IRREDUCIBLE COMPONENTS OF AFFINE DELIGNE–LUSZTIG VARIETIES

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1. Results for affine Deligne–Lusztig varieties

Affine Deligne–Lusztig varieties (ADLV's) were introduced by M. Rapoport in [Rap05], motivated by the study of reductions of Shimura varieties. Fix a local field F, with completed maximal unramified extension $\breve{F}$. Let k be the residue field of $\breve{F}$, which is algebraically closed. In this report we only discuss ADLV's at hyperspecial level. Thus an ADLV is a geometric object $X_\mu(b)$ attached to a triple (G, b, μ), where G is a (connected) reductive group scheme over $\mathcal{O}_F$, b is an element of $G(\breve{F})$, and μ is a cocharacter of $G_{\overline{F}}$ up to conjugacy. At the level of k-points, $X_\mu(b)(k)$ consists of $g \in G(\breve{F})/G(\mathcal{O}_{\breve{F}})$ such that the $G(\mathcal{O}_{\breve{F}})$-double coset of $g^{-1}b\sigma(g)$ corresponds to the conjugacy class of μ under the Cartan decomposition of $G(\breve{F})$. Here σ is the absolute arithmetic Frobenius in $\mathrm{Aut}(\breve{F}/F)$.

An important fact is that $X_\mu(b)$ has the geometric structure of a k-scheme locally of finite type (resp. a perfect k-scheme locally of perfectly finite type) if F has equal (resp. mixed) characteristic. The geometric structure comes from the geometric structure of the affine Grassmannian and the Witt vector affine Grassmannian (constructed by X. Zhu [Zhu17] and Bhatt–Scholze [BS17]) in the two cases. In the mixed characteristic setting, the geometry of ADLV's is closed related to Shimura varieties via the Rapoport–Zink uniformization (cf. [RZ96, HP17, Kim18]). For instance, information about connected components of ADLV's has played an important role in the version of Langlands–Rapoport Conjecture proved in [Kis17] and the subsequent strengthening in [KSZ].[1]

In this report we present results on the set $\Sigma^{\mathrm{top}}(X_\mu(b))$ of top-dimensional irreducible components of $X_\mu(b)$. (It is conjectured [Rap05, Conj. 5.10] that $X_\mu(b)$ is equidimensional, cf. [HV18, §3].) The motivation for studying this set comes from the relation with certain cycles in the basic locus of the special fiber of Shimura varieties, see for instance [XZ17]. The scheme $X_\mu(b)$ is equipped with an action of $J_b(F)$, the F-rational points of the Frobenius-centralizer J_b of b. This induces an action of $J_b(F)$ on $\Sigma^{\mathrm{top}}(X_\mu(b))$. We are interested in the following two questions.

(i) Classify the $J_b(F)$-orbits in $\Sigma^{\mathrm{top}}(X_\mu(b))$.

(ii) For each $Z \in \Sigma^{\mathrm{top}}(X_\mu(b))$, determine the stabilizer of Z in $J_b(F)$.

For question (i), M. Chen and X. Zhu conjectured a natural bijection between $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$ and the Mirkovic–Vilonen basis $\mathbb{MV}_\mu(\lambda_b)$ for the λ_b-weight space of the representation of the dual group $\widehat{G}$ corresponding to μ. Here λ_b is a character on a subtorus of $\widehat{G}$ (which is a maximal torus if G_F is split over F) defined in terms of the two discrete invariants (a la Kottwitz) of the σ-conjugacy class

[1]Strictly speaking, the application to Langlands–Rapoport Conjecture is independent of the Rapoport–Zink uniformization per se, but it follows the same spirit.

of b in $G(\breve{F})$ by a simple recipe. In particular, this conjecture gives an elementary way of computing the cardinality of the finite set $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$. Special cases of this conjecture were proved in [XZ17, HV18, Nie18b] (based on a common idea of reduction to the superbasic case, which goes back to [GHKR06]). The general case was finally proved in the joint work of R. Zhou and the author, and in the work of S. Nie using different methods.

Theorem A ([ZZ20], [Nie18a])**.** *There is a natural bijection* $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b)) \cong \mathbb{MV}_\mu(\lambda_b)$.

For question (ii), it was conjectured by X. Zhu that every stabilizer should be a parahoric subgroup of $J_b(F)$ of maximal volume. In joint work with X. He and R. Zhou we confirm this conjecture. Generalizing the results in [BP89, A.4], we have a distinguished class of parahoric subgroups of $J_b(F)$, called *very special*, which can be read off from the relative local Dynkin diagram of J_b with the decorating integers $d(v)$ as in [Tit79, §§1.8, 1.11]. These subgroups are equivalently characterized as the parahoric subgroups having the maximal volume.

Theorem B ([HZZ21])**.** *The stabilizers for the* $J_b(F)$*-action on* $\Sigma^{\mathrm{top}}(X_\mu(b))$ *are all very special parahoric subgroups of* $J_b(F)$.

For a reductive group H over F with no factors of type C-BC_n, very special parahoric subgroups of $H(F)$ are unique up to conjugation by $H^{\mathrm{ad}}(F)$. In our current setting of hyperspecial level, J_b never has factors of type C-BC_n, so Theorem B determines the stabilizers up to conjugation by $J_b^{\mathrm{ad}}(F)$. It would be an interesting problem to determine the stabilizers up to $J_b(F)$-conjugacy.

2. Application to Shimura varieties

Let $(\mathbf{G}, X)$ be a Shimura datum of Hodge type, and let $\mathrm{K} \subset \mathbf{G}(\mathbb{A}_f)$ be a compact open subgroup. Let $p > 2$ be a prime and assume that $\mathrm{K} = \mathrm{K}_p\mathrm{K}^p$ where K^p is a sufficiently small compact open subgroup of $\mathbf{G}(\mathbb{A}_f^p)$ and $\mathrm{K}_p = G(\mathbb{Z}_p)$ is a hyperspecial subgroup of $\mathbf{G}(\mathbb{Q}_p)$, where G is a reductive model over $\mathbb{Z}_p$ of $\mathbf{G}_{\mathbb{Q}_p}$. By the work of Kisin [Kis10], for any prime v of the reflex field E of (G, X) above p, there is a smooth canonical integral model $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ of $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ over $\mathcal{O}_{E,(v)}$. The geometric special fiber of $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ has a *Newton stratification* indexed by the Kottwitz set $B(G_{\mathbb{Q}_p}, \mu) \subset G(\breve{\mathbb{Q}}_p)/\sigma$-conj, where μ is a Hodge cocharacter of X. The unique basic element $[b]$ of $B(G_{\mathbb{Q}_p}, \mu)$ corresponds to a stratum denoted by $\mathrm{Sh}_{\mathrm{K,bas}}$. This is a generalization of the supersingular locus in the case of a modular curve. Define $X_\mu(b)$ with respect to G. By a "reduced version" of the Rapoport–Zink uniformization (see e.g. [XZ17, Cor. 7.2.16]), the perfection of $\mathrm{Sh}_{\mathrm{K,bas}}$ is isomorphic to

$$I(\mathbb{Q})\backslash X_\mu(b) \times \mathbf{G}(\mathbb{A}_f^p)/\mathrm{K}^p.$$

Here I is a certain reductive group over $\mathbb{Q}$ equipped with isomorphisms $\iota_p : I \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong J_b$ and $\iota^p : I \otimes_{\mathbb{Q}} \mathbb{A}_f^p \cong \mathbf{G} \otimes_{\mathbb{Q}} \mathbb{A}_f^p$, and the left action of $I(\mathbb{Q})$ on $X_\mu(b) \times \mathbf{G}(\mathbb{A}_f^p)$ is defined using ι_p and ι^p. The following result follows from Theorems A and B, and the equidimensionality proved in [HV18, Thm. 3.4].

Theorem C ([HZZ21])**.** *The set* $\Sigma(\mathrm{Sh}_{\mathrm{K,bas}})$ *of irreducible components of* $\mathrm{Sh}_{\mathrm{K,bas}}$ *is in prime-to-p-Hecke-equivariant bijection with*

$$\coprod_{\mathbf{a} \in \mathbb{MV}_\mu(\lambda_b)} I(\mathbb{Q})\backslash I(\mathbb{A}_f)/I_p^{\mathbf{a}} I^p,$$

where $I^p = (\iota^p)^{-1}(\mathrm{K}^p) \subset I(\mathbb{A}_f^p)$ *and* $I_p^{\mathbf{a}}$ *is a very special parahoric subgroup of* $I_p(\mathbb{Q}_p)$ *for each* $\mathbf{a}$*.*

We refer the reader to [VW11, HP17, LT20] for results similar to Theorem C for special cases of Shimura varieties. For certain arithmetic applications, such as the arithmetic level raising in [LT20], Theorem C allows one to interpret functions on $\Sigma(\mathrm{Sh}_{\mathrm{K,bas}})$ as automorphic forms on the group I. The knowledge about $I_p^{\mathbf{a}}$ is thus crucial in controlling the level of these automorphic forms.

In [ZZ20] and [HZZ21] we have also proved generalizations of Theorems A and B beyond the setting of hyperspecial level, to cases where G is a special parahoric group scheme over $\mathcal{O}_F$ and G_F is quasi-split but not necessarily unramified over F. In order to obtain the corresponding generalization of Theorem C, one needs to generalize the above description of the perfection of $\mathrm{Sh}_{\mathrm{K,bas}}$ in terms of $X_\mu(b)$, where $\mathrm{Sh}_{\mathrm{K,bas}}$ is defined using the Kisin–Pappas integral model [KP18]. This is done in [HZZ21].

2.1. **Some features of the proofs.** The key idea in our proof of Theorem A is to use the Lang–Weil bound to relate point counting on $X_\mu(b)$ to the cardinality of $J_b(F)\backslash\Sigma^{\mathrm{top}}(X_\mu(b))$. Based on this idea, we show that this cardinality is related to the asymptotic behavior, as n grows, of a certain twisted orbital integral over the degree n unramified extension of F. We study the latter using explicit methods from local harmonic analysis and representation theory, including the base change fundamental lemma of Clozel and Labesse, and the Kato–Lusztig formula.

In our proof, certain linear combinations of the q-analogue of Kostant partition functions appear, and it is key to estimate their sizes. It seems reasonable to expect that a more thorough study of the combinatorial and geometric properties of these objects could lead to interesting results about ADLV's.

As a byproduct of our proof of Theorem A, we have the following intermediate result: In the "essential cases" that one reaches after some reductions steps, the *average* of the inverses of the volumes of the stabilizers depends only on b, not on μ. This result allows one to reduce Theorem B to the statement that there is at least *one* top-dimensional irreducible component whose stabilizer is very special parahoric. We prove the last statement in [HZZ21] by an explicit construction, employing a refinement of the Deligne–Lusztig reduction method in [He14],

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