I. First-Order Ordinary Differential Equations
1. Introduction to First-Order Equations

C. David Levermore
Department of Mathematics
University of Maryland

August 2, 2022

Contents

©2022 C. David Levermore

Return to Main Webpage

1. Introduction to First-Order Equations

1.1. **Solutions of First-Order Equations.** We now begin our study of first-order ordinary differential equations that involve a single real-valued unknown function $y(t)$. These can always be brought into the form

$$F\left(t, y, \frac{\mathrm{d}y}{\mathrm{d}t}\right) = 0\,.$$

If we try to solve this equation for $\mathrm{d}y/\mathrm{d}t$ in terms of t and y then there might be no solutions or many solutions. For example, equation (c) of (0.3) clearly has no (real) solutions because the sum of nonnegative terms cannot add to -1. On the other hand, equation (d) will be satisfied if either

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1 - 4y^2}\,, \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{1 - 4y^2}\,.$$

To avoid such complications, we restrict ourselves to equations that are already in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)\,. \tag{1.1}$$

Examples (0.1) and (a) of (0.3) are already in this form. Example (j) of (0.3) can easily be brought into this form. And as we saw above, example (d) of (0.3) can be reduced to two equations in this form.

Remark. We will often use t as the independent variable in our differential equations because in many applications the independent variable is time. However, the independent variable in a differential equation need not be time. For example, the independent variable is x in the example discussed above. Similarly, the dependent variable in a differential equation need not be y. Be prepared for the roles of independent and dependent variables in differential equations to be filled by many different letters!

It is important to understand what is meant by a solution of (1.1),

Definition 1.1. *If $Y(t)$ is a function defined for every t in an interval $(t_{\mathrm{L}}, t_{\mathrm{R}})$ then we say $Y(t)$ is a* solution *of (1.1) over $(t_{\mathrm{L}}, \mathbb{R})$ if*

$$\begin{aligned} &\textit{(i)} \quad Y'(t) \textit{ is defined for every } t \textit{ in } (t_{\mathrm{L}}, t_{\mathrm{R}})\,, \\ &\textit{(ii)} \quad f(t, Y(t)) \textit{ is defined for every } t \textit{ in } (t_{\mathrm{L}}, t_{\mathrm{R}})\,, \\ &\textit{(iii)} \quad Y'(t) = f(t, Y(t)) \textit{ for every } t \textit{ in } (t_{\mathrm{L}}, t_{\mathrm{R}})\,. \end{aligned} \tag{1.2}$$

Remark. One can recast condition (i) as "the function Y is differentiable over $(t_{\mathrm{L}}, t_{\mathrm{R}})$." This definition is very natural in that it simply states that (i) the thing on left-hand side of the equation makes sense, (ii) the thing on right-hand side of the equation makes sense, and (iii) the two things are equal. This classical notion of solution will suit our needs now. Later we will see situations arise in which we will want to broaden this notion of solution.

Example. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{t}{y}\,.$$

Show that $y = \sqrt{4 - t^2}$ is a solution of this equation over $(-2, 2)$. Explain why it is not a solution of this equation over $[-2, 2]$.

Solution. We see that $y = \sqrt{4 - t^2}$ is differentiable for every t in $(-2, 2)$ with

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(4 - t^2\big)^{\frac{1}{2}} = \tfrac{1}{2}\big(4 - t^2\big)^{-\frac{1}{2}}(-2t) = -t\big(4 - t^2\big)^{-\frac{1}{2}} .$$

Therefore the left-hand side of the differntial equation makes sense over $(-2, 2)$. We also see that for every t in $(-2, 2)$

$$-\frac{t}{y} = -\frac{t}{\sqrt{4 - t^2}} = -t\big(4 - t^2\big)^{-\frac{1}{2}} .$$

Therefore the right-hand side of the differntial equation also makes sense over $(-2, 2)$. Finally, the two sides are equal over $(-2, 2)$. Therefore $y = \sqrt{4 - t^2}$ is a solution of the differential equation over $(-2, 2)$.

We can see that $y = \sqrt{4 - t^2}$ is a not solution of this equation over $[-2, 2]$ for two reasons. First, we see that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -t\big(4 - t^2\big)^{-\frac{1}{2}} \quad \text{is not defined at } t = \pm 2.$$

Second, when $y = \sqrt{4 - t^2}$ is evaluated at $t = \pm 2$ we obtain $y = 0$. But the right-hand side of the differential equation is not defined when $y = 0$. □

We want to address the following five basic questions about solutions.

- When does equation (1.1) have solutions?
- Under what conditions is a solution unique?
- How can we find analytic expressions for solutions?
- How can we approximate solutions either analytically or numerically?
- How can we visualize solutions graphically?

We will focus on the last three questions. They address practical skills that can be applied when we are faced with a differential equation. The first two questions will be viewed through the lens of the last three. They are important because differential equations that arise in applications are supposed to model or predict something. If an equation either does not have solutions or has more than one solution then it fails to meet this objective. Moreover, in those situations the methods by which we will address the last three questions can give misleading results. Therefore we will study the first two questions with an eye towards avoiding such pitfalls.

Rather than addressing these five questions for a general $f(t, y)$ in (1.1), we will treat special forms $f(t, y)$ of increasing complexity.

1.2. **Review of Explicit Equations.** It is simplest to treat (1.1) when the derivative is given as an explicit function of t. These so-called *explicit* equations have the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t) . \tag{1.3}$$

This case is covered in calculus courses, so we only review it here.

1.2.1. *Recipe for Explicit Equations.* Recall from your study of calculus that a differentiable function F is said to be a *primitive* or *antiderivative* of f if $F' = f$. We thereby see that $y = Y(t)$ is a solution of (1.3) if and only if Y is a primitive of f. Also recall from calculus that if you know one primitive F of f then any other primitive Y of f must have the form $Y(t) = F(t) + c$ for some constant c. We thereby see that if (1.3) has one solution then it has a family of solutions given by the indefinite integral of f — namely, by

$$y = \int f(t)\,\mathrm{d}t = F(t) + c\,, \quad \text{where } F'(t) = f(t) \text{ and } c \text{ is any constant}\,. \tag{1.4}$$

Moreover, there are no other solutions of (1.3). Therefore the family (1.4) is called a *general solution* of the differential equation (1.3). The arbitrary constant c that appears in the family (1.4) is the *parameter* of the general solution.

The above arguments show that the problem of finding a general solution of (1.3) reduces to the problem of finding a primitive F of f. Given such an F, a general solution of (1.3) is given by (1.4). However these arguments do not insure that such a primitive exists. Of course, for sufficiently simple f we can find a primitive analytically.

Example. Find a general solution to the differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}x} = 6x^2 + 1\,.$$

Solution. By (1.4) a general solution is

$$w = \int \big(6x^2 + 1\big)\,\mathrm{d}x = 2x^3 + x + c\,.$$

Members of this family are graphed below in Figure 1.1.

1.2.2. *Initial-Value Problems for Explicit Equations.* In order to pick a unique solution from the family (1.4) we must impose an additional condition that determines c. We do this by imposing a so-called *initial condition* of the form

$$y(t_I) = y_I\,,$$

where t_I is called the *initial time* when time is the independent variable or the *initial point* more generally, while y_I is called the *initial value* or *initial datum*. The combination of equation (1.3) with the above initial condition is the so-called *initial-value problem* given by

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)\,, \qquad y(t_I) = y_I\,. \tag{1.5}$$

If f has a primitive F then by (1.4) every solution of the differential equation in the initial-value problem (1.5) has the form $y = F(t) + c$ for some constant c. By imposing the initial condition in (1.5) upon this family we see that

$$F(t_I) + c = y_I\,,$$

which implies that $c = y_I - F(t_I)$. Therefore the unique solution of initial-value problem (1.5) is given by

$$y = y_I + F(t) - F(t_I)\,. \tag{1.6}$$

The above arguments show that the problem of finding the unique solution of the initial-value problem (1.5) again reduces to the problem of finding a primitive F of f. Given such an F, the unique solition of initial-value problem (1.5) is given by (1.6). These arguments however do not insure that such a primitive exists. Of course, for sufficiently simple f we can find a primitive analytically.

Example. Find the solution to the initial-value problem

$$\frac{\mathrm{d}w}{\mathrm{d}x} = 6x^2 + 1\,, \qquad w(1) = 5\,.$$

Solution. The previous example shows the solution has the form $w = 2x^3 + x + c$ for some constant c. Imposing the initial condition gives $2 \cdot 1^3 + 1 + c = 5$, which implies $c = 2$. Hence, the solution is $w = 2x^3 + x + 2$. This solution is highlighted below in Figure 1.1.

Alternative Solution. By (1.6) with $x_I = 1$, $w_I = 5$, and the primitive $F(x) = 2x^3+x$ we find

$$\begin{aligned} w &= w_I + F(x) - F(x_I) = 5 + F(x) - F(1) \\ &= 5 + \big(2x^3 + x\big) - \big(2 \cdot 1^3 + 1\big) = 2x^3 + x + 2\,. \end{aligned}$$

Remark. As the solutions to the previous example illustrate, when solving an initial-value problem it is often easier to first find a general solution and then evaluate the c from the initial condition rather than to directly apply formula (1.6). With that approach you do not have to memorize formula (1.6).

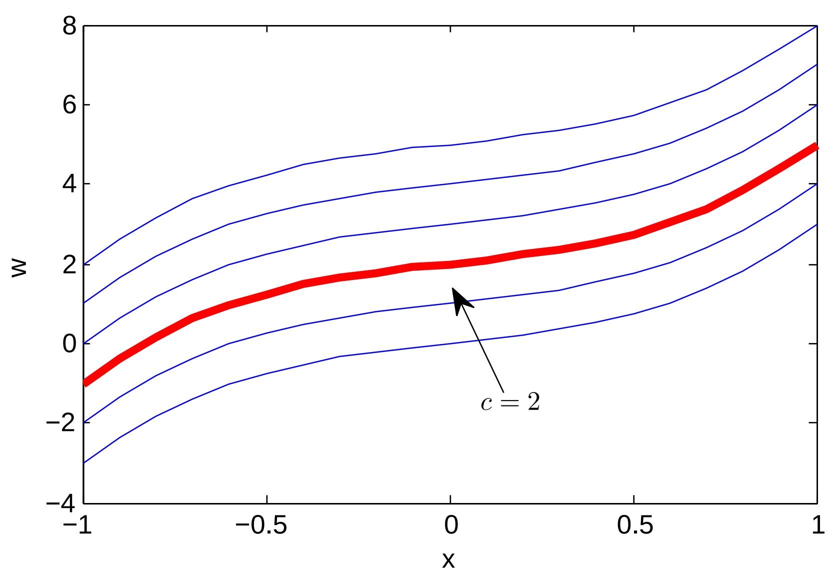

FIGURE 1.1. Plot of the general solution $w = 2x^3 + x + c$ for various values of c. The solution for $c = 2$ is shown in bold. This picture was generated by a computer, but a faithful representation could also have been sketched quickly by hand.

1.2.3. *Theory for Explicit Equations.* Finally, even when we cannot find a primitive analytically, we can show that a solution exists by appealing to the Second Fundamental Theorem of Calculus. It states that if f is continuous over an interval $(t_{\mathrm{L}}, t_{\mathrm{R}})$ then for every t_I in $(t_{\mathrm{L}}, t_{\mathrm{R}})$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_I}^{t} f(s)\,\mathrm{d}s = f(t) \qquad \text{for every } t \text{ in } (t_{\mathrm{L}}, t_{\mathrm{R}})\,.$$

In other words, f has a primitive over $(t_{\mathrm{L}}, t_{\mathrm{R}})$ that can be expressed as a definite integral. Here s is the "dummy" variable of integration in the above definite integral. If t_I is in $(t_{\mathrm{L}}, t_{\mathrm{R}})$ then the First Fundamental Theorem of Calculus implies that formula (1.6) can be expressed as

$$y(t) = y_I + \int_{t_I}^{t} f(s)\,\mathrm{d}s\,. \tag{1.7}$$

This shows that if f is continuous over an interval $(t_{\mathrm{L}}, t_{\mathrm{R}})$ that contains t_I then the initial-value problem (1.5) has a unique solution over $(t_{\mathrm{L}}, t_{\mathrm{R}})$, which is given by formula (1.7).

It is natural to ask if there is a largest time interval over which a solution exists. This information is helpful to know before seeking a numerical approximation to or analyzing the behavior of a solution.

Definition 1.2. *The largest time interval over which a solution exists (in the sense of Definition 1.1) is called its* interval of definition *or* interval of existence *or sometimes* interval of validity*.*

For explicit equations we can usually identify the interval of definition for the solution of the initial-value problem (1.5) by simply looking at $f(t)$. Specifically, if $Y(t)$ is the solution of the initial-value problem (1.5) then its interval of definition will be $(t_{\mathrm{L}}, t_{\mathrm{R}})$ whenever:

- the initial time t_I is in $(t_{\mathrm{L}}, t_{\mathrm{R}})$,
- $f(t)$ is continuous over $(t_{\mathrm{L}}, t_{\mathrm{R}})$,
- $f(t)$ is not defined at both $t = t_{\mathrm{L}}$ and $t = t_{\mathrm{R}}$.

This is because the first two bullets along with the formula (1.7) imply that the interval of definition will be at least $(t_{\mathrm{L}}, t_{\mathrm{R}})$, while the last bullet along with Definition 1.1 of solution imply that the interval of definition can be no bigger than $(t_{\mathrm{L}}, t_{\mathrm{R}})$. This argument works when $t_{\mathrm{L}} = -\infty$ or $t_{\mathrm{R}} = \infty$.

We will now illustrate how easy it is to apply this theory.

Example. Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^t}{4 - t^2}\,, \qquad x(1) = 7\,.$$

Remark. There is no elementary primitive of $f(t) = e^t/(4 - t^2)$, so we will not be able to find explicit elementary solutions of this differential equation.

Solution. The $f(t) = e^t/(4-t^2)$ is not defined at $t = -2$ and $t = 2$, and is continuous everywhere else. Therefore a unique solution exists with interval of definition is $(-2, 2)$ because: $f(t) = e^t/(4-t^2)$ is continuous over this interval; the initial time is $t = 1$, which is in this interval; $f(t) = e^t/(4-t^2)$ is not defined at $t = -2$ and $t = 2$. □

Remark. The next example shows that the interval of definition can be different for the solution of an initial-value problem with the same differential equation as in the previous example but with a different initial time.

Example. Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^t}{4-t^2}\,, \qquad x(5) = 3\,.$$

Solution. The $f(t) = e^t/(4-t^2)$ is not defined at $t = -2$ and $t = 2$, and is continuous everywhere else. Therefore a unique solution exists with interval of definition $(2, \infty)$ because: $f(t) = e^t/(4-t^2)$ is continuous over this interval; the initial time is $t = 5$, which is in this interval; $f(t) = e^t/(4-t^2)$ is not defined at $t = 2$. □

1.2.4. *Numerical and Qualitative Methods.* The theory in the preceding discussion lays the ground work for numerical and graphical methods that can be applied to explicit equations even when no primitive of $f(t)$ can be found. If the solution $y(t)$ of the initial-value problem (1.5) has interval of definition $(t_{\mathrm{L}}, t_{\mathrm{R}})$ then formula (1.7) shows that

$$y(t) = y_I + \int_{t_I}^{t} f(s)\,\mathrm{d}s \quad \text{for every } t \text{ in the interval } (t_{\mathrm{L}}, t_{\mathrm{R}})\,. \tag{1.8}$$

The definite integral on the right-hand side can be approximated numerically for any t in the interval $(t_{\mathrm{L}}, t_{\mathrm{R}})$ by using any of the *numerical methods* covered in calculus courses: the left-hand rule, the right-hand rule, the midpoint rule, the trapezoidal rule, and the Simpson rule. Such a numerical approximation can be graphed to give a visualization of the true solution. We will not discuss these numerical methods further in this review. They will be reviewed in Chapter 7 when we study numerical methods for first-order ordinary differential equations.

However, an approximate solution is not needed to understand how the solution of an initial-value problem behaves. Methods that allow us to understand the behavior of a solution without quantitatively knowing the solution are called *qualitative methods.* We will illustrate this by revisiting the examples in the last subsection.

Example. Consider the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^t}{4-t^2}\,, \qquad x(1) = 7\,.$$

Describe the behavor of its solution.

Remark. Because there is no elementary primitive of $f(t) = e^t/(4-t^2)$, we will not be able to find explicit elementary solutions of this differential equation.

Solution. The theory tells us that this initial-value problem has a unique solution $x(t)$ that has an interval of definition $(-2, 2)$ and that by (1.8) can be expressed as

$$x(t) = 7 + \int_1^t \frac{e^s}{4 - s^2}\,\mathrm{d}s \quad \text{for every } t \text{ in the interval } (-2, 2)\,.$$

Because $x'(t) = e^t/(4 - t^2)$ and because $e^t/(4 - t^2)$ is positive over $(-2, 2)$, we see that $x(t)$ is an increasing function of t. Moreover, by methods from calculus it can be shown that

$$\lim_{t \to -2^+} \int_1^t \frac{e^s}{4 - s^2}\,\mathrm{d}s = -\infty\,, \qquad \text{and} \qquad \lim_{t \to 2^-} \int_1^t \frac{e^s}{4 - s^2}\,\mathrm{d}s = +\infty\,.$$

Therefore we see that $x(t)$ increases from $-\infty$ to $+\infty$ as t increases over $(-2, 2)$. □

Example. Consider the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^t}{4 - t^2}\,, \qquad x(5) = 3\,.$$

Describe the behavor of its solution.

Solution. The theory tells us that this initial-value problem has a unique solution $x(t)$ that has an interval of definition $(2, \infty)$ and that by (1.8) can be expressed as

$$x(t) = 3 + \int_5^t \frac{e^s}{4 - s^2}\,\mathrm{d}s \quad \text{for every } t \text{ in the interval } (2, \infty)\,.$$

Because $x'(t) = e^t/(4 - t^2)$ and because $e^t/(4 - t^2)$ is negative over $(2, \infty)$, we see that $x(t)$ is a decreasing function of t. Moreover, by methods from calculus it can be shown that

$$\lim_{t \to 2^+} \int_5^t \frac{e^s}{4 - s^2}\,\mathrm{d}s = +\infty\,, \qquad \text{and} \qquad \lim_{t \to \infty} \int_5^t \frac{e^s}{4 - s^2}\,\mathrm{d}s = -\infty\,.$$

Therefore $x(t)$ decreases from $+\infty$ to $-\infty$ as t increases over $(2, \infty)$. □

Remark. Qualitative methods can play a big role when analytic approaches fail. However, they can be helpful even when an explicit solution can be found. For example, knowing the behavior of a solution might help you catch an error when you try to compute it analytically.

1.3. **Overview of First-Order Equations.** The foregoing review of first-order explicit equations touches on many elements that we can hope to extend to more general first-order equations (1.1). Specifically, it touches on

- analytic methods for finding explicit solutions,
- theory that tells us when a solution exists and is unique,
- graphical methods to visualize solutions,
- numerical methods by which to approximate solutions,
- qualitative methods that tell us the behavior of a solution.

In addition, there is the fact that first-order explicit equations have many applications. For example, they are used to find the position of a moving object given its velocity, or its velocity given its acceleration. The subsequent chapters will extend all of these elements to broader classes of first-order equations.

We begin by studying **analytic methods** for two of the simplest types of first-order equations: *linear equations*, which can be brought into the form

$$y' + a(t)y = f(t)\,;$$

and *separable equations*, which can be brought into the form

$$y' = f(t)g(y)\,.$$

In each case obtaining an analytic solution generally requires finding two primitives. (Sometimes analytic solutions can be obtained by finding just one primitive.) For linear equations the analytic method will give y as an *explicit* function of t. For separable equations the analytic method generally give an algebraic relation between y and t. Such a so-called *implicit solution* can not always be solved to find y as an *explicit* function of t. We will see that there are many other ways in which linear equations are simpler than separable equations.

We then touch on a **general theory** for a broad class of first-order equations that lays the groundwork for the graphical and numerical methods that we will study.

Next, we explore four **graphical methods** for visualizing solutions of first-order equations. Two of these will require that an analytic solution be found first. The other two are *qualitative* in the sense that they can be applied without finding an analytic solution. One of these methods is so general that it can be applied to the broad class of first-order equations covered by our general theory.

We then use the foregoing analytical and graphical methods to solve problems that arise in **applications**. The new ingredient here is learning how to recast problems stated verbally as problems in differential equations.

Next, we introduce **numerical methods** for obtaining approximations to the values of exact solutions. These methods can be applied when analytic methods fail. In fact, they enable additional graphical methods to be applied when analytic methods fail. They can also be applied when analytic methods become complicated. Most of these methods are so general that they can be applied to the broad class of first-order equations covered by our general theory.

We then cover some new **analytic methods**. First, we show how to obtain implicit solutions to equations that can be recast as something called an *exact differential form*. Once again, obtaining an analytic solution generally requires finding two primitives. We then see how to hunt for an exact differential form by seeking a so-called *integrating factor*. Finally, the last chapter identifies first-order equations with *special forms* that allow them to be recast as a linear or separable equation after a *substitution*.

Exercises on First-Order Equations

(1) Consider the equation $\frac{dy}{dt} = 2\sin(t)\cos(t)$.

(a) Show that $y(t) = \sin^2(t)$ is a solution to this equation for every t in $(-\infty, \infty)$.

(b) Give a general solution to this equation.

Solution

(2) Show that $y_1(t) = \sqrt{t^2 - 1}$ and $y_2(t) = -\sqrt{t^2 - 9}$ are both solutions to

$$\frac{dy}{dt} = \frac{t}{y} \quad \text{for every } t \text{ in } (3, \infty)\,.$$

Solution

In #3-7 find a general solution to the given equation.

(3) $y' = 8x^3 + 10x$

Solution

(4) $\dot{y} = \frac{1}{t^2}$

Solution

(5) $\frac{dw}{dt} = \sec^2(t)$

Solution

(6) $\dot{x} = 6e^{-3t}$

Solution

(7) $y'' = 2t$ (Note: This is a second order equation, so you will have to integrate twice, and your general solution for y will depend on two distinct constants).

Solution

In #8- 12 find the largest interval over which the initial-value problem has a unique solution and find the solution on that interval.

(8) $x' = \log|t|\,, \quad x(1) = 2\,.$

Short Answer

Solution

(9) $y' = \frac{1}{t^2 - 2t}\,, \quad y(1) = 1\,.$

Short Answer

Solution

(10) $\sin(t)y' = \cos(t)\,, \quad y(\frac{\pi}{2}) = 1\,.$

Short Answer

Solution

(11) $y' = 4e^{2x}\,, \quad y(0) = 1\,.$

Short Answer

Solution

(12) $h'' = -9.8\,, \qquad h'(0) = v_0\,, \quad h(0) = h_0\,.$ Note: this is a second-order explicit equation, but it is a first-order explicit equation in h', and using the first initial condition we can get a first order equation in h.

Short Answer

Solution

In the following justify your responses.

(13) $y' = \dfrac{-1}{\sqrt{1-t^2}}\,, \quad y(\tfrac{1}{2}) = \tfrac{2\pi}{3}\,.$

Short Answer

Solution

(14) (a) We know that given an explicit first order equation $y' = f(t)$ we can apply the Fundamental Theorem of Calculus and get a general solution is given by the indefinite integral $y(t) = \int f(t)\,\mathrm{d}t$. Justify why the solution to the initial-value problem $y' = f(t)\,, \quad y(t_0) = y_0$ is given by this expression with a definite integral:

$$y(t) = y_0 + \int_{t_0}^{t} f(u)\mathrm{d}u\,.$$

(b) In a sentence describe how you would find the largest interval where the solution found in part(a) to the initial value problem is unique.

Solution

(15) Every initial value problem can be transformed so that the initial condition is at the origin. This can be obtained via the equivalence: $y' = f(t, y)$ with initial condition $y(t_0) = y_0$ is equivalent to $w' = f(s, w)$ if we set $s = t - t_0$ and $w = y - y_0$ we have the initial equivalent initial condition is $w(0) = 0$. $y(t)$ is a solution to the first IVP iff $w(s)$ is a solution to the second IVP.

Rewrite each of the following IVP as their equivalent IVP centered at the origin:

(a) $y' = 5t\,, \quad y(1) = 2\,.$

(b) $y' = y + 1\,, \quad y(2) = -1\,.$

(c) $y' = yt\,, \quad y(-1) = 1\,.$

Solution

(16) Find a general solution of $y' = \dfrac{1}{(x-\alpha)(x-\beta)}$, where $\alpha \neq \beta$.

Short Answer

Solution

(17) Confirm that $y(t) = 1 + t$ is a solution to $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{y^2 - 1}{t^2 + 2t}\,.$

Short Answer

Solution

Navigation to Other Chapters

This page may not work with every browser-driven pdf viewer. When it does work then it will enable you to link directly to any chapter. When it does not work then you can link to any chapter through the main webpage.