Playing pool on curved surfaces and the wrong way to add fractions

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Public Lecture
20 March 2013
Playing pool on curved surfaces...
Mathematics: a *MOST* exact science

- Natural phenomena understood through quantitative measurements
  - Which are abstracted into mathematics.
  - These abstract ideas can be manipulated rigorously to make predictions.
  - Mathematical statements form a language in which measurements can be processed.
  - Mathematics represents an *ideal* situation which approximates the everyday world.

- For example:
  - Rates of change governed by laws of calculus.
  - Force = Mass \cdot Acceleration.
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Billiards on a square

- A billiard ball starts moving once it is subjected to the initial force, and changes direction when it bounces off the side of a billiard table.
  - Here is an example of a billiard ball on a square billiard table, which follows a periodic path.
  - Here is a longer periodic path. When the slope is rational (a fraction of two whole numbers), the path is periodic.
  - When the slope is irrational, the path never closes up, and eventually fills the whole square.

- Example of the inter-relationship between seemingly different subjects of mathematics: arithmetic (number theory), and differential equations (mechanics).
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Looking for universal patterns

- The same kind of differential equations that govern the motion of a moving ball can govern population growth, financial markets, chemical reactions...
  - Because they exhibit similar patterns.

- Mathematics is scalable:
  - What’s true in the small is true in the large.

- Mathematics is reproducible:
  - Governed only by abstract logic,
  - And does not need special equipment, just working conditions conducive for clear thinking.
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Promote recurring patterns into primitive concepts.
- Break complicated relationships into simpler ones.
- Consolidating definitions creates new concepts.

Sometimes finding the right question is just as important as finding the right answer!

Asking and answering questions about the simpler concepts creates new mathematics.
- And it keeps on going...
- And growing.

More mathematics created in the last 50 years than before.

Challenge: How can you learn enough of what has already been done to create new mathematics?
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Art: beauty in the simplicity of ideas

- Sensing a familiar pattern in an unexpected setting;
- Familiarity is not only reassuring but empowering.
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The Golden Ratio

- The Parthenon is in the proportion of the *Golden Ratio*:
  \[
  \phi = \frac{1 + \sqrt{5}}{2} \approx 1.618033988749894848204586834365638117720309179848622705260462
  \]

- which also appears in the geometry of a seashell.
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A fraction which continues...

- $\phi \approx 1.618 \ldots$ satisfies the algebraic equation

  $$\phi = 1 + \frac{1}{\phi}$$

- Replacing $\phi$ by $1 + \frac{1}{\phi}$ in this expression:

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What does this infinite fraction mean?

- This infinite expression is **meaningless** until we give it meaning!
  - Mathematicians change the questions to fit the answers!
- For example, define it to be the *limit* of the sequence
  
  \[
  1, \ 1 + \frac{1}{1} = 2, \ 1 + \frac{1}{2} = \frac{3}{2}, \ 1 + \frac{1}{3/2} = \frac{5}{3}, \ 1 + \frac{1}{5/3} = \frac{8}{5}, \ 1 + \frac{1}{8/5} = \frac{13}{8}, \ldots
  \]
- Numerators and denominators are *Fibonacci numbers*:

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  \[1 + 1 = 2, 3, 5, 8, 13, 21, 34, \ldots\]
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Approximate \( \phi \) with billiards!
Notice a pattern in the sequence of fractions approximating $\phi$:

\[
\begin{align*}
1 \quad &2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55 \quad 89 \\
1' \quad &1' \quad 2' \quad 3' \quad 5' \quad 8' \quad 13' \quad 21' \quad 34' \quad 55' \quad \cdots
\end{align*}
\]

Each fraction is obtained from the preceding pair by adding numerators and denominators:

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\frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.
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\overline{1}' & \overline{1}' & \overline{2}' & \overline{3}' & \overline{5}' & \overline{8}' & \overline{13}' & \overline{21}' & \overline{34}' & \overline{55}' \\
\end{array}
\]

Each fraction is obtained from the preceding pair by adding numerators and denominators:

\[
\frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.
\]

\[
\begin{array}{cccccccccccc}
  1 & \oplus & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 \\
\overline{1} & \oplus & \overline{1} & \overline{2}' & \overline{3}' & \overline{5}' & \overline{8}' & \overline{13}' & \overline{21}' & \overline{34}' & \overline{55}' \\
\end{array}
\]
The wrong way to add fractions

- Notice a pattern in the sequence of fractions approximating $\phi$:

$$
\begin{array}{cccccccccc}
1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 \\
1' & 1' & 2' & 3' & 5' & 8' & 13' & 21' & 34' & 55' \\
\end{array}
$$

- Each fraction is obtained from the preceding pair by \textit{adding numerators and denominators}:

$$
\frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.
$$

$$
\begin{array}{cccccccccc}
1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 \\
1' & 1' \oplus 2' & = & 3' & 5' & 8' & 13' & 21' & 34' & 55' \\
\end{array}
$$
The wrong way to add fractions

- Notice a pattern in the sequence of fractions approximating $\phi$:

  $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \ldots$

- Each fraction is obtained from the preceding pair by adding numerators and denominators:

  $$\frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.$$ 

  $\frac{1}{1}, \frac{2}{1} \oplus \frac{5}{3} = \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \ldots$
The wrong way to add fractions

- Notice a pattern in the sequence of fractions approximating $\phi$:

  \[
  \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \ldots
  \]

- Each fraction is obtained from the preceding pair by adding numerators and denominators:

  \[
  \frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.
  \]

  \[
  \frac{1}{1}, \frac{1'}{1'}, \frac{3'}{2'}, \frac{5'}{5'}, \frac{8'}{8'}, \frac{13'}{13'}, \frac{21'}{21'}, \frac{34'}{34'}, \frac{55'}{55'}, \ldots
  \]
The wrong way to add fractions

- Notice a pattern in the sequence of fractions approximating φ:

  \[
  \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \ldots
  \]

- Each fraction is obtained from the preceding pair by adding numerators and denominators:

  \[
  \frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.
  \]

  \[
  \frac{1}{1}, \frac{1'}{1'}, \frac{2'}{2'}, \frac{3'}{3'}, \frac{5'}{5'}, \frac{8'}{8'}, \frac{13'}{13'}, \frac{21'}{21'}, \frac{34'}{34'}, \frac{55'}{55'}, \ldots
  \]
The wrong way to add fractions

Notice a pattern in the sequence of fractions approximating $\phi$:

\[
\begin{align*}
1 & \quad 2 & \quad 3 & \quad 5 & \quad 8 & \quad 13 & \quad 21 & \quad 34 & \quad 55 & \quad 89 \\
1' & \quad 1' & \quad 2' & \quad 3' & \quad 5' & \quad 8' & \quad 13' & \quad 21' & \quad 34' & \quad 55' & \cdots
\end{align*}
\]

Each fraction is obtained from the preceding pair by adding numerators and denominators:

\[
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\]

\[
\begin{align*}
1 & \quad 2 & \quad 3 & \quad 5 & \quad 8 & \quad 13 & \quad 21 & \quad 34 & \quad 55 & \quad 89 \\
1' & \quad 1' & \quad 2' & \quad 3' & \quad 5' & \quad 8' & \quad 13' & \quad 21' & \quad 34' & \quad 55' & \cdots
\end{align*}
\]
The wrong way to add fractions

- Notice a pattern in the sequence of fractions approximating $\phi$:

\[
1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55 \quad 89
\]

\[
\frac{1}{1'} \quad \frac{1'}{1'} \quad \frac{2'}{3'} \quad \frac{3'}{5'} \quad \frac{8'}{13'} \quad \frac{13'}{21'} \quad \frac{21'}{34'} \quad \frac{34'}{55'} \quad \cdots
\]

- Each fraction is obtained from the preceding pair by adding numerators and denominators:

\[
\frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.
\]

\[
\frac{1}{1'} \oplus \frac{2'}{3'} = \frac{3}{2} = \frac{34}{21} = \frac{55'}{34'} \quad \cdots
\]
The wrong way to add fractions

- Notice a pattern in the sequence of fractions approximating $\phi$:
  
  \[
  \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \cdots
  \]

- Each fraction is obtained from the preceding pair by *adding numerators and denominators*:
  
  \[
  \frac{a}{b} \oplus \frac{c}{d} = \frac{a + c}{b + d}.
  \]

  \[
  \frac{1 \oplus 1}{1 + 1}, \frac{2 \oplus 2}{1 + 1}, \frac{3 \oplus 3}{2 + 2}, \frac{5 \oplus 5}{3 + 3}, \frac{8 \oplus 8}{5 + 5}, \frac{13 \oplus 13}{8 + 8}, \frac{21 \oplus 21}{13 + 13}, \frac{34 \oplus 34}{21 + 21}, \frac{55 \oplus 55}{34 + 34} = \frac{89}{55}, \cdots
  \]
Farey series

- List the fractions (in order) with denominator \( \leq n \):
- Each fraction is obtained from the two closest ones above by adding *numerators* and *denominators*: \( \frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d} \).

- \( n = 6 \):

\[
0, 1, 1, 1, 2, 1, 3, 2, 3, 4, 5, 1, 7, 6, 5, 4, 7, 3, 8, 5, 7, 9, 11, 2
\]

- John Farey, Sr. (1766–1826), a British geologist, was led to these discoveries through his interest in the mathematics of sound. *(Philosophical Magazine 1816).*
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- \( n = 6 \):
  - \( 0, 1, 1, 1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 1, 7, 6, 5, 4, 7, 3, 8, 5, 7, 9, 11, 2 \)
  - \( \frac{1}{1}, \frac{5}{6}, \frac{4}{3}, \frac{5}{2}, \frac{5}{3}, \frac{4}{5}, \frac{5}{4}, \frac{6}{1}, \frac{2}{5}, \frac{5}{4}, \frac{4}{3}, \frac{5}{2}, \frac{5}{3}, \frac{4}{5}, \frac{6}{1}, 1 \)

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- $n = 6$:
  \[
  \begin{array}{cccccccccccccccc}
  0 & 1 & 1 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 4 & 5 & 1 & 7 & 6 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 9 & 11 & 2 \\
  1 & 5 & 6 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 6 & 1 & 6 & 5 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 6 & 1 \\
  \end{array}
  \]

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- $n = 1$ :
  
  
  \[
  \begin{array}{ccc}
  0 & 1 & 2 \\
  1 & 1 & 1 \\
  \end{array}
  \]

- $n = 6$ :
  
  
  \[
  \begin{array}{cccccccccccccccccccccccc}
  0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 4 & 5 & 1 & 7 & 6 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 9 & 11 & 2 \\
  1 & 5 & 6 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 6 & 1 & 6 & 5 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 6 & 1 \\
  \end{array}
  \]

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- $n = 2$:
  
  $\begin{array}{cccc}
  0 & 1 & 1 & 3 \\
  1 & 1 & 2 & 1 \\
  \end{array}$

- $n = 6$:
  
  $\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
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- \( n = 3 \):

\[
0, 1, 1, 2, 1, 4, 3, 5, 2
\]

\[
1, 3, 2, 3, 1, 3, 2, 3, 2
\]

- \( n = 6 \):

\[
0, 1, 1, 1, 1, 2, 1, 3, 2, 3, 4, 5, 1, 7, 6, 5, 4, 7, 3, 8, 5, 7, 9, 11, 2
\]

\[
1, 5, 6, 4, 3, 5, 2, 5, 3, 4, 5, 6, 1, 6, 5, 4, 3, 5, 2, 5, 3, 4, 5, 6, 1
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- \( n = 4 \):
  
  \[
  0, 1, 1, 2, 3, 1, 5, 4, 3, 5, 7, 2
  \]
  
  \[
  1, 4, 3, 2, 3, 4, 1, 4, 3, 2, 3, 4, 1
  \]
- \( n = 6 \):
  
  \[
  0, 1, 1, 1, 2, 1, 3, 2, 3, 4, 5, 1, 7, 6, 5, 4, 7, 3, 8, 5, 7, 9, 11, 2
  \]
  
  \[
  1, 5, 6, 4, 3, 5, 2, 5, 3, 4, 5, 6, 1, 6, 5, 4, 3, 5, 2, 5, 3, 4, 5, 6, 1
  \]
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Farey series

- List the fractions (in order) with denominator $\leq n$:
- Each fraction is obtained from the two closest ones above by adding *numerators* and *denominators*: $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$.

$n = 5$:

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 4 & 1 & 6 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 9 & 2 \\
1 & 5 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 1 & 6 & 5 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 1
\end{array}
\]

$n = 6$:

\[
\begin{array}{cccccccccccccccccccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 4 & 5 & 1 & 7 & 6 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 9 & 11 & 2 \\
1 & 5 & 6 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 6 & 1 & 6 & 5 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 6 & 1
\end{array}
\]

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- \( n = 6 \):

\[
\frac{0}{1}, \frac{1}{5}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{4}{5}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \frac{11}{6}, \frac{2}{1}
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John Farey, Sr. (1766–1826), a British geologist, was led to these discoveries through his interest in the mathematics of sound. (*Philosophical Magazine* 1816).
How a mathematical concept is created

- A pattern is isolated.
  - Focus on its essential qualities.
- Promote it to a new concept
  - Give it a definition.
- Relate it to already defined concepts through theorems,
  - which must be rigorously proved!
  - The right definitions may make the theorems much easier to prove.
- Similar to art: a human representation of an abstract pattern.
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Challenges to doing mathematics

Its unique nature leads to basic challenges in its teaching, communication, and dissemination, unlike any other intellectual discipline.
A remarkably *successful* discipline

- Mathematics goes back thousands of years, and ...
  - continues to grow.
- Old mathematics is *not* discarded ...
  - but *condensed*.
- Leading to challenges in disseminating, organizing, teaching ...
- As more common relationships are discovered, ideas *generalize* ...
  - and the subject becomes more and more abstract ...
    - And specialized.
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  - And *specialized*. 

Going out of control?

- Too many subdivisions...
  - Despite basic unity, a natural tendency to splinter.
- Specialization must be controlled and resisted as the subject develops.
- Last 30 years: remarkable confluence of mathematical ideas.
  - Making it even harder to learn!

The Tower of Babel
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Investing in mathematics is investing in *people!*

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  - And the developers ...
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Mathematics: A fundamentally *human* activity.

Terrapins work out the equations of straight lines on curved surfaces.
Building communities to promote mathematics

Potomac High School students visit the Experimental Geometry Lab.
Why support mathematics?

- A rapidly changing society needs people who can:
  - Learn and work with abstract ideas,
  - Communicate them effectively
    - ... all in a short period of time...

![Image of a group of people]
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A community activity
Mathematics:

- **A Science**: a rigorous exact discipline which formulates statements modeling natural phenomena.
- **A Language**: a collection of ideas, represented symbolically and organized into units of communication.
- **An art**: an esthetic activity, characterized by elegance and simplicity, despite its innate complexity.
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These three roles complement each other in a unique way.

And the growth of mathematics leads to serious challenges in

- Training,
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Playing pool on curved surfaces...