COMPLEX MULTIPLICATION - COURSE OVERVIEW

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The goal of the first two months of the course is to understand the proof of the main theorem of Complex Multiplication for elliptic curves, following the exposition in Silverman. The first month or so we will spend laying the foundation for the statement and proof, this means giving a reasonable introduction to the theory of Riemann surfaces, elliptic curves, elliptic and modular functions and class field theory.

We will then begin our study of elliptic curves with complex multiplication, and prove the special properties about them such as the algebraicity of the j-invariant. Once the groundwork has been set we will proceed with the proof of the main theorem. This mixes techniques from complex analysis, algebraic geometry and algebraic number theory, however you certainly don't need to be an expert in each of these subjects to understand it! Some familiarity with these subjects would be desirable but part of our aim for this tutorial was to give you small glimpse into each of these areas of mathematics so that when you do come to take a course in them, you will already have built up some intuition as well as having a handy number of examples to play around with.

Once we have finished the proof, there are a number of topics that we could cover (our suggestion would be to study Hecke characters and their associated l-adic representations) but there are other options and we can assess people's interests nearer the time.

The only prerequisite we set for this course is a good understanding of Galois theory and a familiarity with the basic theorems of complex analysis, other than that we hope the course will be self-contained. Having said this, the proofs will need a good understanding of algebraic number theory (especially class field theory) and some basic algebraic geometry. This will all be covered in class but we have included a reading list for those who wish to be extra should prepared for the course. In particular for the results needed from global class field theory can be found in [CF86] and one should consult [Har77] Chapter I for the basics of algebraic geometry.

Tentative schedule

Week 1

Review of Galois theory. Basic number theory. Adeles and Ideles. Motivation of the theory of CM.

Week 2

Riemann surfaces. Classification of dimension 1 complex tori. Weierstrass $\wp$ function. Uniformization theorem.

Week 3

Elliptic and modular functions, inluding their origins in the study of elliptic integrals. j-invariant.

Week 4

Introduction to classical algebraic geometry. Basic theory of algebraic curves over a general field.

Week 5

Elliptic curves as algebraic curves. Elliptic curves over finite fields. Algebraic definition of group law. Isogenies of elliptic curves.

Week 6

Elliptic curves with CM. j-invariant of a CM elliptic curve. The Hilbert class field of a quadratic imaginary field.

Week 7

Global class field theory, statements of main theorems and consequences. Dirichlet's theorem. The maximal abelian extension of a quadratic imaginary field.

Week 8

Statement of the main theorem of CM and its proof. Adelic formulation. t

Week 9

Hecke characters and L-functions. The Hasse-Weil zeta function of elliptic curves. Modularity.

Week 10

The Hecke character associated to a CM elliptic curve. The Hasse-Weil zeta function of CM elliptic curves.

The rest lectures

CM theory of abelian varieties or other topics depending on the interest of the audience.

Some suggested project topics

The canonical models of modular curves and Shimura varieties; Eichler-Shimura theory; Tate's thesis; Honda-Tate theory; p-divisible groups and Hodge-Tate decomposition; Lubin-Tate theory; Integrality of the j-invariant (good reduction proof);

Reading material

For the basic concepts and results of algebraic number theory, the book by Neukirch [Neu99] offers a quite complete and systematic account. The modern introduction to class field theory can be found in many books, among which Cassels-Frohlich [CF86] is still outstanding. For basics in complex analysis, see [Ahl78], which is a standard textbook of the subject that is a real enjoyment to read. For the theory of elliptic curves from various perspectives, see Silverman's book [Sil09]. We will follow Silverman's other book [Sil94] Chap. 2 when we prove the main theorem of CM. For an introduction to modular forms, see the book [DS05] by Diamond and Schurman, the first author Diamond being one of the four collaborators who proved the final version of the celebrated modularity theorem. For the theory of abelian varieties, which are higher dimensional generalizations of elliptic curves, the classical reference is Mumford's book [Mum08]. Also helpful is the book project by van der Geer and Ben Moonen that you can find on their websites. For complex multiplication of abelian varieties, see Lang's book [Lan83], James Milne's online notes, and the online notes of a seminar organized by Brian Conrad. You will probably also find James Milne's online notes on various things (class field theory, abelian varieties, complex multiplication for abelian varieties, etc.) helpful. If you are interested to know how to put this circle of ideas in the context of the Langlands program, a quite friendly introduction is provided in [BCdS$^+$03]. You can even look at the articles in the Corvallis proceedings [BC79a] and [BC79b]. We also mention the book [Cox89], which provided a really motivated introduction to class field theory and complex multiplication.

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