

Ordinary Differential Equations
0. Course Introduction and Overview

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0. COURSE INTRODUCTION AND OVERVIEW

0.1. **What are Differential Equations?** We begin by giving their definition.

Definition 0.1. A differential equation is an algebraic relation that

- involves derivatives of one or more unknown functions with respect to one or more independent variables,
- and possibly involves either the unknown functions themselves or their independent variables.

For example, an unknown function $p(t)$ might satisfy the relation

$$(0.1) \quad \frac{dp}{dt} = 5p.$$

This is a differential equation because it algebraically relates the derivative of the unknown function p to itself. It does not involve the independent variable t .

Similarly, unknown functions $u(x, y)$ and $v(x, y)$ might satisfy the relation

$$(0.2) \quad \partial_x u + \partial_y v = 0,$$

where $\partial_x u$ and $\partial_y v$ denote partial derivatives. This is a differential equation because it algebraically relates some partial derivatives of the unknown functions u and v to each other. It does not involve the values of either u or v , nor does it involve either of the independent variables, x or y .

Here are other examples of differential equations that involve derivatives of a single unknown function:

$$(0.3) \quad \begin{array}{ll} \text{(a)} \quad \frac{dv}{dt} = 9.8 - .05v^2, & \text{(b)} \quad \frac{d^2\theta}{dt^2} + \sin(\theta) = 0, \\ \text{(c)} \quad \left(\frac{dy}{dx}\right)^2 + x^2 + y^2 = -1, & \text{(d)} \quad \left(\frac{dy}{dx}\right)^2 + 4y^2 = 1, \\ \text{(e)} \quad \frac{d^2u}{dr^2} + \frac{2}{r} \frac{du}{dr} + u = 0, & \text{(f)} \quad \frac{d^3u}{dx^3} + u \frac{du}{dx} + \frac{du}{dx} = 0, \\ \text{(g)} \quad \frac{d^2x}{dt^2} + 9x = \cos(2t), & \text{(h)} \quad \frac{d^2r}{dt^2} = \frac{1}{r^{13}} - \frac{1}{r^7}, \\ \text{(i)} \quad \frac{d^2A}{dx^2} + xA = 0, & \text{(j)} \quad W^3 \frac{dW}{dz} + \frac{1}{6}z = 0, \\ \text{(k)} \quad \partial_{tt}h = \partial_{xx}h + \partial_{yy}h, & \text{(l)} \quad \partial_{xx}\phi + \partial_{yy}\phi + \partial_{zz}\phi = 0, \\ \text{(m)} \quad \partial_t u + u \partial_x u = \partial_{xx}u, & \text{(n)} \quad \partial_t T = \partial_x (T^4 \partial_x T). \end{array}$$

In all of these examples except k and l the unknown function itself also appears in the equation. In examples c, e, g, i, and j the independent variable also appears in the equation.

Remark. All of the above examples except one arise in applications. That one is also the only example that has no real-valued solution. Can you spot this example?

0.2. Common Notations for Derivatives. Because derivatives arise in a variety of applications, there are many ways to denote them. Consequently, there are many ways to express every differential equation. For example, in mechanics it is common to denote time derivatives with *dots*, so that if x is a function of time t then

$$\dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2}.$$

In this notation the differential equations a, b, g, and h from (0.3) are expressed as

$$\begin{aligned} \text{(a)} \quad \dot{v} &= 9.8 - .05v^2, & \text{(b)} \quad \ddot{\theta} + \sin(\theta) &= 0, \\ \text{(g)} \quad \ddot{x} + 9x &= \cos(2t), & \text{(h)} \quad \ddot{r} &= \frac{1}{r^{13}} - \frac{1}{r^7}. \end{aligned}$$

In fact, all of these equations arise in mechanics. However, this dot notation is widely used elsewhere in engineering, science, and mathematics. Sometimes it is used to denote derivatives with respect to a variable other than time. In most instances where it is used there is only one independent variable, so it is clear that the dots denote derivatives with respect to that variable. If you have not seen this notation already, you are likely to see it in later courses.

It is also common to denote derivatives of functions of one variable with *primes*, so that if u is a function of x then

$$u' = \frac{du}{dx}, \quad u'' = \frac{d^2u}{dx^2}, \quad u''' = \frac{d^3u}{dx^3}.$$

In this notation the differential equations e, f, i, and j from (0.3) are expressed as

$$\begin{aligned} \text{(e)} \quad u'' + \frac{2}{r}u' + u &= 0, & \text{(f)} \quad u''' + uu' + u' &= 0, \\ \text{(i)} \quad A'' + xA &= 0, & \text{(j)} \quad W^3W' + \frac{1}{6}z &= 0. \end{aligned}$$

In equations e, i, and g the independent variable appears explicitly, so it is understood that the primes denote derivatives with respect to that variable. In most instances where this notation is used there is only one independent variable, so it is clear that the primes denote derivatives with respect to that variable. This prime notation is widely used in engineering, science, and mathematics. If you have not seen it already, you are likely to see it in later courses.

We have already introduced the *subscripted partial* notation for partial derivatives. If h is a function of x and y then

$$\partial_x h = \frac{\partial h}{\partial x}, \quad \partial_y h = \frac{\partial h}{\partial y}, \quad \partial_{xx} h = \frac{\partial^2 h}{\partial x^2}, \quad \partial_{xy} h = \frac{\partial^2 h}{\partial x \partial y}, \quad \partial_{yy} h = \frac{\partial^2 h}{\partial y^2}.$$

Partial derivatives are covered in all multivariable calculus courses and in some science and engineering courses that use vectors. If you have not seen them already, do not panic. We will cover what you need to know about them in this course.

We will employ these common notations throughout the course. Other notations will be introduced as well. This should help prepare you to recognize differential equations no matter how they are expressed.

0.3. Classification. The subject of differential equations is too vast to be studied in one course. In order to describe the kinds of differential equations that we will study in this course, we need to introduce some ways in which differential equations are classified.

Definition 0.2. *A differential equation is called an ordinary differential equation (ODE) if it involves derivatives with respect to only one independent variable. Otherwise, it is called a partial differential equation (PDE).*

Example (0.1) is an ordinary differential equation. Example (0.2) is a partial differential equation. Of the examples in (0.3):

a – j are ordinary differential equations ;
k – n are partial differential equations .

Definition 0.3. *The order of a differential equation is the order of the highest derivative that appears in it. An n^{th} -order differential equation is one whose order is n .*

Examples (0.1) and (0.2) are first-order differential equations. Of the examples in (0.3):

a, c, d, j are first-order differential equations ;
b, e, g, h, i, k, l, m, n are second-order differential equations ;
f is a third-order differential equation .

Definition 0.4. *A differential equation is said to be linear if it can be expressed so that each side of the equation is a sum of terms, each of which is either*

- *a derivative of an unknown function times a factor that is independent of the unknown functions,*
- *an unknown function times a factor that is independent of the unknown functions,*
- *or entirely independent of the unknown functions.*

Otherwise it is said to be nonlinear.

Examples (0.1) and (0.2) are linear differential equations. Of the examples in (0.3):

e, g, i, k, l are linear differential equations ;
a – d, f, h, j, m, n are nonlinear differential equations .

The terms that make equations a – d, f, h, j, m, n nonlinear are

- | | |
|--------------------------------------|---------------------------------------|
| (a) v^2 , | (b) $\sin(\theta)$, |
| (c) all terms on the left-hand side, | (d) all terms on the left-hand side, |
| (f) $u \frac{du}{dx}$, | (h) all terms on the right-hand side, |
| | (j) $w^3 \frac{dw}{dz}$, |
| (m) $u \partial_x u$, | (n) $\partial_x (T^4 \partial_x T)$. |

Remark. Every n^{th} -order, linear ordinary differential equation for a single unknown function y of a single independent variable t can be brought into the form

$$(0.4) \quad p_0(t) \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t) y = r(t),$$

where $p_0(t) \neq 0$ and $p_0(t), p_1(t), \dots, p_n(t)$, and $r(t)$ are independent of the unknown function y . Notice that the left-hand side of this equation is a sum of terms, each of which is y or one of its derivatives times a factor $p_k(t)$ that may or may not depend on t , while the right-hand side is simply $r(t)$, which may or may not depend on t . Of the examples in (0.3), e, g, i are linear ordinary differential equations, all of which are second-order and in the form (0.4).

Remark. Linear differential equations are important because much more can be said about them than can be said about general nonlinear differential equations.

In applications we are often faced with a *system* of coupled differential equations — typically a system of m differential equations for m unknown functions. For example, two unknown functions $p(t)$ and $q(t)$ might satisfy the system

$$(0.5) \quad \frac{dp}{dt} = (6 - 2p - q)p, \quad \frac{dq}{dt} = (8 - 4p - q)q.$$

Similarly, two unknown functions $u(x, y)$ and $v(x, y)$ might satisfy the system

$$(0.6) \quad \partial_x u = \partial_y v, \quad \partial_y u + \partial_x v = 0.$$

Definition 0.5. *A system of differential equation is called a system of ordinary differential equations if it involves derivatives with respect to only one independent variable. Otherwise, it is called a system of partial differential equations.*

The concepts of order, linear, and nonlinear extend to systems.

Definition 0.6. *The order of a system of differential equations is the order of the highest derivative that appears in the entire system. A system is called linear if every equation in it is linear. Otherwise, it is called nonlinear.*

Example (0.5) is a first-order, nonlinear system of ordinary differential equations, while (0.6) is a first-order, linear system of partial differential equations.

Remark. Systems that arise in applications can be extremely large. Systems of over 10^{10} ordinary differential equations are being solved numerically every day in order to approximate solutions of systems of partial differential equations.

0.4. Course Overview. Differential equations arise in mathematics, physics, chemistry, biology, medicine, pharmacology, communications, electronics, finance, economics, aerospace, meteorology, climatology, oil recovery, hydrology, ecology, combustion, image processing, animation, and in many other fields. Partial differential equations are at the heart of most of these applications. However, you should know something about ordinary differential equations before you study partial differential equations.

This course will serve as your introduction to ordinary differential equations. More specifically, we will study four classes of ordinary differential equations. We illustrate these four classes below denoting the independent variable by t .

I. We will begin with single first-order ODEs that can be brought into the form

$$(0.7) \quad \frac{dy}{dt} = f(t, y).$$

You may have seen some of this material in your calculus courses. Some basic facts about partial derivatives will be used.

II. We will next study single n^{th} -order, linear ODEs that can be brought into the form

$$(0.8) \quad \frac{d^n y}{dt^n} + a_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1}(t) \frac{dy}{dt} + a_n(t)y = f(t).$$

This material is the heart of the course. We will learn many new techniques, most of which extend to systems. Some basic facts about linear algebraic systems and determinants will be used.

III. We will then turn towards systems of n first-order, linear ODEs that can be brought into the form

$$(0.9) \quad \begin{aligned} \frac{dy_1}{dt} &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t), \\ \frac{dy_2}{dt} &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t), \\ &\vdots \\ \frac{dy_n}{dt} &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_n(t). \end{aligned}$$

This material builds upon the material covered in part II. Some basic facts about vectors and matrices will be used.

IV. Finally, we will study systems of two first-order, nonlinear ODEs that can be brought into the form

$$(0.10) \quad \frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y).$$

This material builds upon the material in parts I and III. Some basic facts about graphing functions of two variables will be used.

We will study each of these four classes from four perspectives: *analytical* methods, *graphical* methods, *numerical* methods, and in the setting of *word problems*. In addition, these analytical, graphical, and numerical methods will be enhanced by the use of $\text{\textcircled{R}}\textit{Matlab}$ throughout the course. The material from multivariable calculus and linear algebra that we use will be presented, but students who either have already had or are currently taking these courses will be better prepared for this course. This course is far from a complete treatment of the subject. It will however prepare you to learn more about ordinary differential equations or to learn about partial differential equations.

EXERCISES ON CLASSIFICATION

For each of the following ordinary differential equations determine if it is linear or nonlinear and determine its order.

- (1) $\dot{y} + 5y = t^2 + 2$, where y is a function of t .
- (2) $\frac{dx}{dt} = x^2$, where x is a function of t .
- (3) $xy^{(5)} + x^2y^{(3)} + \sin(x)y' - y = e^x$, where y is a function of x .
- (4) $\left(\frac{dy}{dt}\right)\left(\frac{d^2y}{dt^2}\right) + y = 0$, where y is a function of t .
- (5) $x'' + t^2x' + 3t = 0$, where x is a function of t .
- (6) $u \frac{du}{dt} + t^2 = 3u$, where u is a function of t .
- (7) $\sin(t) \frac{d^2x}{dt^2} + \cos(t) \frac{dx}{dt} = \tan(t)$, where x is a function of t .

For each of the following systems of ordinary differential equations determine if it is linear or nonlinear and determine its order.

- (8)

$$\begin{aligned} y' &= y + tx, \\ x' &= y^2 + 3x, \end{aligned}$$
 where x and y are both functions of t .
- (9)

$$\begin{aligned} y'' &= y' + tx', \\ x' &= y + 3x, \end{aligned}$$
 where x and y are both functions of t .
- (10)

$$\begin{aligned} x' &= x + t^2y + 6z, \\ y' &= 3x + y + e^t, \\ z' &= y + tz, \end{aligned}$$
 where x, y and z are functions of t .

For each of the following partial differential equations determine if it is linear or nonlinear and determine its order.

- (11) $\partial_{xx}\phi + \partial_{yy}\phi = 0$, where ϕ is a function of x and y .
- (12) $-\partial_y\phi = (\partial_x\phi)^2/4$ where ϕ is a function of x and y .

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