FLAT AFFINE, PROJECTIVE AND CONFORMAL STRUCTURES ON MANIFOLDS: A HISTORICAL PERSPECTIVE

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Abstract. This historical survey reports on the theory of locally homogeneous geometric structures as initiated in Ehresmann's 1936 paper *Sur les espaces localement homogènes.* Beginning with Euclidean geometry, we describe some highlights of this subject and threads of its evolution. In particular, we discuss the relationship to the subject of discrete subgroups of Lie groups. We emphasize the classification of geometric structures from the point of view of fiber spaces and the later work of Ehresmann on infinitesimal connections. The *holonomy principle,* first isolated by W. Thurston in the late 1970's, relates the classification with the representation variety $\mathsf{Hom}\big(\pi_1(\Sigma), G\big)$. We briefly survey recent results in flat affine, projective, and conformal structures, in particular the tameness of developing maps and uniqueness of structures with given holonomy.

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Introduction

On 23 October 1935, at the Geneva conference "Quelques questions de Geométrie et de Topologie," Charles Ehresmann [57] initiated the study of geometric structures *modeled on a homogeneous space* (X, G), or *locally homogeneous geometric structures* on manifolds. Here G is a Lie group and X a homogeneous space, representing a geometry in the sense of Klein's Erlanger program. A geometric structure is defined by an atlas of coordinate charts mapping into X with coordinate changes locally defined by transformations in G. We call such a structure *a* (G, X)*-structure,* and a manifold equipped with such a structure a (G, X)*-manifold.* A (G, X)-manifold M inherits all of the local geometry of X invariant under G.

These ideas were heavily influenced by Sophus Lie, Felix Klein, Henri Poincaré and Élie Cartan among others. Lie and Klein recognized how a group-theoretic viewpoint unified the disparate *classical geometries:* for them, a *geometry* consists of the properties of a space X upon which a group G acts transitively by symmetries *of that geometry.* Transitivity of the action means that the local geometries at any pair of points are equivalent. For example for *Euclidean geometry,* X is Euclidean space $\mathbb{E}^n$ and G is its group of isometries. Poincaré introduced the fundamental group $\pi_1(\Sigma, x_0)$ of a topological space Σ, consisting of loops based at a fixed (but arbitrary) base point $x_0 \in \Sigma$. Cartan introduced a general notion of *development* along paths, which corresponds to parallel transport of infinitesimal objects (for example, tangent vectors and frames) along paths. Development for an Ehresmann structure M modeled on a geometry (G, X) defines a homomorphism $\pi_1(M) \longrightarrow G$ compatible with a local homeomorphism $\widetilde{M} \longrightarrow X$.

Ehresmann begins with Riemannian manifolds of constant curvature, which he calls *Clifford-Klein space forms.* Such manifolds are locally modeled on Euclidean space $\mathbb{E}^n$, the sphere S^n, or hyperbolic space H^n, depending on whether the curvature is zero, positive or negative, respectively. Indeed, for these geometries, a (G, X)-structure is completely equivalent to a Riemannian metric of constant sectional curvature. The key property upon which he focuses is that any two points in such a space possess open neighborhoods which are isometric, that is, they have the *same local geometries.* He considers the more general situation of a manifold X with a transitive left action of a Lie group G; choosing a point $x \in X$,

$$\begin{aligned} G &\longrightarrow X \\ g &\longmapsto g(x) \end{aligned}$$

maps G (with its simply transitive group of left-multiplications) G-equivariantly to X. Furthermore this map passes down to a isomorphism of left G-spaces $\mathsf{Stab}(G, x)\backslash G \longrightarrow X$, where

$$\mathsf{Stab}(G, x) := \{g \in G \mid g(x) = x\}$$

is the *stabilizer* of x in G. In modern parlance, X is a *homogeneous space* of G.

He then defines a *locally homogeneous space* to be a manifold M (having the same dimension as X) which is *locally modeled* on the G-invariant geometry of X. Specifically, M is covered by open neighborhoods, *coordinate patches,* U (which Ehresmann calls "elementary neighborhoods") equipped with homeomorphisms, *coordinate charts,* $U \xrightarrow{\psi} X$. The coordinate charts transfer the local G-invariant geometry of X to U.

Coordinate patches U and U' with corresponding charts ψ and ψ' respectively define possibly competing geometries on the intersection

$U \cap U'$. Thus we require that ψ and ψ' define the same local geometry on $U \cap U'$: that is, each $p \in U \cap U'$ possesses an open neighborhood $V \subset U \cap U'$ such that $\psi'|_V = g \circ \psi|_V$ for some $g \in G$. If we require that G acts effectively on X, then g will be uniquely determined. Furthermore g only depends on the connected component of $U \cap U'$ containing p.

This is what he calls a *locally homogeneous space of Lie,* to distinguish it from a *homogeneous space of Lie,* and he observes that every homogeneous space is locally homogeneous. The main question addressed in the paper is to what extent the converse holds.

He begins with the observation that a locally homogeneous space is a real analytic manifold. Although at the time, the global notion of a Lie group had not been popularized (Chevalley's book [36] would not be published for at least another decade), Ehresmann spends some time clarifying the relation between local groups of transformations on X.

In [57], Ehresmann calls M a *Clifford form* of X.

Ehresmann structures modeled on a Lie group G and its group of left-translations arise from discrete subgroups of G, at least under the assumption that the developing map is a covering space. Certainly when M is compact, such a (G, X)-structure corresponds to a discrete subgroup $\tilde{\Gamma} \subset \tilde{G}$ and an isomorphism $M \cong \tilde{\Gamma} \backslash \tilde{G}$. (For more information see [78].)

In the literature, "locally homogeneous spaces" sometimes refer to biquotients $\Gamma \backslash G/H$, since "homogeneous space" may refer to the quotient G/H. (Here $\Gamma \subset G$ is a discrete subgroup and $H \subset G$ is a closed subgroup, so that G/H is Hausdorff. Furthermore Γ is assumed to act *properly* on G/H, which, unless H is compact, is a nontrivial assumption on Γ.) If, in addition, Γ is asssumed to act freely on G/H, then the double coset space $\Gamma \backslash G/H$ admits the natural structure of a (G, X)-manifold M, where $X = G/H$. Under various completeness assumptions (see below), "locally homogeneous" in our sense will imply that the geometric manifold is indeed a double coset space.

The case when H is compact implies that X carries a G-invariant Riemannian metric which is necessarily *geodesically complete,* and the Hopf-Rinow theorem implies that a *closed* (G, X)-manifold is a double coset space in the above sense. These basic examples are particularly tame, although nonetheless extremely rich.

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1. Euclidean manifolds

The most familiar geometry is *Euclidean geometry.* Euclidean geometry includes relations between points, lines, planes, and measurements such as distance, angle, area and volume. A more sophisticated aspect of Euclidean geometry is the theory of harmonic functions and Laplace's equation. The key property is that these objects, and the relations between them, are invariant under the transitive action of the isometry group $G = \mathsf{Isom}(\mathbb{E}^n)$ of Euclidean space $\mathbb{E}^n$.

The model space $X = \mathbb{E}^n$ is defined as the vector space $\mathbb{R}^n$ (or more accurately the affine space A^n, where the special significance of the additive identity $\mathbf{0} \in \mathbb{R}^n$ is removed). The isometry group is generated by the *group* of translations (identified as the vector space $\mathbb{R}^n$), and the group $\mathsf{O}(n)$ of orthogonal linear automorphisms of $\mathbb{R}^n$. Thus $G \cong \mathbb{R}^n \rtimes \mathsf{O}(n)$ and $X = \mathbb{E}^n$ identifies with the homogeneous space G/H where $H := \mathsf{Stab}(G, \mathbf{0}) = \mathsf{O}(n)$ is the stabilizer of the origin $\mathbf{0} \in \mathbb{R}^n$.

1.1. **Riemannian geometry.**

Euclidean structures are *flat Riemannian structures,* that is, Riemannian structures whose curvature tensor vanishes. We adopt the viewpoint that the Riemannian structure is the geometry defined by a Riemannian metric (tensor), which then leads to notions of *speed* and *length* of smooth curves, and finally the structure of a metric space.

The key point is that for $X = \mathbb{E}^n$, a positive innner product on the associated vector space $\mathbb{V} = \mathbb{R}^n$ extends to a $G = \mathsf{Isom}(\mathbb{E}^n)$-invariant metric tensor on X. (The G-invariant tensor is uniquely determined up to scaling by a nonzero constant.) This infinitesimal structure makes X into a metric space upon which G acts isometrically.

If M is a manifold, then an Ehresmann structure on M modeled on Euclidean geometry is essentially equivalent to a metric space locally isometric to a Euclidean structure. (We say "essentially" because the distance function is determined up to scaling by positive constant.) Equivalently, this is just a Riemannian metric which is *flat,* that is, one whose Riemann curvature tensor vanishes. Other Ehresmann structures can be defined in a similar way, using an infinitesimal form (*Cartan connections*) such that an object generalizing the curvature tensor vanishes. (Compare Sharpe [132].)

However, the theory of Euclidean manifolds really goes back much earlier, to crystallography and the theory of regular tilings of Euclidean space. Once the abstract notion of a *group of transformations* was formulated, late nineteenth-century crystallographers such as Schoenflies and Fedorov classified *crystallographic groups,* namely symmetry groups of tilings of $\mathbb{E}^3$ by compact polyhedra. These are the mathematical abstractions of *crystals*. In arbitrary dimension a *Euclidean crystallographic group* is a subgroup $\Gamma \subset \mathsf{Isom}(\mathbb{E}^n)$ acting properly discontinuously on $\mathbb{E}^n$ with compact quotient (equivalently, a compact fundamental domain).

A compact flat Riemannian manifold M (that is, a *Euclidean manifold*) determines a crystallographic group. The Hopf-Rinow theorem (see §5 below) implies that the universal covering space $\widetilde{M}$ is isometric to $\mathbb{E}^n$, and the group $\pi_1(M)$ of deck transformations acts properly and isometrically on $\mathbb{E}^n$. Conversely, if $\Gamma \subset \mathsf{Isom}(\mathbb{E}^n)$ is discrete, then it acts properly and isometrically on E^n. If, furthermore, Γ is torsion-free, it acts freely on $\mathbb{E}^n$ and the quotient $M := \Gamma\backslash\mathbb{E}^n$ is a manifold. In particular M identifies with a double coset space $\Gamma\backslash G/H$. Since $H = \mathsf{O}(n)$ is compact, the homogeneous space $\Gamma\backslash G$ is compact if and only if the locally homogeneous space $\Gamma\backslash G/H$ is compact. Crystallographic groups, then, are just discrete subgroups $\Gamma \subset \mathsf{Isom}(\mathbb{E}^n)$ which are *cocompact,* that is, when $\Gamma\backslash\mathsf{Isom}(\mathbb{E}^n)$ is compact. In the case that $G = \mathsf{Isom}(\mathbb{E}^n)$, this is equivalent to Γ being a *lattice* in G, namely a discrete subgroup such that $\Gamma\backslash G$ has finite Haar measure. Thus the classification of crystallographic groups is equivalent to the classification of lattices in $\mathsf{Isom}(\mathbb{E}^n)$. (See Milnor [123] and the references cited there for an excellent exposition of these ideas and their histroical motivation.)

1.2. **The Bieberbach theorems.** In 1911, Bieberbach proved a *Structure Theorem* for crystallographic groups. Namely, the *linear holonomy* $\Gamma \xrightarrow{\mathbb{L}} \mathsf{O}(n)$ defined by the (constant) derivative of the isometry $\gamma \in \Gamma$ has finite image. Its kernel consists of all translations in Γ, and $\mathsf{Ker}(\mathbb{L}) = \Gamma \cap \mathbb{R}^n$ is a lattice Λ in $\mathbb{R}^n$ (the additive group spanned by a basis of $\mathbb{R}^n$). The geometric version is that *a compact Euclidean manifold* admits a finite covering space whose total space is a *flat torus* $\mathbb{R}^n/\Lambda$.

He also proved a *Rigidity Theorem* and a *Finiteness Theorem* for Euclidean manifolds. Euclidean manifolds are *rigid* in the following sense: Every isomorphism $\Gamma_1 \to \Gamma_2$ of crystallographic groups extends to an *affine* automorphism of $\mathbb{E}^n$ conjugating Γ_1 to Γ_2. Observe that the rigidity is up to *affine equivalence,* not *Euclidean isometry.* While *isometry* classes of marked Euclidean n-manifolds comprise a deformation space with rich geometry $\big($identifying with $\mathsf{GL}(\mathbb{R}^2)/\mathsf{O}(2)\big)$, the deformation space of affine equivalence classes of marked Euclidean n-manifolds is a point.

Finally any n admits finitely many isomorphism classes of crystallographic subgroups $\Gamma \subset \mathsf{Isom}(\mathbb{E}^n)$. For $n = 2$, only the torus and Klein bottle have Euclidean structures. For $n = 3$ only six orientable 3-manifolds admit Euclidean structures.

These three theorems provide a satisfactory qualitative picture of Euclidean structures on closed manifolds. Compare Wolf [148], Raghunathan [129], and Thurston [138].

By the Rigidity Theorem above, it seems natural to consider more general *affine crystallographic groups,* namely discrete subgroups $\Gamma \subset \mathsf{Aff}(\mathsf{A}^n)$ such that Γ acts properly on A^n, and the quotient $\Gamma\backslash\mathsf{A}^n$ is compact. (Our notation emphasizes *context:* A^n denotes the *affine space* underlying $\mathbb{E}^n$: that is, A^n "is" $\mathbb{E}^n$, but without the special structure defined by the Euclidean inner product. Similarly A^n "is" $\mathbb{R}^n$, but without the special structure given by the additive identity $\mathbf{0} \in \mathbb{R}^n$.)

1.3. **Affine crystallographic groups.** In dimension three, all three of Bieberbach's theorem fail for affine crystallographic groups: the image of the linear holonomy is generally infinite, there are generally infinitely many affine isomorphism types in a given topological type, and there are infinitely many topological types (Auslander [7]). Auslander and Markus [6] construct 3-dimensional flat *Lorentzian* manifolds which are *geodesically complete:* these are quotients $M := \Gamma\backslash\mathsf{A}^n$ by discrete subgroups $\Gamma \subset G = \mathsf{Aff}(\mathsf{A}^n)$ which act properly on A^n with compact quotient. The situation is now much more tricky, since M is a biquotient $\Gamma\backslash G/H$, where $G = \mathsf{Aff}(\mathsf{A}^n)$. However, since $H = \mathsf{Stab}(G, \mathbf{0}) = \mathsf{GL}(\mathbb{R}^n)$ is nocompact, generally discrete subgroups of G will *not* act properly on $X = \mathsf{A}^n$.

A structure theorem analogous to the Bieberbach's theorem may hold in this context, but presently is not known in general. This is the famous "Auslander Conjecture," since it was erroneously claimed in Auslander [8]. The assertion is that the fundamental group (or affine holonomy group) Γ is necessarily virtually solvable. This was proved in dimension three by Fried-Goldman[66], and Abels–Margulis–Soifer [2] have proved this in all dimensions < 6.

The Auslander Conjecture implies the following Structure Theorem: If $\Gamma \subset \mathsf{Aff}(\mathsf{A}^n)$ is an affine crystallographic group, then there exists a subgroup $G \subset \mathsf{Aff}(\mathsf{A}^n)$ such that

- G has finitely many connected components; $\Gamma \subset G$ is a lattice;
- The identity component G^0 acts simply transitively on A^n.

The last condition means that G inherits a left-invariant complete affine structure, and a finite-sheeted covering space of the complete affine manifold $M = \mathsf{A}^n/\Gamma$ identifies with the homogeneous space $\Gamma\backslash G^0$. The group G replaces the group of translations in the Euclidean (Bieberbach) case. For details, see the first section of [66]; we call Γ a *crystallographic hull,* but generally is not unique.

When M is not required to be compact, then many examples are now known where Γ is not virtually solvable. The first ones were constructed in the late 1970's by Margulis [118, 117], where Γ is a nonabelian free group, and $n = 3$. For more information, see Abels [1], Charette-Drumm-Goldman-Morrill [32], Fried-Goldman [66], Milnor [126] and [79].

The three-dimensional examples found by Auslander [7] and Auslander-Markus [6] have special significance. Such a 3–manifold M^3 is a 2-torus bundle over S^1, and is a mapping torus of a linear automorphism of a flat torus T^2. We can identify T^2 as $\mathbb{R}^2/\mathbb{Z}^2$, and the automorphism corresponds to $A \in \mathsf{GL}(2,\mathbb{Z})$. The monodromy A is periodic if and only if M^3 is a Euclidean manifold, in which case the fundamental group Γ is a classical crystallographic group. When A is parablolic, then Γ is nilpotent and nonabelian, its crystallographic hull is the 3-dimensional Heisenberg group Nil, and M^3 is a nilmanifold.

The most interesting case arises when A is hyperbolic. In that case, the crystallographic hull G is the semidirect product $\mathbb{R}^2 \ltimes \mathbb{R}$ where $\mathbb{R}$ acts on $\mathbb{R}^2$ by the hyperbolic one-parameter group

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathsf{GL}(\mathbb{R}^2) \\ t &\longmapsto \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}. \end{aligned}$$

Γ is a cocompact lattice in G, and the quotient $M^3 = \Gamma\backslash G$ is a 3-dimensional *solvmanifold.* We denote this group by Sol. Geometrically G is the identity component of the group of Lorentzian isometries of flat Minsowski 2-space, and this interpretation easily yields the flat Lorentzian structure on M.

2. Geometrization of 3-manifolds

Ehresmann's viewpoint set the context for Thurston's geometrization program for 3-manifolds, and revolutionized the subject.

Every closed 2-manifold Σ admits a Riemannian metric of *constant curvature,* and hence a (G, X)-structure where X is a model space of constant curvature (the 2-sphere S^2, Euclidean space $\mathbb{E}^2$, or the hyperbolic plane H^2) and $G = \mathsf{Isom}(X)$. Which geometry is supported arises from the topology of Σ: if $\chi(\Sigma) < 0$ $\big($respectively $\chi(\Sigma) = 0$, $\chi(\Sigma) > 0\big)$, then Σ admits hyperbolic structures (respectively, Euclidean structures, spherical structures). The deformation spaces of these structures (equivalent to the Teichmüller spaces of Σ) are a powerful tool for understanding the topology of Σ.

In 1976, Thurston proposed that 3-manifolds possess a suggestive natural structure in terms of *canonical decompositions* into pieces which have locally homogeneous Riemannian structures. There are eight local models for such Riemannian structures, including the three constant curvature geometries (spherical, Euclidean and hyperbolic) as well as certain product and local-product geometries (such as $S^2 \times S^1$, $H^2 \times S^1$, the Heisenberg group and the solvable group $\mathsf{Isom}(\mathbb{R}^{1,1})$ and the unit tangent bundle $T_1(H^2) \cong \mathsf{PSL}(2, \mathbb{R})$). As these are metric structures, the Hopf-Rinow theorem implies that the developing maps are tame, so (at least when one passes to a simply-connected model space X) the structures are all quotient structures by discrete subgroups of G.

The tools for the decomposition existed at the time, due to earlier work of Seifert, Dehn, Kneser, Milnor, Haken, Waldhausen, Jaco, Shalen, Johannsen and many others; Thurston realized that these topological results gave an intimate and suggestive relationship between topology and differential geometry, *in dimension three.* The importance of these insights cannot be overestimated. See Scott [131], Bonahon [26] and Thurston [138] for further details.

Three of the geometries correspond to the Euclidean manifolds, nilmanifolds and solvmanifolds above. Namely, Euclidean geometry lives on quotients of flat 3-tori by finite groups. *Nilgeometry* lives on quotients of the Heisenberg nilpotent, and is defined as the geometry of a left-invariant metric on the Heisenberg group Nil. *Solvgeometry* lives on quotients of the solvable Lie group Sol described above, and is defined as the geometry of a left-invariant metric on Sol.

3. Ehresmann structures

Now we return to Ehresmann's vision, as outlined in his paper [57] and the later paper [58], in the context of locally homogeneous structures which are *not necessarily* Riemannian.

In the later paper [58], he associates to a (G, X)-structure on M a fiber bundle with structure group G, fiber X and a section corresponding to the developing map. More generally, this structure corresponds to what is now called a *Cartan connection* on M, and the locally homogeneous structures described in [57] are precisely those Cartan connections which are *flat.* A flat Cartan connection is one for which the *curvature* vanishes. Sharpe [132] is a particularly readable exposition of this general theory. He calls a *Cartan geometry* a Cartan connection, and a flat one *Klein geometry.*

The local triviality of these structures implies that the study of such structures is essentially topological, and in particular closely related to the fundamental group and the universal covering space. Specifically, suppose M is a connected (G,X)=manifold with basepoint $p_0 \in M$. Let $U \subset M$ a coordinate patch containing p_0, with a coordinate chart $U \xrightarrow{\psi} X$. Let $\tilde{M} \xrightarrow{\Pi} M$ denote the corresponding universal covering space with covering group $\pi = \pi_1(M, p_0)$. Then ψ extends to a unique map $\tilde{M} \xrightarrow{\mathsf{dev}} X$ which is compatible with the (G,X)-atlas; it is a (G,X)-map, a morphism in the category whose objects are (G,X)-manifolds. As the restrictions of dev to coordinate patches are locally compositions of coordinate charts with transformations from G, the *developing map* dev is a local real-analytic diffeomorphism. (Since the action of G on X is real-analytic, a (G,X)-atlas determines a unique real-analytic structure.) Furthermore the group π of deck transformations acts by (G,X)-automorphisms of $\tilde{M}$, and therefore defines a homomorphism $\pi \xrightarrow{\rho} G$ such that

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow \\ \tilde{M} & \longrightarrow & X \end{array}$$

commutes.

This process of development originated with Élie Cartan and generalizes the notion of a developable surface in $\mathbb{E}^3$. If $S \hookrightarrow \mathbb{E}^3$ is an embedded surface of zero Gaussian curvature, then for each $p \in S$, the exponential map at p defines an isometry of a neighborhood of 0 in the tangent plane T_pS, and corresponds to rolling the tangent plane $\mathsf{A}_p(S)$ on S without slipping. In particular every curve in S starting at p lifts to a curve in T_pS starting at $0 \in T_pS$. For a Euclidean manifold, this globalizes to a local isometry of the universal covering $\widetilde{S} \longrightarrow \mathbb{E}^2$, called by Élie Cartan the *development* of the surface (along the curve). The metric structure is actually subordinate to the affine connection, as this notion of development really only involves the construction of *parallel transport.*

Later this was incorporated into the notion of a *fiber space,* as discussed in the 1950 conference [136]. The collection of coordinate changes of a (G,X)-manifold M defines a fiber bundle $\mathcal{E}_M \longrightarrow M$ with fiber X and structure group G. The fiber over $p \in M$ of the associated principal bundle

$$\mathcal{P}_M \xrightarrow{\Pi_{\mathcal{P}}} M$$

consists of all possible germs of (G, X)-coordinate charts at p. The fiber over $p \in M$ of $\mathcal{E}_M$ consists of all possible *values* of (G, X)-coordinate charts at p. Assigning to the germ at p of a coordinate chart $U \xrightarrow{\psi} X$ its value

$$x = \psi(p) \in X$$

defines a mapping

$$(\mathcal{P}_M)_p \longrightarrow (\mathcal{E}_M)_p.$$

Working in a local chart, the fiber over a point in $(\mathcal{E}_M)_p$ corresponding to $x \in X$ consists of all the different germs of coordinate charts ψ taking $p \in M$ to $x \in X$. This mapping identifies with the quotient mapping of the natural action of the stabilizer $\mathsf{Stab}(G, x) \subset G$ of $x \in X$ on the set of germs.

For Euclidean manifolds, $(\mathcal{P}_M)_p$ consists of all *affine orthonormal frames,* that is, pairs (x, F) where $x \in \mathbb{E}^n$ is a point and F is an orthonormal basis of the tangent space $T_x\mathbb{E}^n \cong \mathbb{R}^n$. For an affine manifold, $(\mathcal{P}_M)_p$ consists of all *affine frames:* pairs (x, F) where now F is *any* basis of $\mathbb{R}^n$.

The coordinate atlas/developing map defines a section of $\mathcal{E}_M \to M$ which is transverse to the two complementary foliations of $\mathcal{E}_M$:

- As a section, it is necessarily transverse to the foliation of $\mathcal{E}_M$ by fibers;
- The nonsingularity of the coordinate charts/developing map implies this section is transverse to the horizontal foliation $\mathfrak{F}_M$ of $\mathcal{E}_M$ defining the flat structure.

The differential of this section is the *solder form* of the corresponding Cartan connection.

3.1. **Properties of the developing map.** Ehresmann [57] proves several basic facts about the development/holonomy pair:

Suppose that M is compact and $\pi_1(M)$ is finite.

- X must be compact and $\pi_1(X)$ is finite.
- The universal covering $\tilde{M}$ of M is (G, X)-isomorphic to the universal covering of X.

He defines a structure to be *normal* if and only if the developing map is a covering space. Structures on closed manifolds with finite fundamental group are normal.

3.2. **Hierarchy of structures.** One may pass between different local models. We may define a *category of homogeneous spaces,* whose objects are pairs (G, X) where G is a Lie group and X is a manifold with a transitive action of G. A morphism $(G, X) \longrightarrow (G', X')$ is defined by a pair of maps $h : G \longrightarrow G'$ and $f : X \longrightarrow X'$, where h is a homomorphism and f is a local diffeomorphism which is h-equivariant, that is, for all $g \in G$,

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h(g)} \\ X & \xrightarrow{f} & X' \end{array}$$

commutes. Such a morphism induces a mapping from (G, X)-manifolds to (G', X')-manifolds.

Particularly interesting is the case when h is a local isomorphism of Lie groups. In this case the pseudogroups defined by (G, X) and (G', X') are identical, and the two categories of locally homogeneous structures identify. In this case Ehresmann calls X' a *Klein form* of X.

Here is another point of view concerning morphisms $(G, X) \longrightarrow (G', X')$. There is a unique (G', X')-structure on X such that $X \xrightarrow{f} X'$ is a (G', X')-map. Since the transformations of X defined by G are f-related to transformations of G', the action of G on X preserves this structure. In particular, for a given homogeneous space (G, X), a morphism $(G, X) \longrightarrow (G', X')$ is equivalent to a G-invariant (G', X')-structure on X.

In the special case that X is a Lie group and G is the group of left-multiplications, we see that a left-invariant (G', X')-structure on G is equivalent to a representation $G \longrightarrow G'$, together with an open orbit in X' which has discrete isotropy.

In many cases, the classification of geometric structures on a fixed topology proceeds by showing that the structures can be refined to certain subgeometries.

A particularly interesting and nontrivial example is Fried's classification of similarity structures on closed manifolds [63], whereby a compact manifold modeled on Euclidean similarity geometry is either a Euclidean manifold, or a finite quotient of a Hopf manifold. (See also Reischer-Vaisman [141] for a much different proof of the classification of closed similarity manifolds. This was first announced by Kuiper [106], but he implicitly assumed that the developing map was a covering-space onto its image.)

3.3. **The Ehresmann-Weil-Thurston holonomy principle.**

Fundamental in the deformation theory of locally homogeneous (Ehresmann) structures is the following principle, first observed in this generality by Thurston [137]:

Theorfem 3.1. *Let X be a manifold upon which a Lie group G acts transitively. Let M be a compact (G, X)-manifold with holonomy representation $\pi_1(M) \xrightarrow{\rho} G$.*

(1) *Suppose that ρ' is sufficiently near ρ in the representation variety $\mathsf{Hom}(\pi_1(M), G)$. Then there exists a (nearby) (G, X)-structure on M with holonomy representation ρ'.*
(2) *If M' is a (G, X)-manifold near M having the same holonomy ρ, then M' is isomorphic to M by an isomorphism isotopic to the identity.*

Here the topology on marked (G, X)-manifolds is defined in terms of the atlases of coordinate charts, or equivalently in terms of developing maps, or developing sections. In particular one can define a *deformation space* $\mathsf{Def}_{(G,X)}(\Sigma)$ whose points correspond to equivalence classes of marked (G, X)-structures on Σ. One might *like* to say the holonomy map

$$\mathsf{Def}_{(G,X)}(\Sigma) \xrightarrow{\mathsf{hol}} \mathsf{Hom}\big(\pi_1(\Sigma), G\big)/\mathsf{Inn}(G)$$

is a local homeomorphism, with respect to the quotient topology on $\mathsf{Hom}\big(\pi_1(\Sigma), G\big)/\mathsf{Inn}(G)$ induced from the classical topology on the $\mathbb{R}$-analytic set $\mathsf{Hom}\big(\pi_1(\Sigma), G\big)$. In many cases this is true (see below) but misstated in [78]. However, Kapovich [92] and Baues [13] observed that this is not quite true, because local isotropy groups acting on $\mathsf{Hom}\big(\pi_1(\Sigma), G\big)$ may not fix marked structures in the corresponding fibers.

In any case, these ideas have an important consequence:

Corollary 3.2. *Let M be a closed manifold. The set of holonomy representations of (G, X)-structures on M is open in $\mathsf{Hom}(\pi_1(M), G)$ (with respect to the classical topology).*

One can define a space of flat (G, X)-bundles (defined by a fiber bundle $\mathcal{E}_M$ having X as fiber and G as structure group) and the foliation $\mathcal{F}$ transverse to the fibration $\mathcal{E}_M \longrightarrow M$. The foliation $\mathcal{F}$ is equivalent to a reduction of the structure group of the bundle from G with the classical topology to G with the discrete topology. This set of flat (G, X)-bundles over Σ identifies with the quotient of the $\mathbb{R}$-analytic set $\mathsf{Hom}(\pi_1(\Sigma), G)$ by the action of the group $\mathsf{Inn}(G)$ of inner automorphisms action by left-composition on homomorphisms $\pi_1(\Sigma) \to G$.

Conversely, if two nearby structures on a compact manifold M have the same holonomy, they are equivalent. The (G, X)-structures are topologized as follows. Let $\Sigma \longrightarrow M$ be a marked (G, X)-manifold, that is, a diffeomorphism from a fixed model manifold Σ to a (G, X)-manifold M. Fix a universal covering $\widetilde{\Sigma} \longrightarrow \Sigma$ and let $\pi = \pi_1(\Sigma)$ be its group of deck transformations. Choose a holonomy homomorphism $\pi \xrightarrow{\rho} G$ and a developing map $\widetilde{\Sigma} \xrightarrow{\mathsf{dev}} X$.

In the nicest cases, this means that under the natural topology on flat (G,X)-bundles $(X_\rho, \mathcal{F}_\rho)$ over M, the holonomy map hol is a local homeomorphism. Indeed, for many important cases such as hyperbolic geometry (or when the structures correspond to geodesically complete affine connections), hol is actually an embedding.

3.4. **Historical remarks.** Thurston's holonomy principle has a long and interesting history.

The first application is the theorem of Weil [147] that the set of *discrete embeddings* of the fundamental group $\pi = \pi_1(\Sigma)$ of a closed surface Σ in $G = \mathsf{PSL}(2,\mathbb{R})$ is open in the quotient space $\mathsf{Hom}(\pi, G)/G$. Indeed, a discrete embedding $\pi \hookrightarrow G$ is exactly a holonomy representation of a *hyperbolic structure* on Σ. The corresponding subset of $\mathsf{Hom}(\pi, G)/G$ is called the *Fricke space* $\mathfrak{F}(\Sigma)$ of Σ, and will be discussed more fully in §6.3. Weil's results are clearly and carefully expounded in Raghunathan [129], (see Theorem 6.19), and extended in Bergeron-Gelander [25].

In the context of $\mathbb{C}\mathsf{P}^1$-structures, this is due to Hejhal [85, 84]; see also Earle [56] and Hubbard [88]. This venerable subject originated with conformal mapping and the work of Schwarz, and closely relates to the theory of second order (Schwarzian) differential equations on Riemann surfaces. In this case, where $X = \mathbb{C}\mathsf{P}^1$ and $G = \mathsf{PSL}(2,\mathbb{C})$, we denote the deformation space $\mathsf{Def}_{(G,X)}(\Sigma)$ simply by $\mathbb{C}\mathsf{P}^1(\Sigma)$. See Dumas [53] and §7.1 below.

Thurston sketches the intuitive ideas for Theorem 3.1 in his notes [137]. The first detailed proofs of this fact are Lok [115], Canary-Epstein-Green [30], and Goldman [76] (the proof in [76] was worked out with M. Hirsch, and were independently found by A. Haefliger). The ideas in these proofs may be traced to Ehresmann [58], although he didn't express them in terms of moduli of structures. Corollary 3.2 was noted by Koszul [103], Chapter IV, §3, Theorem 3; compare also the discussion in Kapovich [93], Theorem 7.2.

4. Example: One real dimension

We illustrate these ideas in dimension 1. One-dimensional geometry (in our narrow *locally homogeneous* sense) reduces to projective geometry (where $X = \mathbb{R}\mathsf{P}^1$ and $G = \mathsf{PGL}(2,\mathbb{R})$). Let Σ be a compact connected 1-manifold (that is, a circle). Denote the deformation space $\mathsf{Def}_{(G,X)}(\Sigma)$ of $\mathsf{PGL}(2,\mathbb{R})$-equivalence classes of marked $\mathbb{R}\mathsf{P}^1$-structures on Σ by $\mathbb{R}\mathsf{P}^1(\Sigma)$. Denote the universal covering group of $\mathsf{SL}(2,\mathbb{R})$ by

$\widetilde{\mathsf{SL}(2,\mathbb{R})}$); say that two elements $a, b \in \widetilde{\mathsf{SL}(2,\mathbb{R})}$) are *equivalent* if a is conjugate to b or b^{-1}.

The classification of $\mathbb{R}\mathsf{P}^1$-manifolds is due to Kuiper [109] and the following succinct description is due to Goldman [72].

Theorfem 4.1. *The deformation space* $\mathbb{R}\mathsf{P}^1(\Sigma)$ *identifies with the space of equivalence classes of nontrivial elements of the universal covering group* $\widetilde{\mathsf{SL}(2,\mathbb{R})}$).

In other words,

$$\mathbb{R}\mathsf{P}^1(\Sigma) = \Big(\widetilde{\mathsf{SL}(2,\mathbb{R})}) \setminus \{1\}\Big)/\sim .$$

This space is a non-Hausdorff space containing several copies of $\mathbb{R}$, one corresponding to the lifts of an elliptic one-parameter subgroup, and others corresponding to cosets of a hyperbolic one-parameter subgroup. We describe the corresponding structures in detail below in §4.2.

4.1. **Noncompact manifolds.** Let X be a 1-dimensional homogeneous space, and G its transitive group of automorphisms. A connected 1-manifold M is homeomorphic to either $\mathbb{R}$ or $S^1 \approx \mathbb{R}/\mathbb{Z}$. If $M \approx \mathbb{R}$, then it is simply connected, and a structure modeled on X is just an immersion $M \looparrowright X$. If X is not already simply connected, replace it by its universal cover $\tilde{X}$. Since $\tilde{X} \approx \mathbb{R}$, then a structure on M is an immersion, which must be an embedding. Such an embedding corresponds to a monotone function $\mathbb{R} \longrightarrow \mathbb{R}$. By choosing compatible orientations on M and X, we may assume that this monotone function is increasing. Such an increasing function is determined (up to the appropriate relation of isotopy) by the endpoints of the closure, that is a pair (a, b) where

$$-\infty \leq a < b \leq \infty.$$

4.2. **Compact manifolds.** Now consider the case M is compact. In that case $M \approx S^1$, which we realize topologically as the quotient space of a closed interval $[a, b] \subset \mathbb{R}$ by the equivalence relation defined by identifying its two endpoints a, b. Denote the common image of the endpoints by $p_0 \in M$. The total space for the universal covering $\widetilde{M}$ is the quotient space of $[a, b] \times \mathbb{Z}$ by the equivalence relation defined by:

$$(b, n) \sim (a, n+1)$$

for $n \in \mathbb{Z}$. The group $\pi_1(M)$ is the cyclic group $\langle\mu\rangle \cong \mathbb{Z}$ acting on $\widetilde{M}$ by:

$$\begin{aligned} [a,b] \times \mathbb{Z} &\xrightarrow{\mu^m} [a,b] \times \mathbb{Z} \\ (u,n) &\longmapsto (u, n+m) \end{aligned}$$

where $u \in [a,b]$ and μ denotes the generator of $\pi_1(M)$.

Now we construct a developing map $\widetilde{M} \xrightarrow{\mathsf{dev}} X$. The developing map is determined by two pieces of information:

- Its restriction to $[a,b] \subset \widetilde{M}$ (corresponding to the subset $[a,b] \times \{0\} \subset [a,b] \times \mathbb{Z}$), which is an immersion
$$[a,b] \overset{f}{\looparrowright} X;$$
- A holonomy transformation $\eta = \rho(\mu) : X \to X$ such that $\eta f(b) = f(a)$.

Then f extends to the developing map by defining:

$$\mathsf{dev}(u,n) := \eta^n f(u)$$

for an arbitrary element $[(u,n)] \in \widetilde{M}$.

As above, it is convenient to lift f to the universal covering $\tilde{X} \approx \mathbb{R}$ and using a diffeomorphism $\widetilde{M} \approx \mathbb{R}$, identify f with a monotone function $\mathbb{R} \longrightarrow \mathbb{R}$.

4.3. **Euclidean manifolds.** The first example arises when f is the embedding of $[a,b]$ as the unit interval $[0,1] \subset \mathbb{E}^1$ and η is unit translation. Then M identifies naturally with the quotient $\mathbb{R}/\mathbb{Z}$. Its natural structure is that of a compact flat Riemannian manifold of total length 1.

More generally, for any $l > 0$, the quotient $\mathbb{R}/l\mathbb{Z}$ (where η is translation by l) is a Euclidean manifold of length l. Different values of l give non-isometric Euclidean structures, but homotheties define isomorphisms as *affine manifolds:*

$$\begin{aligned} \mathbb{R}/\mathbb{Z} &\xrightarrow{\cong} \mathbb{R}/l\mathbb{Z} \\ [x] &\longmapsto [lx] \end{aligned}$$

Also observe that these structures are *homogeneous:* the group of translations acts transitively on M. Indeed, this defines a bi-invariant (Euclidean) geometric structure on the circle group.

4.4. **Incomplete affine structures.** Any $\lambda > 1$ generates a lattice inside the multiplicative group $\mathbb{R}_+$, which acts affinely on A^1. The quotient $\mathbb{R}_+/\langle\lambda\rangle$ also defines an affine structure on M, which is not a Euclidean structure since dilation by λ is not an isometry. Explcitly, take f to be a diffeomorphism onto the interval $[1,\lambda] \subset \mathbb{R} \approx \mathsf{A}^1$, so that dev is a diffeomorphism of $\widetilde{M}$ onto $(0,\infty) = \mathbb{R}_+ \subset \mathsf{A}^1$.

Like the preceding example, this affine structure is also bi-invariant with respect to the natural Lie group structure on $\mathbb{R}_+/\langle\lambda\rangle$.

Observe that, since the exponential map

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R}_+ \\ x &\longmapsto e^x \end{aligned}$$

converts addition (translation) to multiplication (dilation), it defines a diffeomorphism between two quotients

$$\mathbb{R}/l\mathbb{Z} \longrightarrow \mathbb{R}_+/\langle\lambda\rangle$$

where $l := \log(\lambda)$. This map also defines a (non-affine) analytic isomorphism between the corresponding Lie groups.

These structures are *geodesically incomplete,* and in fact model *incomplete closed geodesics* on affine manifolds. Namely, the geodesic on A^1 defined by

$$t \longmapsto 1 + t(\lambda^{-1} - 1)$$

begins at 1 and in time

$$t_\infty := 1 + \lambda^{-1} + \lambda^{-2} + \cdots = (1 - \lambda^{-1})^{-1} > 0$$

reaches 0. It defines a closed incomplete closed geodesic $p(t)$ on M starting at $p(0) = p_0$. The lift

$$(-\infty, t_\infty) \xrightarrow{\tilde{p}} \widetilde{M}$$

satisfies

$$\mathsf{dev}\tilde{p}(t) = 1 + t(\lambda^{-1} - 1),$$

which uniquely specifies the geodesic $p(t)$ on M. It is a geodesic since its velocity $p'(t) = (\lambda^{-1} - 1)\partial_x$ is constant (parallel). However $p(t_n) = p_0$ for

$$t_n := \frac{1 - \lambda^{-n}}{1 - \lambda^{-1}} = 1 + \lambda^{-1} + \cdots + \lambda^{1-n}$$

and as viewed in M, seems to go "faster and faster" through each cycle. By time $t_\infty = \lim_{n\to\infty} t_n$, it seems to "run off the manifold:" the geodesic is only defined for $t < t_\infty$. The apparent paradox is that $p(t)$ has *zero acceleration:* it would have "constant speed" if "speed" were only defined.

The model space X is the real projective line $\mathbb{R}\mathsf{P}^1$ and the structure group G is the group $\mathsf{PGL}(2,\mathbb{R})$ of collineations of $\mathbb{R}\mathsf{P}^1$. The fixed reference topology Σ is the *circle,* which we understand as a the identification space of a closed interval $[a,b]$ with its two endpoints a,b identified. We identify the universal covering $\widetilde{\Sigma} \longrightarrow \Sigma$ as the quotient of the Cartesian product $[a,b]\times\mathbb{Z}$, with identifications

$$(b,n) \sim (a,n+1)$$

for $n\in\mathbb{Z}$. The group $\pi_1(\Sigma)\cong Z$ of deck transformations acts by:

$$(x,n) \longmapsto (x,n+m)$$

for $m\in\mathbb{Z}$. An $\mathbb{R}\mathsf{P}^1$-structure on Σ is defined by the restriction $\mathsf{dev}|_{[a,b]}$.

Choose a transformation $\gamma\in G$ to serve as the generator of the holonomy group. Specifically, define the presumptive holonomy homomorphism ρ by:

$$\begin{aligned} \mathbb{Z} \cong \pi_1(\Sigma) &\xrightarrow{\rho} G \\ m &\longmapsto \gamma^m \end{aligned}$$

Any immersion $f : [a,b] \looparrowright X$ such that $f(b)=\gamma f(a)$ extends to a ρ-equivariant immersion $\widetilde{\Sigma}\longrightarrow X$.

If $G\cong\mathbb{R}$, left-invariant structures form three equivalence classes, corresponding to the three conjugacy classes of one-parameter subgroups in G':

- An elliptic one-parameter subgroup acts simply transitively on all of X'. The deck transformation τ is a discrete subgroup of this one-parameter group.
- A parabolic one-parameter subgroup acts simply transitively on the open interval between a point $x'\in X'$ and its image $\tau(x')$.
- A hyperbolic one-parameter subgroup acts simply transitively on the open interval between two points $x',y'\in X'$.

Of these structures, the last two are affine structures, since $X\setminus\{x'\}$ is an affine line and its stabilizer in G' is the affine group. For example, taking x',y' to be lifts of $0,\infty\in\mathbb{R}\mathsf{P}^1$ and g to be scalar multiplication by $\lambda>1$, we obtain a structure, which we call a *Hopf structure*

$$\mathbb{R}_+/\lambda^n \mid n\in\mathbb{Z} \tag{4.1}$$

since it is a special case of the construction of Hopf manifolds below.

In 1953, Kuiper [109] classified all such structures. In particular he found structures which are not homogeneous. These occur only if the holonomy group is parabolic or hyperbolic. Let $g\in G$ be either parabolic or hyperbolic, and let τ be the postive generator of the center of G' as above. Let $n>0$. Choose a point $x\in X$ *not* fixed under g.

Let $J \subset X'$ be a positively oriented interval going from x to $\tau^n g(x)$. A homeomorphism $[0,1] \xrightarrow{h} J$ extends to the $\mathbb{Z}$-equivariant homeomorphism defined by:

$$\begin{aligned} \mathbb{R} &\xrightarrow{\tilde{h}} X' \\ t &\longmapsto \tau^n h(t-n), \end{aligned}$$

(where $n = \lfloor t \rfloor$), which is a developing map for an $\mathbb{R}\mathsf{P}^1$-structure on the closed 1-manifold $\mathbb{R}/\mathbb{Z}$.

Kuiper [109] showed these comprise all the equivalence classes of structures. In particular the developing map is always a homeomorphism of the universal covering $\tilde{M}$ to either:

- all of X' (any structure with trivial or elliptic holonomy, or an inhomogeneous structure with parabolic or hyperbolic holonomy);
- the lift of the complement of one point (homogeneous structures with parabolic holonomy);
- the lift of the complement of two points (homogeneous structures with hyperbolic holonomy).

5. Geodesics on affine manifolds

5.1. **Geodesic completeness.** The next examples we discuss are *affine structures.* In this case the affine group G' preserves the Euclidean connection on affine space (although not the Euclidean metric). A smooth vector field along a smooth curve γ has a well-defined *covariant derivative,* which is another vector field along γ. The *acceleration* γ'' of γ is the covariant derivative of the *veclocity vector field* γ', and γ is a *geodesic* if it has zero acceleration. If M is a manifold with an affine structure, and $(x, v) \in TM$ is a tangent vector, then there exists $\epsilon > 0$, and a unique geodesic $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ with $\gamma(0) = x$ and $\gamma'(0) = v$.

If M is a Euclidean manifold with its underlying affine structure (or more generally if (M, g) is a Riemannian manifold with its Levi-Civita connection), then geodesic completeness is equivalent to the more intuitive notion of *metric completeness* of the associated metric space. This is the *Hopf-Rinow theorem,* and plays a fundamental role in controlling the developing map of flat structures.

As compact metric spaces are complete, a compact Riemannian manifold is geodesically complete. This also follows from the fact that the geodesic local flow of a compact Riemannian manifold reduces to local flows on the *energy hypersurfaces*

$$E_R(M) := \{(v, x) \in TM \times M \mid v \in T_x M, g(v, v) = R\},$$

which are compact. The complete integrability of these vector fields on $E_R(M)$ (for $R > 0$) implies geodesic completeness of (M, g).

5.2. **Hopf manifolds.** In 1948, H. Hopf [87] constructed a compact complex manifold which is not Kähler. His construction also yields compact affine manifolds which are *geodesically incomplete.* Indeed, the examples in §4.4 above are the simplest case of Hopf's construction.

An affine manifold is *complete* if it is geodesically complete in the above sense, that is, for every initial location and velocity (x, v) there is a geodesic $\mathbb{R} \xrightarrow{\gamma} M$ with $\gamma(0) = x$ and $\gamma'(0) = v$. Auslander and Markus [5] showed that completeness is equivalent to bijectivity of the developing map. That is, a complete affine manifold is affinely isomorphic to a quotient of $\mathbb{R}^m$ by a discrete group of affine transformations acting freely and properly.

6. Surfaces and 3-manifolds

6.1. **Affine structures on surfaces.** For a detailed survey of this subject, see Baues [15]. The first results are due to Kuiper [110], who listed the affine structures on 2-dimensional tori which are *convex,* that is, the developing map is an embedding onto a convex domain $\Omega \subset \mathsf{A}^2$. Either the structure is complete (which Kuiper also calls "normal"), in which case $\Omega = \mathsf{A}^2$, or Ω is either a half-plane or a quadrant.

The classification was completed in the 1970's by, independently, Nagano-Yagi [127] and Arrowsmith-Furness [67] (see also the classification of Klein bottles in [4]). In the remaining (*nonconvex*) cases after Kuiper, M is a quotient of the complement of a point in A^2. These are special cases of *radiant affine structures,* which we now describe.

6.1.1. *Radiant affine structures.* Affine manifolds which are quotients of the complement of a point $p \in \mathsf{A}^n$ have special properties, which deserve special attention. Necessarily p is fixed under the affine holonomy group $\Gamma \subset \mathsf{Aff}(\mathsf{A}^n)$. By applying a translation, we may conveniently assume that p is the origin $\mathbf{0} \in \mathbb{R}^n$. Since the stabilizer $\mathsf{Stab}\big(\mathsf{Aff}(\mathsf{A}^n), \mathbf{0}\big) = \mathsf{GL}(\mathbb{R}^n)$, the *affine* holonomy group is actually *linear.* Such affine structures are called *radiant* in [65], since they are characterized by the existence of a *radiant vector field* which generates a homothetic flow (that is, scalar multiplications on the vector space $\mathbb{R}^2$).

Nagano-Yagi [127] observed that on a closed radiant affine manifold M, the developing image $\mathsf{dev}(\widetilde{M})$ is disjoint from the set $\mathsf{Fix}(\Gamma)$ of fixed points of Γ. Thus every radiant vector field on M is nonsingular. A purely topological consequence is that $\chi(M) = 0$. Another topological consequence is that M cannot have parallel volume (see §7.3.1 below), and therefore the first Betti number $\beta(M) \geq 1$.

In his 1960 unpublished lecture notes, L. Markus observed that the known examples of compact affine manifolds which are geodesically complete are precisely the known examples of compact affine manifolds with parallel volume.

These manifolds can be understood by a *suspension* construction of mapping tori of automorphisms of the compact $\mathbb{R}\mathsf{P}^1$-manifolds discussed in §4. This is due to Benzécri [24], who proved that, any $\mathbb{R}\mathsf{P}^n$-manifold M admits a double covering $\hat{M}$ such that the Cartesian product $\hat{M} \times S^1$ admits a radiant affine structure, where the radiant flow is the flow in the S^1-factor.

For example, all radiant affine 2-manifolds arise in this way. Furthermore the affine holonomy group contains a linear expansion of $\mathbb{R}^2$. The simplest example is the Hopf manifold described above.

A radiant affine manifold (M, ξ) is a radiant suspension if and only if the flow ρ admits a cross-section. David Fried [62, 64] constructed a closed affine 6-manifold with diagonal holonomy whose radiant flow admits no cross-section. Choi [41] (using work of Barbot [12]) proves that every radiant affine 3-manifold is a radiant suspension, and therefore is either a Seifert 3-manifold covered by a product $F \times S^1$, where F is a closed surface, a nilmanifold or a hyperbolic torus bundle.

In dimensions 1 and 2 all closed radiant manifolds are radiant suspensions. Together with Kuiper's list of convex structures, these comprise all closed affine 2-manifolds, since a closed surface M admits an affine structure if and only if $\chi(M) = 0$ (Benzécri [23]). That is, either M is diffeomorphic to a Klein bottle (in which case its orientable double covering is diffeomorphic to a torus) or M is diffeomorphic to a torus. (Benzécri's theorem inspired Milnor's generalization [124] to flat oriented rank two vector bundles over surfaces; see [79] for a more detailed account of these developments.)

6.2. **Complete affine structures.** The complete structures on T^2 are all *affine Lie groups,* that is, an affine structure on a Lie group invariant under both left- and right-multiplications. For example, the Euclidean structures are all quotients $\mathbb{R}^2/\Lambda$, where $\Lambda \subset \mathbb{R}^2$ is a lattice. The other structures are obtained by *polynomial deformations* of Euclidean structures, namely the diffeomorphism

$$\begin{aligned} \mathbb{R}^2 &\xrightarrow{F_\epsilon} \mathbb{R}^2 \\ (x, y) &\longmapsto (x + \epsilon y^2, y) \end{aligned}$$

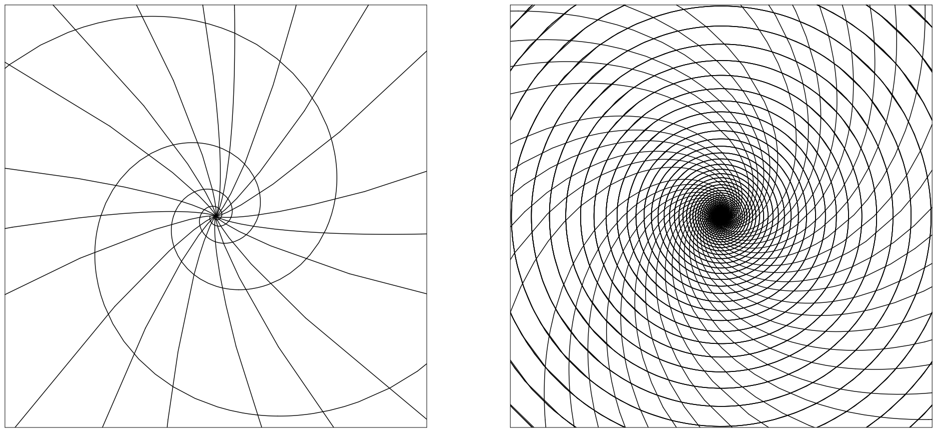

FIGURE 1. Incomplete complex-affine structures on the 2-torus are radiant suspensions. The two examples depicted are suspensions of rotations (isometries) of the circle.

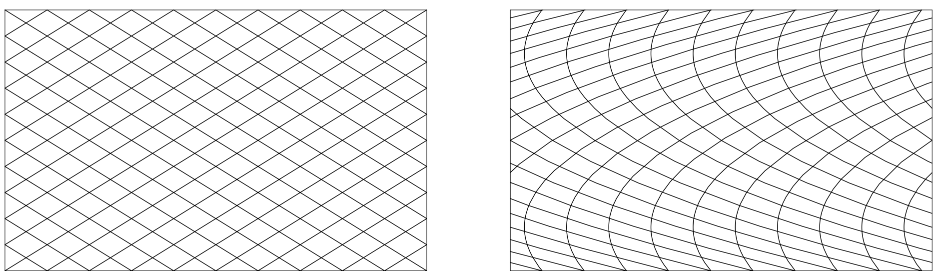

FIGURE 2. Complete affine structures on the 2-torus

conjugates translation by (s,t) to the affine transformation

$$(x,y) \longmapsto (x + 2\epsilon yt + (s + \epsilon^2 t^2), y + t)$$

and the quotient space

$$M = \mathbb{R}^2 / F_\epsilon \Lambda F_\epsilon^{-1}$$

is a non-Euclidean complete affine torus. Figure 6.2 illustrates a Euclidean torus and a polynomial deformation.

Baues [14] showed that the deformation space of complete affine structures on T^2 is homeomorphic to $\mathbb{R}^2$ (see also Baues-Goldman [16]), with the origin $(0,0)$ corresponding to the single equivalence class corresponding to Euclidean structures. The action of $\mathsf{Mod}(T^2) = \mathsf{GL}(2,\mathbb{Z})$ on this deformation space identifies with the usual linear action of

$\mathsf{GL}(2,\mathbb{Z})$ on $\mathbb{R}^2$. This action is highly chaotic: it is topologically mixing, and every continuous invariant function is constant.

Complete affine structures on *compact* 3-manifolds were classified by Fried–Goldman [66]. A closed 3-manifold admits a complete affine structure if and only if it is finitely covered by a 2-torus bundle over the circle; in other words, it is Euclidean, Heisenberg, or Sol in the Thurston geometrization [131, 26, 138]. For more information, see Abels [1], Baues [14, 13] and Goldman [78, 79].

Noncompact complete affine 3-manifolds are considerably more complicated, due to Margulis's discovery [118] of proper isometric actions of nonabelian free groups on Minkowski $2+1$-space. The structure is much better understood now, and they all arise from a Schottky-group construction using *crooked planes* invented by Drumm [51, 52]. We refer to [35, 27, 34, 33, 42, 50, 49] for more current information.

6.3. **Hyperbolic structures on surfaces.** A paradigm for this theory is the classification of hyperbolic structures on surfaces.

The Fricke space $\mathfrak{F}(\Sigma)$ embeds in $\mathsf{Hom}(\Gamma, G)/G$, where

$$G = \mathsf{Isom}(\mathsf{H}^2) \cong \mathsf{SO}(2,1),$$

and indeed defines a connected component in this space. As noted above, openness follows from Corollary 3.2.

Closedness is more special. If Γ is not virtually nilpotent and G is semisimple, then the discrete embeddings $\Gamma \hookrightarrow G$ form a *closed* subset of $\mathsf{Hom}(\Gamma, G)$ in the classical topology. (A proof of this well-known statement can be found in Goldman-Millson [80], although it was known much earlier.) In this special case, it is originally due to Chuckrow [46]. In general closedness follows from Kazhdan-Margulis uniform discreteness [95], see Chapter VIII of [129] , §4.12 of Kapovich [93], or §4.1 of Thurston [138].

It remains to see that Fricke space $\mathfrak{F}(\Sigma)$ is connected. In general the discrete embeddings $\Gamma \hookrightarrow G$ fall into many connected components. Counting the components is an interesting and difficult general problem. In this particular case, one can use the direct hyperbolic-geometry parametrization of $\mathfrak{F}(\Sigma)$ by Fenchel-Nielsen coordinates, whereby

$$\mathfrak{F}(\Sigma) \approx \mathbb{R}^{6g-6+3b}$$

is connected, completing the proof that $\mathfrak{F}(\Sigma)$ is a connected component of $\mathsf{Hom}(\Gamma, G)/G$. (See, for example, Buser [29], Abikoff [3], Ratcliffe [130], Theorem 9.7.4, §4.6 of Thurston [138], or Wolpert [150] for accessible accounts of the Fenchel-Nielsen parametrization of $\mathfrak{F}(\Sigma)$.

Another proof uses the Uniformization Theorem. First identify $\mathfrak{F}(\Sigma)$ with the Teichmüller space $\mathfrak{T}(\Sigma)$ of Σ (uniformization). Now apply Teichmüller's theorem to identify $\mathfrak{T}(\Sigma)$ with the unit ball in the vector space $Q(M)$ of holomorphic quadratic differentials on a Riemann surface M homeomorphic to Σ. (For details see Hubbard [89], Theorem 7.2.1.) Alternatively, following Wolf [149], identify $\mathfrak{T}(\Sigma)$ with all of $Q(M)$ using the Hopf differentials of harmonic maps from M to an arbitrary Riemann surface homeomorphic to Σ.

The components of $\mathsf{Hom}(\pi, G)/G$ were classified in [77] in terms of the Euler class. In particular $\mathfrak{F}(\Sigma)$ identifies with the component [77] *maximizing* this characteristic class. In terms of the foliated (G, X)-bundle X_ρ associated to a representation ρ, this result means that the necessary topological conditions for X_ρ to admit a transverse section are sufficient. See Goldman [78, 79] for further details and discussion.

Toledo [140] considered surface group representations when G acts on a Hermitian symmetric space X. He defined a characteristic number which includes the Euler class of flat $\mathsf{PSL}(2, \mathbb{R})$-bundes when $X = \mathsf{H}^2$. In particular Toledo's invariant is bounded by topological invariants of M . Representations maximizing Toledo's invariant have many properties of Fuchsian representations , in particular forming connected components consisting entirely of discrete embeddings. In a diffrerent direction, when G is a split $\mathbb{R}$-form, Hitchin [86] found components which naturally contain compositions of Fuchsian $\mathsf{SL}(2, \mathbb{R})$-representations with the *Kostant principal representation* $\mathsf{SL}(2, \mathbb{R}) \to G$. When $G = \mathsf{PGL}(3, \mathbb{R})$, the component identifies with the deformation space of convex $\mathbb{R}\mathsf{P}^2$-surfaces [43] , as discussed in §7.2 below. Hitchin proved that these components are topologically open cells. Labourie [113] characterized Hitchin's representations dynamically, and proved that they are all quasi-isometric (and hence discrete) embeddings. (From a somewhat different viewpoint, these representations were also studied by Fock-Goncharov [61, 60] who found coordinates on these compoents.) This subject, sometimes called "higher Teichmüller theory," is surveyed in Burger-Iozzi-Wienhard [28] (with background expounded in Labourie [114]), to which we refer for turther details.

7. Projective and conformal structures

Finally we discuss Ehresmann structures modeled on compact homogeneous spaces, such as the sphere and projective space. Although this subject dates back to the nineteenth century, in the context of second order linear differential equations on Riemann surfaces and conformal mapping, many mysteries remain, and the subject is fundamental in the broader hierarchy of geometries. We then discuss real-projective structures on surfaces, for which a complete classification is known [43]. We then briefly discuss several results about flat conformal structures and real-projective structures in higher dimensions.

7.1. **Projective structures on Riemann surfaces.** This rich subject promises to be fundamental in the theory of Ehresmann locally homogeneous structures. I find it rather striking that although the algebraic theory of the character variety is less pathological, the geometric theory is exceedingly profound and difficult. The parametrization of the deformation space $\mathbb{C}\mathsf{P}^1(\Sigma)$ as an affine bundle whose underlying vector bundle is $T^*\mathfrak{T}(\Sigma)$ is the (holomorphic) cotangent bundle of the Teichmüller space $\mathfrak{T}(\Sigma)$ is rather "soft" but the geometric theory of $\mathbb{C}\mathsf{P}^1$-manifolds is extremely subtle, involving some of the most technically difficult aspects of the "modern" theory of hyperbolic 3-manifolds and Kleinian groups (see Marden [116]). We only concentrate on the properties of the holonomy mapping, referring to the excellent survey article [53] by David Dumas. See also Gunning [81, 82] for background.

Gallo-Kapovich-Marden [68] answered a question first raised by Gunning [83]:

Theorem. *Let Σ be a closed orientable surface of $\chi(\Sigma) < 0$. Denote the deformation space of marked $\mathbb{C}\mathsf{P}^1$-structures on Σ by $\mathbb{C}\mathsf{P}^1(\Sigma)$. The image of the holonomy mapping*

$$\mathbb{C}\mathsf{P}^1(\Sigma) \xrightarrow{\mathsf{hol}} \mathsf{Hom}\big(\pi_1(\Sigma)$$

consists of equivalence classes of representations ρ for which:

- *ρ lifts to a representation $\pi_1(\Sigma) \xrightarrow{\tilde{\rho}} \mathsf{SL}(2,\mathbb{C})$;*
- *the image Γ of ρ fixes no point on hyperbolic space H^3, fixes no point in the boundary $\partial\mathsf{H}^3$, and leaves invariant no geodesic in H^3.*

The first condition means that ρ lies in the connected component of $\mathsf{Hom}\big(\pi_1(\Sigma), G\big)$ containing the trivial representation (Goldman [77]). The second condition means that ρ is *nonelementary,* and is equivalent to numerous other conditions. For example, it is equivalent to the real Zariski closure of Γ being $\mathsf{PSL}(2,\mathbb{C})$ or conjugate to $\mathsf{PGL}(2,\mathbb{R})$. Another equivalent condition is that the image of ρ is *unbounded* (having noncompact closure) and non-solvable. Yet another condition is that the holonomy group Γ is *not amenable.*

Although W. Thurston announced this and communicated the outline of the proof to the author in the late 1970's, many details were missing. The full proof (following Thurston's outline) was completed by Gallo-Kapovich-Marden [68]. (An incorrect proof, but with an extremely interesting approach, can be found in [94].) See also Kamishima-Tan [91].

The injectivity of the holonomy mapping is also quite fundamental and mysterious. Goldman [75], using ideas inspired by the Thurston parametrization (see §7.4 below), computed the inverse image $\mathsf{hol}^{-1}(\mathfrak{F}_\Sigma)$ in terms of a *grafting construction,* first developed by Hejhal [85] and Maskit [121] (Theorem 5) and Sullivan-Thurston [135].

The main result is that, over the inverse image of the quasi-Fuchsian subset of $\mathsf{Hom}\big(\pi_1(\Sigma), G\big)$, the holonomy map hol is a covering space and the fiber admits an explicit topological description in terms of grafting. In this case, X decomposes into two subdomains Ω_+ and Ω_- along their common boundary which is the *limit set* of the holonomy group Γ. The geometric manifold M with this holonomy then admits a corresponding decomposition $M = M_+ \cup M_-$, and under the assumption that the holonomy homomorphism is an isomorphism $\pi_1(M) \xrightarrow{\cong} \Gamma$, one of $M_\pm$ is a union of annuli.

However, as pointed out by M. Kapovich and S. Choi, the proof of a key lemma (Theorem 2.2) of Goldman [75] is flawed. (A similar problem can be found in Faltings [59]). See Choi-Lee [45] for a corrected proof and extensive discussion. One would like to control the developing map by decomposing the geometric manifold into open submanifolds modeled on holonomy-invariant subdomains $\Omega \subset X$ where the holonomy Γ preserves a complete Riemannian structure g_Ω. However, even if M is compact, the induced metric on $\mathsf{dev}^{-1}(\Omega)$ may be incomplete. One needs a sharper argument involving the asymptotics of Γ, as in Kuiper [108]. (Indeed, the Sullivan-Thurston-Smillie examples discussed in §7.2 and depicted in Figure 7.1 provide counterexamples to Theorem 2.2 of [75].)

Shinpei Baba's work [11, 10, 9] describes $\mathbb{C}\mathsf{P}^1$-structures with Schottky holonomy in terms of a similar grafting construction. Although developing maps for general $\mathbb{C}\mathsf{P}^1$-structures are intractable, under the assumption of Schottky holonomy, Baba obtains sharp results on decomposing the developing map into basic pieces.

7.2. **Real-projective structures on surfaces.** When $X = \mathbb{R}\mathsf{P}^2$ and

$$G = \mathsf{Aut}(X) \cong \mathsf{PGL}(3,\mathbb{R}) \cong \mathsf{SL}(3,\mathbb{R})$$

is its group of collineations, we denote the deformation space $\mathsf{Def}_{(G,X)}(\Sigma)$ simply by $\mathbb{R}\mathsf{P}^2(\Sigma)$. Curiously, the case when $\chi(\Sigma) = 0$ has a much more complicated general picture than when $\chi(\Sigma) < 0$.

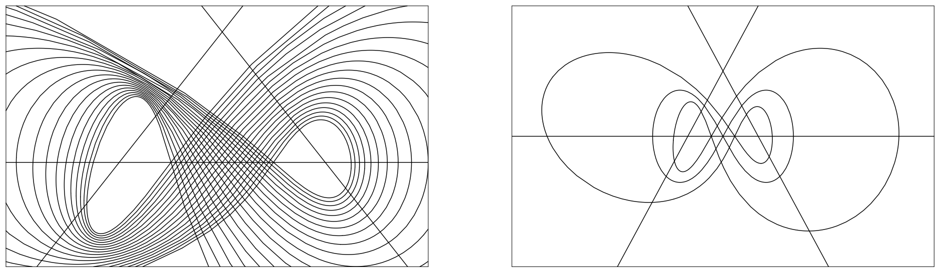

FIGURE 3. Developing maps of $\mathbb{R}\mathsf{P}^2$-structures which are not covering spaces of their images, due to Sullivan-Thurston and Smillie.

When $\chi(\Sigma) = 0$, then Σ admits affine structures, discussed in §6.1. The remaining $\mathbb{R}\mathsf{P}^2$-structures on T^2 (and the Klein bottle) were classified in [71]. The new examples involve a surgery construction (due to Sullivan-Thurston [135] and independently Smillie [134]) analogous to the $\mathbb{C}\mathsf{P}^1$-grafting construction of Hejhal [85], Maskit [121] and Sullivan-Thurston [135].

Here one starts with a collineation γ of $\mathbb{R}\mathsf{P}^2$ described by a diagonal 3×3-matrix with distinct positive eigenvalues. The coordinate axes define the three fixed points of γ on $\mathbb{R}\mathsf{P}^2$, which lie outside the developing image. The coordinate planes define invariant lines, whose complements are γ-invariant affine patches in $\mathbb{R}\mathsf{P}^2$. Corresponding to the maximum (respectively minimum) eigenvalue are two affine patches in which γ acts by a linear expansion (respectively linear contraction). There corresponds two Hopf manifolds with $\mathbb{R}\mathsf{P}^2$-structures, which can be glued together along closed geodesics (corresponding to the coordinate plane with the middle eigenvalue) to form new $\mathbb{R}\mathsf{P}^2$-surfaces with the "same" holonomy representation. (See [135, 75] for details.) The Sullivan-Thurston-Smillie example is illustrated in Figure 7.1.

About ten years later, the combined work [43] of the author [69] and Suhyoung Choi's dissertation [38, 39] (and its extensions in [40, 41]) completely described the deformation space $\mathbb{R}\mathsf{P}^2(\Sigma)$. Choi shows that if Σ is closed and $\chi(\Sigma) < 0$, then any $\mathbb{R}\mathsf{P}^2$-surface decomposes *canonically* into annuli bounded by closed geodesics and convex surfaces with totally geodesic boundary. $\mathbb{R}\mathsf{P}^2(\Sigma)$ is a countable disjoint union of copies of the deformation space of *convex* $\mathbb{R}\mathsf{P}^2$-structures (shown in [69] to be a cell of dimension $-8\chi(\Sigma)$). The components are parametrized by a discrete invariant involving the multicurve on Σ controlling Choi's convex decomposition.

Much more recently Choi observed that these grafted $\mathbb{R}\mathsf{P}^2$-structures with Schottky $\mathsf{SO}(2,1)$-holonomy *compactify* Margulis spacetimes [42].

7.3. **Incomplete affine structures on closed 3-manifolds.** The classification of incomplete affine structures in dimension 3 is largely unkown, except under rather strong assumptions on the fundamental group. Smillie's work [134] on closed affine manifolds with abelian holonomy was generalized by Fried-Goldman-Hirsch [65] to nilpotent holonomy, and leads to a classification of closed 3-manifolds with nilpotent fundamental group. Serge Dupont [55] gave a beautiful classification of affine structures on hyperbolic 3-manifolds, which we briefly describe below.

7.3.1. *Parallel volume.* An affine manifold has *parallel volume* if and only if its linear holonomy preserves volume (up to sign). Equivalently the linear holonomy has determinant ± 1. Another equivalent condition is the existence of a coordinate atlas whose coordinate changes preserve volume.

The obstruction to parallel volume is the class in $H^1(M;\mathbb{R})$ defined by the homomorphism

$$\begin{aligned} \pi_1(M) &\longrightarrow \mathbb{R} \\ \gamma &\mapsto \log|\det L(\gamma)|. \end{aligned}$$

When the first Betti number vanishes, every affine structure must admit *parallel volume.*

Then the results of §7.3.1 below apply to give nonexistence of affine structures on certain closed 3-manifolds.

Theorfem 7.1 (Smillie)**.** *Let M be a closed affine manifold with a parallel exterior differential k-form which has nontrivial de Rham cohomology class. Suppose $\mathcal{U}$ is an open covering of M such that for each $U \in \mathcal{U}$, the affine structure induced on U is radiant. Then* $\dim \mathcal{U} \geq k$*; that is, there exist $k+1$ distinct open sets*

$$U_1, \dots, U_{k+1} \in \mathcal{U}$$

such that the intersection

$$U_1 \cap \cdots \cap U_{k+1} \neq \emptyset.$$

(Equivalently the nerve of $\mathcal{U}$ has dimension at least k.)

A published proof of this theorem can be found in Goldman-Hirsch [70].

Using these ideas, Carrière, d'Albo and Meignez [31] have proved that a nontrivial Seifert 3-manifold with hyperbolic base cannot have an affine structure with parallel volume. This implies that the 3-dimensional Brieskorn manifolds $M(p, q, r)$ with

$$p^{-1} + q^{-1} + r^{-1} < 1$$

admit no affine structure whatsoever. (Compare Milnor [125].)

7.3.2. *Hyperbolicity.* The opposite of geodesic completeness is *hyperbolicity* in the sense of Vey [143] and Kobayashi [98, 97], which is equivalent to the following notion: Say that an affine manifold M is *completely incomplete* if there exists no affine map $\mathbb{R} \longrightarrow M$, that is, M admits *no* complete geodesic. Similarly, an $\mathbb{R}\mathsf{P}^n$-manifold is *completely incomplete* if there exists no projective map $\mathbb{R} \longrightarrow M$. As noted by the author (see Kobayashi [98]), the combined results of Kobayashi [98], Wu [151], and Vey [144] imply:

Theorem. *Let M be a closed hyperbolic affine manifold. Then M is a quotient of a sharp convex cone.*

In particular M is radiant. Moreover M fibers over S^1 (which implies that $\chi(M) = 0$ and $b_1(M) > 0$.

For projective manifolds, taking the radiant suspension of a hyperbolic projective structure yields a radiant affine structure, which one easily sees is hyperbolic. Applying the above theorem imples that M is a quotient of a sharp convex cone.

This striking characterization of hyperbolicity uses *intrinsic metrics* in the category of affine and projective manifolds, developed by Vey [143] and Kobayashi [98, 97]. Their constructions were inspired by the intrinsic metrics of Carathéodory and Kobayashi in the category of complex manifolds.

Denote by I the open unit interval $\big(-1, 1\big)$ and

$$g_I := \frac{4}{(1-u^2)^2} du^2$$

its *Poincaré metric.*

For projective manifolds M, one defines a "universal" pseudo-metric $M \times M \xrightarrow{d_M} \mathbb{R}$ such that affine (respectively projective) maps $I \to M$ are distance non-increasing with respect to g_I.

The definition of d_M enforces the triangle inequality by taking the infimum of g_I-distances over sequences $x_0 = x, x_1, \dots, x_m = y$ where x_i and x_{i+1} are "close" in the following sense: there are projective maps $I \xrightarrow{f_i} M$ such that $x_i = f_i(a_i)$ and $x_{i+1} = f_i(b_i$ for $-1 < a_i < b_i < 1$. Then define $d_M(x, y)$ as the infimum over all such sequences (f_i, a_i, b_i)

of

$$\sum_{i=0}^{m-1} d_I(a_i, b_i)$$

where d_I is the distance function on the Riemannian 1-manifold $\big(I, g_{[-1,1]}\big)$.

That is, $d_M(x, y)$ is the infimum of

$$\int_a^b f\big(\gamma'(t)\big)\,dt$$

over all piecewise C^1 paths $[a, b] \xrightarrow{\gamma} M$ with $\gamma(a) = x, \gamma(b) = y$.

This function has an infinitesimal form, defined by a nonnegative upper-semicontinuous function $TM \xrightarrow{\phi} \mathbb{R}$. For affine manifolds, completeness is equivalent to $f \equiv 0$.

Following Kobayashi and Vey, M is *projectively hyperbolic* if and only if d_M is a *metric,* that is, if $d_M(x, y) > 0$ for $x \neq y$. Then d_M is a *Finsler metric* and equals the Hilbert metric on the convex domain $\widetilde{M}$.

When M is affine, then Vey [144] proves that M is a quotient of a sharp convex cone. In that case there is (in addition to the Hilbert metric), a natural Riemannian metric introduced by Vinberg [145], Koszul [102, 101, 104, 105] and Vesentini [142]. In particular Koszul and Vinberg observe that this Riemannian structure is the covariant differential $\nabla\omega$ of a closed 1-form ω. In particular ω is everywhere nonzero, so by Tischler [139], M fibers over S^1.

7.3.3. *Hessian manifolds.* Hyperbolic affine manifolds are closely related to *Hessian manifolds.* If ω is a closed 1-form, then its covariant differential $\nabla\omega$ is a symmetric 2-form. Since closed forms are locally exact, $\omega = df$ for some function; in that case $\nabla\omega$ equals the *Hessian* $d^2 f$. Koszul [104] showed that hyperbolicity is equivalent to the existence of a closed 1-form ω whose covariant differential $\nabla\omega$ is positive definite, that is, a Riemannian metric. More generally, Shima [133] considered Riemannian metrics on an affine manifold which are locally Hessians of functions, and proved that such a closed *Hessian* manifold is a quotient of a convex domain, thus generalizing Koszul's result.

7.3.4. *Hyperbolic torus bundles.* Although the class of affine structures on closed 3-manifolds with *nilpotent* holonomy are understood, the general case of *solvable* holonomy remains mysterious. However, Serge Dupont [55] completely classifies affine structures on 3-manifolds with solvable *fundamental group.* (Compare also Dupont [54].) In terms of the Thurston geometrization, these are the geometric 3-manifolds modeled on Sol, that is, 3-manifolds finitely covered by *hyperbolic torus bundles:* mapping tori (suspensions) of hyperbolic elements of $\mathsf{GL}(2,\mathbb{Z})$. Dupont shows that all such structures arise from left-invariant affine structures on the corresponding Lie group G, which is the semidirect product of $\mathbb{R}^2$ by $\mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ as a unimodular hyperbolic one-parameter subgroup (explicitly, G is isomorphic to the identity component in the group of Lorentz isometries of the Minkowski plane).

Two structures are particularly interesting for the behavior of geodesics in light of the results of Vey [144]. A properly convex domain $\Omega \in \mathsf{A}^n$ is said to be *divisible* if Ω admits a discrete group Γ of projective automorphisms acting properly on Ω such that Ω/Γ is compact. (Equivalently, the quotient space Ω/Γ by a discrete subgroup $\Gamma \subset \mathsf{Aut}(\Omega)$ is compact and Hausdorff.) Vey proved that a divisible domain is a cone. However, dropping the properness of the action of Γ on Ω allows counterexamples: the *parabolic cylinder*

$$\Omega := \{(x,y) \in \mathsf{A}^2 \mid y > x^2\}$$

is a properly convex domain which is not a cone, but admits a group Γ of automorphisms such that Ω/Γ is compact but not Hausdorff.

Now take the product $\Omega \times \mathbb{R} \subset \mathsf{A}^3$. The author [73] found a discrete subgroup $\Gamma \subset \mathsf{Aff}(\mathsf{A}^3)$ acting properly on $\Omega \times \mathbb{R}$ such that:

- The quotient $M = (\Omega \times \mathbb{R})/\Gamma$ is a hyperbolic torus bundle (and in particular compact and Hausdorff);
- $\Omega \times \mathbb{R}$ is not a cone.

Clearly $\Omega \times \mathbb{R}$ is not properly convex, showing that Vey's result is sharp. The Kobayashi pseudometric degenerates along a 1-dimensional foliation of M, and defines the hyperbolic structure transverse to this foliation discussed by Thurston [137], Chapter 4.

7.4. **Flat conformal structures in higher dimensions.** Flat conformal structures generalize $\mathbb{C}\mathsf{P}^1$-structures, under the identification of $\mathbb{C}\mathsf{P}^1$ with S^2 with its usual conformal structure. In general the group of conformal transformations of S^n is the group $\mathsf{PO}(n+1,1)$, where $S^n \hookrightarrow \mathbb{R}\mathsf{P}^{n+1}$ embeds as a quadric invariant under the projectivized Lorentz group $\mathsf{PO}(n+1,1)$. See Matsumoto [122] for an excellent survey.

Flat conformal structures arise in Riemannian geometry. Specifically, a Riemannian manifold (M,g) is *conformally flat* if and only if every point $p \in M$ possesses an open neighborhood U and a smooth coordinate chart $U \xrightarrow{\psi} \mathbb{E}^n$ such that the Riemannian structure on U induced by g is conformally equivalent to the Euclidean structure. (Sometimes this condition is called *locally conformally flat.*) When

$n > 2$, a flat conformal structure in our sense is then equivalent to a conformal equivalence class of conformally flat Riemannian structures. See Kulkarni [111, 112] for more background.

Kuiper [107] initiated the subject of flat conformal structures, and in [108], classified those with abelian fundamental group. Kulkarni [111] defined connected sum operation between flat conformal manifolds. The construction is based on the fact that a conformal inversion interchanges the two components of the complement of a hypersphere in S^n. In other words, the inside and the outside of a hypersphere in S^n are *conformally equivalent.* Thus a closed 3-manifold need not be *geometric* in Thurston's sense to admit a flat conformal structure. However, as shown by the author [74], 3-manifolds with nilgeometric and solvgeometric structures do *not* admit flat conformal structures. These were the first examples of 3-manifolds *without* flat conformal structures.

7.5. **Real projective structures in higher dimensions.** In a series of papers [17, 19, 18, 20, 21], Yves Benoist developed a vast theory of convex $\mathbb{R}\mathsf{P}^n$-structures on compact manifolds. (See Benoist [22] for a survey.) In particular he analyzed the boundary and showed that strict convexity is equivalent to hyperbolicity in various contexts.

All of these studies involve the projectively invariant *Hilbert metric* on a properly convex domain. When the domain is bounded by a quadratic, this metric is just the hyperbolic metric in the Beltrami-Klein projective model of hyperbolic space. See Marquis [120], and in general the collection [128] for surveys of Hilbert geometry on such manifolds. Recently Benoist's theory of convex $\mathbb{R}\mathsf{P}^n$-structures on compact manifolds has been extended to the analog of finite volume hyperbolic manifolds. In particular we mention the work of Cooper-Long-Tillmann [47] on cusped $\mathbb{R}\mathsf{P}^n$-manifolds, as well as Choi [37], Choi-Lee-Marquis [44] and Marquis [119].

Kapovich [94] gave examples of convex $\mathbb{R}\mathsf{P}^n$-structures on compact negatively curved Riemannian manifolds which admit no locally symmetric Riemannian metric.

Which closed 3-manifolds admit $\mathbb{R}\mathsf{P}^3$-structures is an interesting and difficult question. Unlike flat conformal structures, the topology of $\mathbb{R}\mathsf{P}^3$ precludes any inversion such as the Steiner inversion facilitating the Kulkarni connected-sum operation. (Indeed, the two components of the complement of a projective hyperplane in projective space are not even *topologically* equivalent.) In this direction, Weiqiang Wu [152] showed that any compact $\mathbb{R}\mathsf{P}^n$-structure bounded by a sphere on its convex side must be a disc — as noted above, this rigidity phenomenon is evidently absent for flat conformal manifolds. In particular it seems notoriously difficult to construct an $\mathbb{R}\mathsf{P}^3$-structure on a connected sum. In this vein, Cooper-Goldman [48] showed thatthe connected sum $\mathbb{R}\mathsf{P}^3 \# \mathbb{R}\mathsf{P}^3$ fails to admit an $\mathbb{R}\mathsf{P}^3$-structure; as of yet we know very few obstructions for a 3-manifold *not* to admit a flat projective structure.

7.6. **Complex projective structures in higher dimensions.** In a different direction, Klingler [96] classified (holomorphic) projective structures on complex surfaces, following earlier work by Vitter [146] and Kobayashi-Ochiai [90, 99, 100]. Every closed $\mathbb{C}\mathsf{P}^2$-manifold is finitely covered by a manifold of one of the following types:

- the complex projective plane $\mathbb{C}\mathsf{P}^2$;
- complex hyperbolic manfiolds;
- *complex solvmanifolds,* that is, homogeneous spaces $\Gamma\backslash G$ where G is a 4-dimensional (real) Lie group with left-invariant complex structure and $\Gamma \subset G$ is a lattice;
- Hopf manifolds $\mathbb{C}^2 \setminus \{0\}/\Gamma$, where Γ is a cyclic group of linear expansions.
- elliptic surfaces over $\mathbb{C}\mathsf{P}^1$-manifolds, that is, holomorphic fibrations by elliptic curves over a Riemann surface with a projective structure.

These two latter classes are affine structures.

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