complex numbers.

Thus $\mathbb{H}$ is a *division algebra* (or *noncommutative field.*

The quaternions generalize complex numbers, built from the field $\mathbb{R}$ of real numbers by adjoining *one* root $\hat{\imath}$ of the equation $z^2 = 1$. Note that by adjoining one $\sqrt{-1}$, there is *automatically* a *second* one, namely $-\sqrt{-1}$. This is a special case of the *Fundamental Theorem of Algebra,* that (counting mulitiplicities) a polynomial equation of degree n admits n complex roots.

However, there is no ordering on the field $\mathbb{C}$ of complex numbers, that is, there is no meaningful sense of a "positive" or "negative" complex number. Thus there is no essential difference between $\hat{\imath}$ and $-\hat{\imath}$. This *algebraic symmetry* gives rise to the field automorphism of *complex conjugation:*

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \bar{z} \end{aligned}$$

The quaternions arise by adjoining *three* values of $\sqrt{-1}$, each in one of the coordinate directions of $\mathbb{R}^3$. Thus we obtain 6 values of $\sqrt{-1}$, but in fact there are *infinitely many* square-roots of -1, one in *every* direction in $\mathbb{R}^3$.

However, these basic quaternion don't commute, but rather *anti-commute:*

$$\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath} = \hat{k}$$
$$\hat{\jmath}\hat{k} = -\hat{k}\hat{\jmath} = \hat{\imath}$$
$$\hat{k}\hat{\imath} = -\hat{\imath}\hat{k} = \hat{\jmath}$$

Recall that (multi)linear maps of vector spaces can be uniquely determined by their values on a basis. These can be succinctly expressed in terms of *tables* as follows. Multiplication tables for the dot and cross products of vectors in $\mathbb{R}^3 = \mathbb{H}_0$ are:

$\cdot$	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\hat{\imath}$	1	0	0
$\hat{\jmath}$	0	1	0
$\hat{k}$	0	0	1

$\times$	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\hat{\imath}$	0	$\hat{k}$	$-\hat{\jmath}$
$\hat{\jmath}$	$-\hat{k}$	0	$\hat{\imath}$
$\hat{k}$	$\hat{\jmath}$	-$\hat{\imath}$	0

We can describe quaternion operations by their *tables* as they are multilinear. For example, quaternion conjugation is described in the basis $(\mathbf{1}, \hat{\imath}, \hat{\jmath}, \hat{k})$ as:

$\mathbf{1}$	$\mathbf{1}$
$\hat{\imath}$	-$\hat{\imath}$
$\hat{\jmath}$	-$\hat{\jmath}$
$\hat{k}$	-$\hat{k}$

Here is the multiplication table for quaternion multiplication:

	$\mathbf{1}$	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\mathbf{1}$	$\mathbf{1}$	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\hat{\imath}$	$\hat{\imath}$	$-\mathbf{1}$	$\hat{k}$	$-\hat{\jmath}$
$\hat{\jmath}$	$\hat{\jmath}$	$-\hat{k}$	$-\mathbf{1}$	$\hat{\imath}$
$\hat{k}$	$\hat{k}$	$\hat{\jmath}$	$-\hat{\imath}$	$-\mathbf{1}$

A *unit quaternion* is a quaternion $\mathbf{q} \in \mathbb{H}$ such that $\|\mathbf{q}\| = 1$. Unit quaternions form the *unit* 3*-sphere* $S^3 \subset \mathbb{R}^4$. The *imaginary unit quaternions* $\mathbb{H}_1$ form a 2-sphere

$$S^2 \subset \mathbb{H}_0 = \mathbb{R}^3.$$

Note that if $\mathsf{u} \in \mathbb{H}_1$ is an imaginary unit quaternion then $\mathsf{u}^2 = -1$. This gives the *infinitely many* square-roots of -1 promised earlier. Furthermore, since

$$\mathsf{u}^n = \begin{cases} 1 & \text{if } n \equiv 0(\mathsf{mod}4) \\ \mathsf{u} & \text{if } n \equiv 1(\mathsf{mod}4) \\ -1 & \text{if } n \equiv 2(\mathsf{mod}4) \\ -\mathsf{u} & \text{if } n \equiv 3(\mathsf{mod}4) \end{cases}$$

the usual calculation with power series implies:

$$\begin{aligned} \exp(\theta\mathsf{u}) &= \sum_{n=0}^{\infty} \frac{1}{n!}(\theta\mathsf{u})^n \\ &= \cos(\theta)\mathbf{1} + \sin(\theta)\mathsf{u} \end{aligned}$$

just like $e^{\hat{\imath}\theta} = \cos(\theta) + \hat{\imath}\sin\theta$ for complex numbers.

Futhermore, if $\mathsf{v} \in \mathbb{H}_0$ represents a vector in $\mathbb{R}^3$, then rotation in the unit vector u by angle θ is:

$$\mathsf{v} \xrightarrow{\mathsf{Rot}_{\mathsf{u}}^{\theta}} \exp(\theta/2\mathsf{u})\ \mathsf{v}\ \exp(-\theta/2\mathsf{u})$$

The usual Euclidean inner product on $\mathbb{R}^4$ is given in terms of quaternions $\mathbb{H} \cong \mathbb{R}^4$ by:

$$\mathsf{v} \cdot \mathsf{w} = \mathsf{Re}(\mathsf{v}\bar{\mathsf{w}}),$$

again, just like the analogous formula for complex numbers.