STABILIZATION OF THE TRACE FORMULA FOR ORTHOGONAL SHIMURA VARIETIES

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Let (G, X) be a Shimura datum. Let $\mathrm{Sh}_K = \mathrm{Sh}_K(G, X)$ be the associated Shimura variety for a neat open compact subgroup $K \subset G(\mathbb{A}_f)$. It is smooth and quasi-projective over the reflex field E. To simplify notation we assume $E = \mathbb{Q}$ in the following.

We temporarily assume that Sh_K is proper. With the aid of the (conjectural) formalism of Arthur parameters, the local Euler factors of the Hasse-Weil zeta function $\zeta(\mathrm{Sh}_K, s)$ are given conjecturally as follows:

Conjecture A (Kottwitz, cf. [7, §10], [5, Conj. 5.2]). *For almost all primes p, the following local Euler factors are equal*

$$\zeta_p(\mathrm{Sh}_K, s) = \prod_{i=0}^{2\dim \mathrm{Sh}_K} \prod_\psi \prod_{\pi_f} \prod_\nu L_p\left(s - \frac{i}{2}, \ \psi^i(\nu)\right)^{(-1)^{i+1} m(\psi, \nu, \pi_f)}.$$

Here ψ runs through the Arthur parameters that are cohomological at infinity, π_f runs through representations of $G(\mathbb{A}_f)$ belonging to ψ, and ν runs through suitable characters of the group S_ψ of self-equivalences of ψ. For given (ψ, π_f, ν), the expression $L_p(\cdot, \psi^i(\nu))$ is the local Euler factor of an automorphic L-function, and $m(\psi, \nu, \pi_f) \in \mathbb{Z}$ is determined by a certain recipe.

To approach Conjecture A, Langlands and Kottwitz have developed the strategy of comparing Grothendieck-Lefschetz trace formulas and stable Arthur-Selberg trace formulas, summarized in Conjecture B below. We drop the assumption that Sh_K is proper. Let $\overline{\mathrm{Sh}_K}$ be the Baily-Borel compactification of Sh_K, which is a normal projective variety over E compactifying Sh_K. Let $\mathbf{IH}^*$ denote the formal alternating sum of the (ℓ-adic) intersection cohomology of $(\overline{\mathrm{Sh}_K})_{\bar{\mathbb{Q}}}$. (For the motivation of considering $\mathbf{IH}^*$ cf. [12, §2] or the introduction of [17])

Conjecture B ([7, §10]). *Let $p > 2$ be a hyperspecial prime for K, in the sense that $K = K_p K^p$ with $K_p \subset G(\mathbb{Q}_p)$ hyperspecial and $K^p \subset G(\mathbb{A}_f^p)$. Assume $\ell \neq p$ in the definition of $\mathbf{IH}^*$. We have a commuting action of the Hecke algebra $\mathcal{H}(G(\mathbb{A}_f^p)//K^p)$ and $\mathrm{Gal}(\bar{E}/E)$ on $\mathbf{IH}^*$. Let $f^{p,\infty} \in \mathcal{H}(G(\mathbb{A}_f^p)//K^p)$, and let $\Phi_p \in \mathrm{Gal}(\bar{E}/E)$ be any lift of the geometric Frobenius at p. We have, at least for $j \in \mathbb{N}$ large enough,*

$$(1) \qquad \mathrm{Tr}(f^{p,\infty} \times \Phi_p^j \mid \mathbf{IH}^*) = \sum_H \iota(G, H) ST^H(f^H),$$

where in the summation H runs through the elliptic endoscopic data of G. For each H the test function $f^H : H(\mathbb{A}) \to \mathbb{C}$ is prescribed in terms of $f^{p,\infty}, (G, X), j$ by a

certain recipe [1], *and $ST^H(\cdot)$ is the geometric side of the stable trace formula for H. The constants $\iota(G, H)$ are explicitly defined.*

Theorem 1 ([17])**.** *Conjecture B is true for the orthogonal Shimura varieties in Definition 1 below, for almost all p.*

Definition 1. The *orthogonal Shimura varieties* are associated to Shimura data (G, X) as follows: $G = \mathrm{SO}(V)$ where V is a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$, and X is the space of oriented negative definite planes in $V_{\mathbb{R}}$. For simplicity we assume $n \geq 5$.

Remark 1. $(n, 2)$ is the only possible signature for $G = \mathrm{SO}(V)$ to have a Shimura datum. The orthogonal Shimura varieties are non-compact and not of PEL type. They are examples of abelian type Shimura varieties of types B and D.

Remark 2. In the proof of Theorem 1 we have used the simplified formulas for $ST^H(f^H)$ due to Kottwitz. It seems that a proof that they coincide with Arthur's stabilization [2][1][3] has not appeared in the published literature. In view of that, Theorem 1 should be regarded as a variant of Conjecture B, with each $ST^H(f^H)$ re-defined by Kottwitz's formulas.

Remark 3. The reason that Theorem 1 is only stated for almost all primes is due to a lack of the construction of integral models of the Baily-Borel and toroidal compactifications of abelian type (even at hyperspecial level) in the literature.

Remark 4. Theorem 1 is the analogue of Morel's earlier work [13] [14] for unitary and Siegel Shimura varieties.

Remark 5. In principle, one could deduce Conjecture A from Conjecture B and Arthur's conjectures. For some choices of G in Definition 1, the relevant Arthur's conjectures have been proved by Taïbi [16], generalizing Arthur's work on quasi-split classical groups [4]. In these cases we will pursue in a future work the proof of a version of Conjecture A based on Theorem 1.

We now present the idea of the proof of Theorem 1.

The work of Morel ([10] [11] [13], also cf. [12]), together with a theorem of Pink ([15]), allows one to compute the LHS of (1) in terms of compact support cohomology of the strata in $\overline{\mathrm{Sh}_K}$. The contribution from the compact support cohomology of Sh_K itself is computed and stabilized in an ongoing joint work with Kisin and Shin [9]. The boundary terms can be organized as a summation over standard proper Levi subgroups $M \subsetneq G_{\mathbb{Q}}$:

$$\text{Boundary terms on the LHS of } (1) = \sum_{M \subsetneq G} \mathrm{Tr}_M.$$

Thus the problem is to stabilize the above expression, i.e. to prove:

$$(2) \qquad \sum_{M \subsetneq G} \mathrm{Tr}_M = \sum_H \iota(G, H) \sum_{M' \subsetneq H} ST^H_{M'}(f^H).$$

Here M' runs through the standard proper Levi subgroups of $H_{\mathbb{Q}}$. On the RHS of (2), each term $ST^H_{M'}(f^H)$ has a relatively simple formula due to Kottwitz (Remark 2). On the LHS of (2), each term Tr_M is a mixture of the following:

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[1]The definition of f^H assumes the Langlands-Shelstad transfer and the fundamental lemma (at primes away from p and ∞).

- Kottwitz's fixed point formula [7][8].
- The topological Lefschetz formula of Goresky-Kottwitz-MacPherson [6].
- *Kostant-Weyl traces.* By this we mean Weyl character formulas for certain algebraic $\tilde{M}$-sub-representations of $\mathbf{H}^*(\text{Lie}\tilde{N}, \mathbb{C})$, where $\tilde{P} = \tilde{M}\tilde{N}$ is a standard parabolic subgroup of G containing M. These sub-representations are defined by truncation in terms of the weights of certain central cocharacters of $\tilde{M}$.

We highlight some key ingredients in the proof of (2).

Archimedean comparison. On the RHS of (2), the archimedean contributions consist of values of stable discrete series characters. On the LHS of (2), the archimedean contributions are understood to be the Kostant-Weyl traces discussed above. After explicit computation we prove identities between these two kinds of objects.

Proving cancellations. We have to prove cancellations among a lot of terms on the RHS of (2). These include the terms with a "bad" factor at p, and also the terms indexed by (H, M') with M' not transferring to G.

Comparing transfer factors. Signs are important to ensure the correct cancellation of terms. One important source of signs comes from comparing different normalizations of transfer factors for real endoscopy. We compute the difference between the normalization $\Delta_{j,B}$ (introduced in e.g. [7, §7]) and the Whittaker normalizations. In case there are more than one Whittaker data, we also compare the resulting different Whittaker normalizations.

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