I. First-Order Ordinary Differential Equations
2. Linear Equations

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2. LINEAR EQUATIONS

The simplest class of first-order equations to treat beyond the explicit ones is that of linear equations. It was remarked around equation 0.4 of Chapter 0 that every linear first-order ODE for a single unknown function $y(t)$ can be brought into the form

$$p(t)\frac{\mathrm{d}y}{\mathrm{d}t} + q(t)y = r(t)\,, \tag{2.1}$$

where $p(t)$, $q(t)$, and $r(t)$ are given functions of t such that $p(t) \neq 0$ for those t over which the equation is considered. The functions $p(t)$ and $q(t)$ are called coefficients while the function $r(t)$ is called the forcing or driving. Equation (2.1) is called homogeneous when the forcing is absent (i.e. when $r(t) = 0$ for all t), and is called nonhomogeneous or inhomogeneous otherwise.

2.1. Linear Normal Form.

Because $p(t) \neq 0$ for those t over which equation (2.1) is being considered, we can divide by $p(t)$ to bring equation (2.1) into its so-called normal or standard form

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = f(t)\,, \tag{2.2}$$

where

$$a(t) = \frac{q(t)}{p(t)}\,, \qquad f(t) = \frac{r(t)}{p(t)}\,.$$

Equation (2.2) is explicit when the coefficient is absent (i.e. when $a(t) = 0$ for all t). Equation (2.2) is homogeneous when the forcing is absent (i.e. when $f(t) = 0$ for all t), and is nonhomogeneous otherwise.

Remark. The linear normal form (2.2) can be put into the form 1.1 by simply solving for the derivative of y as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t) - a(t)y\,.$$

The derivative of the unknown function y thereby is given as a linear function of y whose coefficients are functions of t. We could have chosen this as the normal form for first-order linear equations, but picked (2.2) instead because it is the restriction to first-order of the normal form for higher-order linear equations that we will use later.

The linear normal form (2.2) will be the starting point for all of the methods and theory we will develop for linear equations. Therefore you should get into the habit of putting every linear equation into its normal form. Because we may not be given a linear equation in form (2.1), we can often discover it is linear by trying to put it into the normal form (2.2).

Example. Consider the equation

$$e^t\frac{\mathrm{d}z}{\mathrm{d}t} + t^2 = \frac{2t + z}{1 + t^2}\,.$$

Show that this equation is linear and put it into the normal form (2.2).

Solution. By grouping all the terms involving either the z or its derivative on the left-hand side, while grouping all the other terms on the right-hand side, we obtain

$$e^t \frac{\mathrm{d}z}{\mathrm{d}t} - \frac{1}{1+t^2}\, z = \frac{2t}{1+t^2} - t^2 \,.$$

This can be transformed into the normal form (2.2) by multiplying both sides by e^{-t}.

2.2. **Recipe for Homogeneous Linear Equations.** When a linear equation is homogeneous, the problem of finding an analytic solution reduces to that of finding one primitive. We begin by putting the equation into the linear normal form (2.2), which is simply

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = 0 \,. \tag{2.3}$$

We will show that a general solution of this equation is given by

$$y = e^{-A(t)}c \,, \qquad \text{where } A'(t) = a(t) \text{ and } c \text{ is any constant} \,. \tag{2.4}$$

Hence, for homogenous linear equations the recipe for solution only requires finding a primitive of $a(t)$. This means that for simple enough $a(t)$ you should be able to write down a general solution immediately. We illustrate this with a few examples.

Example. Find a general solution of the equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 5p \,.$$

Solution. After the equation is put into linear normal form, we see that $a(t) = -5$. We can set $A(t) = -5t$. A general solution is then

$$p = e^{5t}c \,, \qquad \text{where } c \text{ is an arbitrary constant} \,.$$

Example. Find a general solution of the equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} + t^2 w = 0 \,.$$

Solution. The equation is already in linear normal form. Because $a(t) = t^2$, we can set $A(t) = \frac{1}{3}t^3$. A general solution is then

$$w = e^{-\frac{1}{3}t^3}c \,, \qquad \text{where } c \text{ is an arbitrary constant} \,.$$

Example. Find a general solution of the equation

$$(1+t^2)\frac{\mathrm{d}z}{\mathrm{d}t} + 4tz = 0 \,.$$

Solution. First put the equation into its linear normal form

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{4t}{1+t^2}\, z = 0 \,.$$

Because $a(t) = (4t)/(1+t^2)$, we can set $A(t) = 2\log(1+t^2)$. A general solution is then

$$z = e^{-2\log(1+t^2)}c = \frac{c}{(1+t^2)^2} \,, \qquad \text{where } c \text{ is an arbitrary constant} \,.$$

We will see that recipe (2.4) is a special case of the recipe for solving nonhomogeneous linear equations. Therefore we will justify it when we justify that more general recipe.

2.3. **Recipe for Nonhomogeneous Linear Equations.** When a linear equation is nonhomogeneous, the problem of finding an analytic solution reduces to that of finding two primitives. We begin by putting the equation into the normal form (2.2), which is

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = f(t)\,. \tag{2.5}$$

Below we will show that this is equivalent to the so-called *integrating factor form*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{A(t)}y\Big) = e^{A(t)}f(t)\,, \qquad \text{where } A'(t) = a(t)\,. \tag{2.6}$$

This is an explicit equation for the derivative of $e^{A(t)}y$ that can be integrated to obtain

$$e^{A(t)}y = \int e^{A(t)}f(t)\,\mathrm{d}t = B(t) + c\,, \qquad \text{where } B'(t) = e^{A(t)}f(t) \text{ and } c \text{ is any constant}\,. \tag{2.7}$$

Therefore a general solution of (2.5) is given by the family

$$y = e^{-A(t)}B(t) + e^{-A(t)}c\,. \tag{2.8}$$

Notice that because $B'(t) = e^{A(t)}f(t)$, for a homogeneous equation we can set $B(t) = 0$. In that case this recipe recovers the recipe for homogeneous linear equations given by (2.4).

The key to understanding recipe (2.8) is to understand the equivalence of the normal form (2.5) and integrating factor form (2.6). This equivalence follows from the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{A(t)}y\Big) = e^{A(t)}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{A(t)}\Big)y = e^{A(t)}\frac{\mathrm{d}y}{\mathrm{d}t} + e^{A(t)}A'(t)y = e^{A(t)}\left(\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y\right).$$

This calculation shows that equation (2.6) is simply equation (2.5) multiplied by $e^{A(t)}$. Because the factor $e^{A(t)}$ is always positive, the equations are equivalent. We call $e^{A(t)}$ an *integrating factor* of equation (2.5) because after multiplying both sides of (2.5) by $e^{A(t)}$ the left-hand side can be written as the derivative of $e^{A(t)}y$. An integrating factor thereby allows us to reduce the linear equation (2.5) to the explicit equation (2.6).

Remark. The integrating factor $e^{A(t)}$ is just the reciprocal of a solution given by (2.4) to the associated homogeneous problem. The general solution (2.8) thereby has the form

$$y = Y_P(t) + Y_H(t)\,,$$

where $Y_P(t) = e^{-A(t)}B(t)$ is the particular solution of (2.5) obtained by setting $c = 0$ in (2.8) while $Y_H(t) = e^{-A(t)}c$ is the general solution given by (2.4) of the associated homogeneous problem. General solutions of higher-order nonhomogeneous linear equations share this structure.

Rather than using formula (2.8), we should find general solutions of first-order linear ordinary differential equations by simply retracing the steps by which (2.8) was derived. We illustrate this approach with the following examples.

Example. Find a general solution to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x + e^{2t}\,.$$

Solution. First bring the equation into the normal form

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = e^{2t}\,.$$

An integrating factor is $e^{A(t)}$ where $A'(t) = 3$. By setting $A(t) = 3t$, we then bring the equation into the integrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{3t}x\Big) = e^{3t}e^{2t} = e^{5t}\,.$$

By integrating both sides of this equation we obtain

$$e^{3t}x = \int e^{5t}\,\mathrm{d}t = \tfrac{1}{5}e^{5t} + c\,.$$

Therefore a general solution is given by

$$x = \tfrac{1}{5}e^{2t} + e^{-3t}c\,.$$

Members of this family are illustrated in Figure 2.1 below.

Example. Find a general solution to

$$(1+t^2)\frac{\mathrm{d}z}{\mathrm{d}t} + 4tz = \frac{1}{\big(1+t^2\big)^2}\,.$$

Solution. First bring the equation into the normal form

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{4t}{1+t^2}z = \frac{1}{\big(1+t^2\big)^3}\,.$$

An integrating factor is $e^{A(t)}$ where $A'(t) = 4t/(1+t^2)$. By setting $A(t) = 2\log(1+t^2)$, we see that

$$e^{A(t)} = e^{2\log(1+t^2)} = \Big(e^{\log(1+t^2)}\Big)^2 = \big(1+t^2\big)^2\,.$$

We then bring the differential equation into the integrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big((1+t^2)^2 z\Big) = \big(1+t^2\big)^2\frac{1}{\big(1+t^2\big)^3} = \frac{1}{1+t^2}\,.$$

By integrating both sides of this equation we obtain

$$(1+t^2)^2 z = \int \frac{1}{1+t^2}\,\mathrm{d}t = \tan^{-1}(t) + c\,.$$

Therefore a general solution is given by

$$z = \frac{\tan^{-1}(t)}{\big(1+t^2\big)^2} + \frac{c}{\big(1+t^2\big)^2}\,.$$

Members of this family are illustrated in Figure 2.2 below.

2.4. **Linear Initial-Value Problems.** In order to pick a unique solution from the family (2.8) we must impose an additional condition that determines c. We do this by again imposing an *initial condition* of the form

$$y(t_I) = y_I \,,$$

where t_I is called the *initial time* or *initial point* while y_I is called the *initial value* or *initial datum*. The combination of the differential equation (2.2) with this initial condition is

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = f(t) \,, \qquad y(t_I) = y_I \,. \tag{2.9}$$

This is a so-called *initial-value problem*. By imposing the initial condition upon the family (2.8) we see that

$$y(t_I) = e^{-A(t_I)}B(t_I) + e^{-A(t_I)}c = y_I \,,$$

whereby $c = e^{A(t_I)}y_I - B(t_I)$. Therefore if the primitives $A(t)$ and $B(t)$ exist then the unique solution of initial-value problem (2.9) is given by

$$y = e^{-A(t)+A(t_I)}y_I + e^{-A(t)}\big(B(t) - B(t_I)\big) \,. \tag{2.10}$$

Rather than using formula (2.10), we will solve initial-value problems by simply retracing the steps by which it was derived — namely, by first finding a general solution as we did in the previous section, and then by imposing the initial condition to evaluate c. Formula (2.10) shows us that this approach will always yield the solution. We illustrate this with the following examples.

Example. Solve the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x + e^{2t} \,, \qquad x(0) = 2 \,.$$

Solution. We showed above that a general solution of the differential equation is

$$x = \tfrac{1}{5}e^{2t} + e^{-3t}c \,.$$

By imposing the initial condition we find that

$$x(0) = \tfrac{1}{5}e^{0} + e^{0}c = \tfrac{1}{5} + c = 2 \,,$$

whereby $c = \frac{9}{5}$. Therefore the solution of the initial-value problem is

$$x = \tfrac{1}{5}e^{2t} + \tfrac{9}{5}e^{-3t} \,.$$

This solution is shown in Figure 2.1 below.

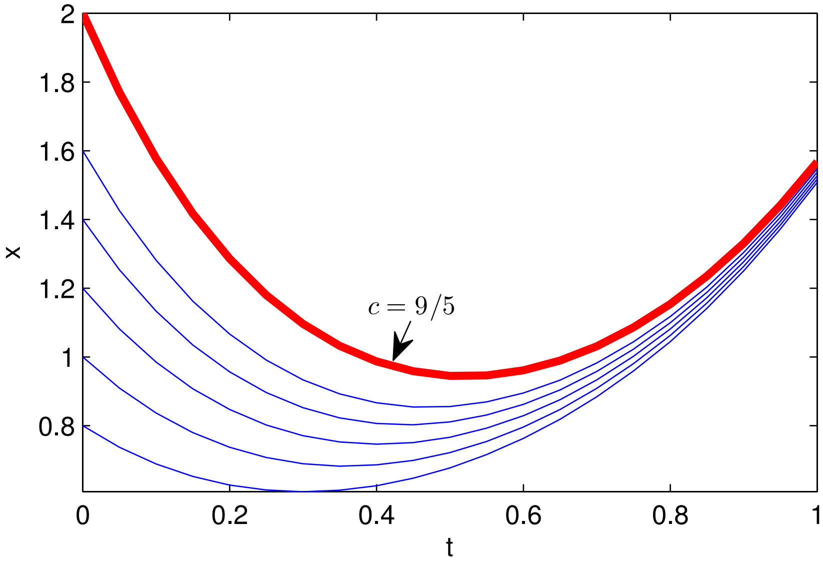

FIGURE 2.1. Plot of the general solution to the differential equation in the above example for various values of c. The solution for $c = \frac{9}{5}$ is shown in bold.

Example. Solve the initial-value problem

$$(1+t^2)\frac{\mathrm{d}z}{\mathrm{d}t} + 4tz = \frac{1}{\left(1+t^2\right)^2}\,, \qquad z(1) = \pi\,.$$

Solution. We showed above that a general solution of the differential equation is

$$z = \frac{\tan^{-1}(t)}{\left(1+t^2\right)^2} + \frac{c}{\left(1+t^2\right)^2}\,.$$

By imposing the initial condition we find that

$$z(1) = \frac{\tan^{-1}(1)}{\left(1+1^2\right)^2} + \frac{c}{\left(1+1^2\right)^2} = \frac{\frac{\pi}{4}}{2^2} + \frac{c}{2^2} = \frac{\pi}{16} + \frac{c}{4} = \pi\,,$$

whereby $c = \frac{15}{4}\pi$. Therefore the solution of the initial-value problem is

$$z = \frac{\tan^{-1}(t)}{\left(1+t^2\right)^2} + \frac{\frac{15}{4}\pi}{\left(1+t^2\right)^2}\,.$$

This solution is shown in Figure 2.2 below.

Remark. The hardest part of finding an explicit analytic solution of a first-order nonhomogeneous linear equation is usually finding a primitive $B(t)$ of $e^{A(t)}f(t)$. The above examples do not make this fact evident because they were chosen to make the task of finding $B(t)$ easy.

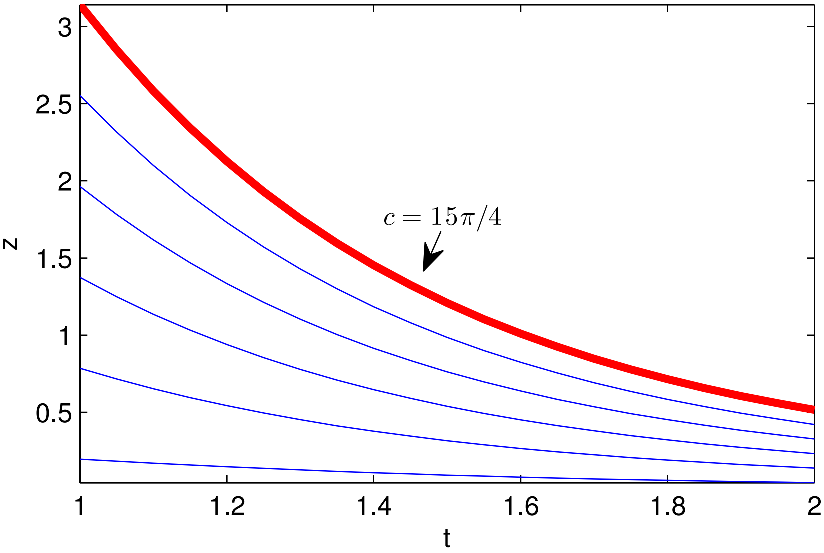

FIGURE 2.2. Plot of the general solution to the differential equation in the above example for various values of c. The solution for $c = \frac{15}{4}\pi$ is shown in bold.

2.5. **Theory for Linear Equations.** Even when we cannot find primitives $A(t)$ and $B(t)$ analytically, we can show that a solution exists whenever $a(t)$ and $f(t)$ are continuous over an interval (t_L, t_R) that contains the initial time t_I. In that case we can appeal to the Second Fundamental Theorem of Calculus to express $A(t)$ and $B(t)$ as the definite integrals

$$A(t) = \int_{t_I}^{t} a(r)\,\mathrm{d}r\,, \qquad B(t) = \int_{t_I}^{t} e^{A(s)} f(s)\,\mathrm{d}s\,.$$

For this choice of $A(t)$ and $B(t)$ we have $A(t_I) = B(t_I) = 0$, whereby formula (2.10) becomes

$$y = e^{-A(t)} y_I + e^{-A(t)} B(t) = e^{-A(t)} y_I + \int_{t_I}^{t} e^{-A(t)+A(s)} f(s)\,\mathrm{d}s\,. \tag{2.11}$$

The First Fundamental Theorem of Calculus implies that

$$A(t) - A(s) = \int_{s}^{t} a(r)\,\mathrm{d}r\,,$$

whereby formula (2.11) can be expressed as

$$y = \exp\biggl(-\int_{t_I}^{t} a(r)\,\mathrm{d}r\biggr) y_I + \int_{t_I}^{t} \exp\biggl(-\int_{s}^{t} a(r)\,\mathrm{d}r\biggr) f(s)\,\mathrm{d}s\,. \tag{2.12}$$

This shows that if a and f are continuous over an interval (t_L, t_R) that contains t_I then the initial-value problem (2.9) has a unique solution over (t_L, t_R) given by formula (2.12). We state this result as a theorem.

Theorem 2.1. *Let $a(t)$ and $f(t)$ be functions defined over the open interval (t_L, t_R) that are also continuous over (t_L, t_R).*

Then for every inital time t_I in (t_L, t_R), and every initial value y_I there exists a unique solution $y = Y(t)$ to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = f(t)\,, \qquad y(t_I) = y_I\,, \tag{2.13}$$

that is defined over (t_L, t_R).

Moreover, this solution is continuously differentiable and is given by formula (2.12).

When $a(t)$ and $f(t)$ are given by elementary functions, they will be continuous wherever they are defined. In that case we can identify the interval of definition for the solution of the initial-value problem (2.13) by simply looking at $a(t)$ and $f(t)$. This is because if $Y(t)$ is the solution of the initial value problem (2.13) then its interval of definition will be (t_L, t_R) whenever:

- the initial time t_I is in (t_L, t_R);
- the coefficient $a(t)$ and forcing $f(t)$ are continuous over (t_L, t_R);
- either the coefficient $a(t)$ or the forcing $f(t)$ is not defined at $t = t_L$;
- either the coefficient $a(t)$ or the forcing $f(t)$ is not defined at $t = t_R$.

The first two bullets along with the formula (2.12) imply that the interval of definition will be at least (t_L, t_R). The last two bullets along with the Definition 1.1 of a solution imply that the interval of definition can be no bigger than (t_L, t_R) because the equation is not defined at both $t = t_L$ and $t = t_R$. This argument can be applied when either $t_L = -\infty$ or $t_R = \infty$.

Example. Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \cot(t)\,z = \frac{1}{\log(t^2)}\,, \qquad z(4) = 3\,.$$

Solution. This is a nonhomogeneous linear equation that is already in normal form. Its coefficient $\cot(t)$ is undefined at $t = n\pi$ where n is any integer, and is continuous elsewhere. Its forcing $1/\log(t^2)$ is undefined at $t = 0$ and $t = \pm 1$, and is continuous elsewhere. Therefore the interval of definition for the solution of the initial-value problem is $(\pi, 2\pi)$ because:

- the initial time is $t = 4$, which is in $(\pi, 2\pi)$;
- both $\cot(t)$ and $1/\log(t^2)$ are continuous over $(\pi, 2\pi)$;
- $\cot(t)$ is not defined at $t = \pi$;
- $\cot(t)$ is not defined at $t = 2\pi$.

Remark. The solution of this initial-value problem is graphed in Figure 2.3. There you will see that the solution blows up as t either decreases towards π or increases towards 2π.

Example. Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \cot(t)\,z = \frac{1}{\log(t^2)}\,, \qquad z(2) = 3\,.$$

Solution. This is the same nonhomogeneous linear equation that appeared in the previous example. The interval of definition for the solution of the initial-value problem is $(1, \pi)$ because:

- the initial time is $t = 2$, which is in $(1, \pi)$;
- both $\cot(t)$ and $1/\log(t^2)$ are continuous over $(1, \pi)$;
- $1/\log(t^2)$ is not defined at $t = 1$;
- $\cot(t)$ is not defined at $t = \pi$.

Remark. The solution of this initial-value problem is graphed in Figure 2.3. There you will see that the derivative of the solution blows up as t decreases towards 1 and that the solution itself blows up as t increases towards π.

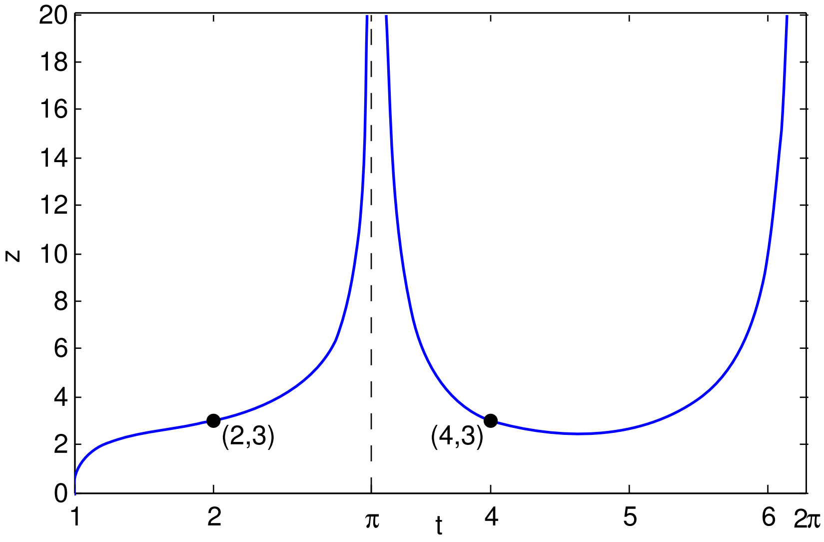

Figure 2.3. Solutions of $z' + \cot(t)z = 1/\log(t^2)$ that satisfy the initial conditions $z(2) = 3$ and $z(4) = 3$. The interval of definitions are $(1, \pi)$ and $(\pi, 2\pi)$ respectively. Notice that $z(t)$ blows up as t approaches π and 2π, while $z(t)$ does not blow up but $z'(t)$ does blow up as t approaches 1.

Remark. If $y = Y(t)$ is a solution of (2.8) whose interval of definition is (t_L, t_R) then this does not mean that $Y(t)$ will become undefined at either $t = t_L$ or $t = t_R$ when those endpoints are finite. For example, $y = t^4$ solves the initial-value problem

$$t\,\frac{\mathrm{d}y}{\mathrm{d}t} - 4y = 0\,, \qquad y(1) = 1\,,$$

and is defined for every t. However, the interval of definition is just $(0, \infty)$ because the initial time is $t = 1$ and normal form of the equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{4}{t}\,y = 0\,,$$

the coefficient of which is undefined at $t = 0$.

Remark. It is natural to ask why we do not extend our definition of solutions so that $y = t^4$ is considered a solution of the initial-value problem in the preceding remark for every t. For example, we might say that $y = Y(t)$ is a solution provided it is differentiable and satisfies the above equation rather than its normal form. However by this definition the function

$$Y(t) = \begin{cases} t^4 & \text{for } t \geq 0 \\ ct^4 & \text{for } t < 0 \end{cases}$$

also solves the initial-value problem for any c. This shows that because the equation breaks down at $t = 0$, there are many ways to extend the solution $y = t^4$ to $t < 0$. We avoid such complications by requiring the normal form of the equation to be defined.

EXERCISES ON LINEAR EQUATIONS

(1) Which of the following equations are linear? For the ones that aren't, explain where the problem lies. For the ones that are, put them in linear normal form if they aren't already.
  (a) $y' = ty - 7$
  (b) $ty' + y^2 = e^t$
  (c) $yy' = 5t - \sin(y)$
  (d) $\frac{y'}{t} + \tan(t)y = 1$
  (e) $y' + \frac{t}{y} = t^2$
  (f)
$$\sin(t)y' - \frac{\log(t^2) + \frac{y}{t}}{1 + e^{2t}} = y'$$

Solution

For #2–#5, find a general solution for the following homogeneous differential equations.

(2) $\dot{y} + y = 0$

Short Answer
Solution

(3) $tw' + 2w = 0$

Short Answer
Solution

(4) $\cos(x)y' = -\sin(x)y$

Short Answer
Solution

(5) $(t+1)y' + 3y = 0$

Short Answer
Solution

For #6 – #11, find a general solution for the following nonhomogeneous differential equations.

(6) $w' - 4w = 7$

Short Answer
Solution

(7) $\dot{y} + \frac{1}{t} \cdot y = 3$

Short Answer
Solution

(8) $y' - 2y = xe^{2x}$

Short Answer
Solution

(9) $y' + y = (1+t)\sin(t) - ty'$

Short Answer
Solution

(10) $t\log(t)z' + z = 2\log(t)$

Short Answer
Solution

(11) $y' + p(t)y = 1$

Short Answer
Solution

For #12 – #18, Solve the following initial value problems. What is the largest interval containing the initial conditions on which your solution is defined?

(12) $y' + 3y = e^t, \;\; y(0) = 4$

Short Answer
Solution

(13)

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{z}{\sin(t)} = \frac{1}{t^2 - 25} \text{ where } z(4) = 2$$

Short Answer
Solution

(14)

$$\tan(t)y' - y = \frac{1}{1-t} \text{ where } y(2) = 1$$

Short Answer
Solution

(15) $y' + 2ty = e^{-t^2}, \;\; y(0) = 1$

Short Answer
Solution

(16) $\theta' + \frac{1}{t}\theta = \frac{1}{t^2 - 4}, \;\; \theta(\sqrt{5}) = 2$

Short Answer
Solution

(17)

$$x' - x/(e^{-t} - 1) = \tan(t) \text{ where } x(1) = 3$$

Short Answer
Solution

(18) $\frac{y'}{\cos(t)} + y = 1, \;\; y(\pi) = 3$

Short Answer
Solution

(19) $v' + \frac{1}{t-1}v = \frac{1}{t^3 - t}, \;\; v(\frac{1}{2}) = -4$

Short Answer

Solution

(20) This exercise is a check that the recipe given does indeed give solutions to first order linear differential equations. The equation we want to verify is

$$y'(t) + a(t)y(t) = b(t),$$

and the solution the recipe gives us, for an arbitary constant c, is

$$y(t) = e^{-\int a(t)\,dt} \int b(t)e^{\int a(t)\,dt}\,dt + ce^{-\int a(t)\,dt}.$$

[*Hint.* If you're a go-getter, it is possible to check this by computing y' and ay, then adding them to see that everything cancels except b. It might be less onerous if you define

$$\mu(t) = e^{\int a(t)\,dt},$$

which makes the solution look a little bit less daunting:

$$y(t) = \frac{1}{\mu(t)} \int b(t)\mu(t)\,dt + \frac{c}{\mu(t)}.$$

Figure out what $\mu'(t)$ is as a first step.]

Solution

(21) Suppose that $y_1(t)$ is a solution to the differential equation $y' + a(t)y = 0$ and that $y_2(t)$ is a solution to $y' + a(t)y = b(t)$. Check that $y_1(t) + y_2(t)$ is also a solution to $y' + a(t)y = b(t)$. This is an important feature of linear equations: solutions can be built up from smaller pieces in some circumstances.

Solution

(22) As noted, the previous problem heavily relies on the fact that the differential equation $y' + a(t)y = b(t)$ is a linear equation. Consider the differential equation

$$(y(t))^2 \cdot y''(t) + (y'(t))^3 = 0.$$

Show that both $y_1(t) = 1$ and $y_2(t) = e^{-t}$ are solutions, but nonetheless $1 + e^{-t}$ is not a solution. (Of course, this equation is wildly nonlinear.)

Solution

(23) The recipe given for first-order differential equations doesn't really require us to find antiderivatives at all the intermediate steps, which is good news because there are not many functions $f(t)$ for which $\int \exp\big(\int f(t)\,dt\big)\,dt$ can be calculated. Find a general solution to the differential equation

$$e^t\, y'(t) - 5t^2\, y(t) = \log(t + 3).$$

Your answer will involve indefinite integrals that you shouldn't be able to evaluate.

Solution

(24) Some higher-order differential equations are actually first-order linear if you look at them in the right way. Use the substitutions suggested to solve the following equations. (The solution to the linear equation will be an explicit equation which then can be solved. You will have multiple constants.)

(a) $y'' + 5y' = t$; use $w = y'$
(b) $y''' + \frac{2}{t}y'' = 6$; use $v = y'$ and $w = v'$ and rewrite in terms of w

Solution

(25) $y' + (\alpha - 1)y = 4$

Answer the following for the above differential equation:

(a) Find a general solution.

(b) For what values of α does a solution exist?

(c) For what values of α will the solution tend to infinity as $t \to \infty$?

(d) If $\alpha = 1$ why do we not get $y = 4t$?

Short Answer
Solution

(26) What is

$$x(t) = \frac{-1}{3} + 3e^{3t}$$

a solution to? What is the integrating factor?

Short Answer
Solution

NAVIGATION TO OTHER CHAPTERS

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Ordinary Differential Equations