1. Quaternions

1.1. Generalizing complex numbers. Let \( \mathbb{H} \) denote a four-dimensional vector space with basis denoted \( 1, \hat{i}, \hat{j}, \hat{k} \). Let \( \mathbb{H}_0 \) be the 3-dimensional vector space based on \( \hat{i}, \hat{j}, \hat{k} \), regarded as vectors \( v \in \mathbb{R}^3 \). The bilinear map \( \mathbb{H}_0 \times \mathbb{H}_0 \to \mathbb{H} \), \( (v_1, v_2) \mapsto v_1 v_2 := -(v_1 \cdot v_2)1 + (v_1 \times v_2) \), is called quaternion multiplication. Now (bilinearly) extend quaternion multiplication to \( \mathbb{H} = \mathbb{R}1 \oplus \mathbb{H}_0 \) by “making” \( 1 \) a two-sided identity element for quaternion multiplication (\( q = 1q = q1 \)):

\[
(r_1 1 + v_1)(r_2 1 + v_2) := (r_1 r_2 - v_1 \cdot v_2)1 + (r_1 v_2 + r_2 v_1 + v_1 \times v_2)
\]

(Remember that quaternion multiplication is not commutative!)

Quaternion conjugation is the linear map:

\[
\mathbb{H} \to \mathbb{H}
\]

\[
q := r 1 + q_0 \mapsto \bar{q} := r 1 - q_0
\]

where \( r \in \mathbb{R} \) is the real part \( \text{Re}(q) \) and \( q_0 \in \mathbb{H}_0 \) is the imaginary part \( \text{Im}(q) \). Then

\[
q \bar{q} = \|q\|^2 = r^2 + \|q_0\|^2 \geq 0
\]

and equals zero if and only if \( q = 0 \). Thus if \( q \in \mathbb{H} \) is nonzero, then it is multiplicatively invertible, with its inverse defined by:

\[
q^{-1} := \|q\|^{-2} \bar{q}
\]

just like for complex numbers.

Thus \( \mathbb{H} \) is a division algebra (or noncommutative field).
The quaternions generalize complex numbers, built from the field \( \mathbb{R} \) of real numbers by adjoining one root \( i \) of the equation \( z^2 = 1 \). Note that by adjoining one \( \sqrt{-1} \), there is automatically a second one, namely \( -\sqrt{-1} \). This is a special case of the Fundamental Theorem of Algebra, that (counting multiplicities) a polynomial equation of degree \( n \) admits \( n \) complex roots.

However, there is no ordering on the field \( \mathbb{C} \) of complex numbers, that is, there is no meaningful sense of a “positive” or “negative” complex number. Thus there is no essential difference between \( i \) and \( -i \). This algebraic symmetry gives rise to the field automorphism of complex conjugation:

\[
\mathbb{C} \longrightarrow \mathbb{C} \\
z \longmapsto \bar{z}
\]

The quaternions arise by adjoining three values of \( \sqrt{-1} \), each in one of the coordinate directions of \( \mathbb{R}^3 \). Thus we obtain 6 values of \( \sqrt{-1} \), but in fact there are infinitely many square-roots of \(-1\), one in every direction in \( \mathbb{R}^3 \). This enhances the algebraic symmetry of \( \mathbb{C} \) since there is no essential difference between \( i \) and any other unit imaginary quaternion \( u \in \mathbb{H}_1 \).

1.2. Noncommutativity. However, these basic quaternion don’t commute, but rather anti-commute:

\[
\hat{i} \hat{j} = -\hat{j} \hat{i} = \hat{k} \\
\hat{j} \hat{k} = -\hat{k} \hat{j} = \hat{i} \\
\hat{k} \hat{i} = -\hat{i} \hat{k} = \hat{j}
\]

Recall that (multi)linear maps of vector spaces can be uniquely determined by their values on a basis. These can be succinctly expressed in terms of tables as follows. Multiplication tables for the dot and cross products of vectors in \( \mathbb{R}^3 = \mathbb{H}_0 \) are:

\[
\begin{array}{c|ccc}
& \hat{i} & \hat{j} & \hat{k} \\
\hline
\hat{i} & 1 & 0 & 0 \\
\hat{j} & 0 & 1 & 0 \\
\hat{k} & 0 & 0 & 1 \\
\end{array}
\]
We can describe quaternion operations by their tables as they are multilinear. For example, quaternion conjugation is described in the basis $(1, \hat{i}, \hat{j}, \hat{k})$ as:

<table>
<thead>
<tr>
<th>$\times$</th>
<th>$\hat{i}$</th>
<th>$\hat{j}$</th>
<th>$\hat{k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{i}$</td>
<td>0</td>
<td>$\hat{k}$</td>
<td>$-\hat{j}$</td>
</tr>
<tr>
<td>$\hat{j}$</td>
<td>$-\hat{k}$</td>
<td>0</td>
<td>$\hat{i}$</td>
</tr>
<tr>
<td>$\hat{k}$</td>
<td>$\hat{j}$</td>
<td>$-\hat{i}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Here is the multiplication table for quaternion multiplication:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$\hat{i}$</th>
<th>$\hat{j}$</th>
<th>$\hat{k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$\hat{i}$</td>
<td>$\hat{j}$</td>
<td>$\hat{k}$</td>
</tr>
<tr>
<td>$\hat{i}$</td>
<td>$\hat{i}$</td>
<td>$-1$</td>
<td>$\hat{k}$</td>
<td>$-\hat{j}$</td>
</tr>
<tr>
<td>$\hat{j}$</td>
<td>$\hat{j}$</td>
<td>$-\hat{k}$</td>
<td>$-1$</td>
<td>$\hat{i}$</td>
</tr>
<tr>
<td>$\hat{k}$</td>
<td>$\hat{k}$</td>
<td>$\hat{j}$</td>
<td>$-\hat{i}$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

1.3. **The rotation formula.** A unit quaternion is a quaternion $q \in \mathbb{H}$ such that $\|q\| = 1$. Unit quaternions form the unit 3-sphere $S^3 \subset \mathbb{R}^4$. The unit imaginary quaternions $\mathbb{H}_1$ form a 2-sphere $S^2 \subset \mathbb{H}_0 = \mathbb{R}^3$.

Note that if $u \in \mathbb{H}_1$ is an unit imaginary quaternion then $u^2 = -1$. This gives the infinitely many square-roots of $-1$ promised earlier. Furthermore, since

$$u^n = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{4} \\ u & \text{if } n \equiv 1 \pmod{4} \\ -1 & \text{if } n \equiv 2 \pmod{4} \\ -u & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

the usual calculation with power series implies:

$$\exp(\theta u) = \sum_{n=0}^{\infty} \frac{1}{n!} (\theta u)^n = \cos(\theta) 1 + \sin(\theta) u$$
just like \( e^{i\theta} = \cos(\theta) + i\sin(\theta) \) for complex numbers. In fact, for each \( u \in \mathbb{H}_1 \), the 2-dimensional subspace \( \mathbb{R}1 + \mathbb{R}u \subset \mathbb{H} \) is a subalgebra isomorphic to the field \( \mathbb{C} \) (a “copy” of \( \mathbb{C} \) inside \( \mathbb{H} \)).

Furthermore, if \( v \in \mathbb{H}_0 \) represents a vector in \( \mathbb{R}^3 \), then rotation in the unit vector \( u \) by angle \( \theta \) is:

\[
(1) \quad v \xrightarrow{\text{Rot}^\theta_u} \exp\left(\frac{\theta}{2}u\right)v \exp\left(-\frac{\theta}{2}u\right)
\]

Notice that, if multiplication were commutative, (1) would degenerate and just give the identity map \( v \mapsto v \).

1.4. Euler angles. The classical terms pitch, yaw and roll correspond to rotations about the \( x \)-axis, \( y \)-axis, and \( z \)-axis, respectively:

\[
\text{Rot}^\theta_x = \begin{bmatrix} \cos(\theta_x) & -\sin(\theta_x) & 0 \\ \sin(\theta_x) & \cos(\theta_x) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{Rot}^\theta_y = \begin{bmatrix} \cos(\theta_y) & 0 & -\sin(\theta_y) \\ 0 & 1 & 0 \\ \sin(\theta_y) & 0 & \cos(\theta_y) \end{bmatrix},
\]

\[
\text{Rot}^\theta_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_z) & -\sin(\theta_z) \\ 0 & \sin(\theta_z) & \cos(\theta_z) \end{bmatrix}
\]

An arbitrary rotation may be expressed as a composition

\[
\text{Rot}^\theta_x \text{Rot}^\theta_y \text{Rot}^\theta_z
\]

but the expression fails to be unique. (This problem is fundamental, due to the complicated topology of the group \( \text{SO}(3) \) of rotations.) For example,

\[
\text{Rot}^\theta_x \text{Rot}^{\pi/2}_y \text{Rot}^\theta_z = \begin{bmatrix} 0 & 0 & -1 \\ \sin(\theta_x - \theta_z) & \cos(\theta_x - \theta_z) & 0 \\ \cos(\theta_x - \theta_z) & -\sin(\theta_x - \theta_z) & 0 \end{bmatrix}
\]

so the angles \( \theta_x, \theta_z \) are not independent — this loss of one degree of freedom is called gimbal lock. (Marsh, p. 51) Indeed, the group \( \text{SO}(3) \) of rotations is homeomorphic to projective space \( \mathbb{P}^3 \)!
2. Problems

(1) Solve the quaternion equations
\[ (-1 + 2i - 3j + 4k)q = (5 + 4i + 3j + 2k) \]
\[ q(-1 + 2i - 3j + 4k) = (5 + 4i + 3j + 2k) \]
for \( q = r + xi + yj + zk \). Explicitly find the two sets of values for \( r, x, y, z \in \mathbb{R} \) for \( q \).

(2) We explore the rotation formula (1). Taking \( u \) to be a unit imaginary quaternion (so it represents a unit vector in \( S^2 \subset \mathbb{R}^3 \)), show that the mapping \( \mathbb{H} \xrightarrow{\varphi} \mathbb{H} \)
\[ q \mapsto \exp \left( \frac{\theta}{2} u \right) q \exp \left( -\frac{\theta}{2} u \right) \]
satisfies
\[ \varphi(r1 + q_0) = r1 + \varphi(q_0). \]
In particular it defines a transformation of \( \mathbb{H}_0 = \mathbb{R}^3 \) to itself.

(3) What happens when \( u = k \)? Write \( q_0 = q_1 + zk \), where \( q_1 = xi + yj \) can be considered a vector in the plane. Compute \( \varphi(q_0) \)
explicitly in terms of its components \( q_1 = xi + yj \) and \( zk \).

(4) Explain how using ideas of symmetry, this proves the rotation formula in general.

(5) Find all quaternion solutions \( x \in \mathbb{H} \) of the equations \( x^2 = q \),
where \( q = 2, -2 \) and \( k \).

(6) Prove or disprove: A quaternion \( q \) commutes with every quaternion if and only if \( q \in \mathbb{R} \).

(7) Find a general formula for \( \exp(tq) \) where \( q \) is a quaternion (not necessarily unit imaginary) and \( t \in \mathbb{R} \).

(8) Prove or disprove: Suppose \( q \in \mathbb{H} \) is a quaternion and \( \exp(tq) \) is real for all \( t \in \mathbb{R} \). Then \( q \) is real.

(9) Prove or disprove: Suppose \( q \in \mathbb{H} \) is a quaternion and \( \exp(tq) \)
is real for some \( t \in \mathbb{R} \). then \( x \) is real.

(10) (Bonus problem) Describe a homeomorphism of \( SO(3) \) with \( \mathbb{P}^3 \) as the closed ball \( B \) of radius \( \pi \) in \( \mathbb{R}^3 \) with its boundary identified by the antipodal map. Namely, associate to a vector \( \theta u \) (where \( u \in S^2 \) is a unit vector, \(-\pi \leq \theta \leq \pi \), the rotation \( \text{Rot}^\theta_u \).
Explain the identifications
\[ \text{Rot}^\theta_u \leftrightarrow \text{Rot}_{-\theta}^{-u}, \quad \text{Rot}^{\pi}_{-u} \leftrightarrow \text{Rot}^\pi_{-u}. \]